Boundary Entropy of One-Dimensional Quantum Systems at Low Temperature

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The boundary β function generates the renormalization group acting on the universality classes of one-dimensional quantum systems with boundary which are critical in the bulk but not critical at the boundary. We prove a gradient formula for the boundary β function, expressing it as the gradient of the boundary entropy s at fixed nonzero temperature. The gradient formula implies that s decreases under renormalization, except at critical points (where it stays constant). At a critical point, the number $\exp(s)$ is the "ground-state degeneracy," g, of Affleck and Ludwig, so we have proved their long-standing conjecture that g decreases under renormalization, from critical point to critical point. The gradient formula also implies that s decreases with temperature, except at critical points, where it is independent of temperature. It remains open whether the boundary entropy is always bounded below.

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For a one-dimensional quantum critical system with a boundary, $\ln Z = \ln \mathrm{Tr} \exp(-\beta H)$ takes the universal form [1] $(c\pi/6)(L/\beta) + \ln g$, where H is the Hamiltonian, $\beta = 1/T$ is the inverse temperature, $L \gg \beta$ is the length, c is the numerical coefficient of the bulk conformal anomaly, and g is the "universal noninteger ground-state degeneracy" at the boundary (using natural units in which $\hbar = k = v = 1$, v being the velocity of "light"). This formula applies in the limit of large L. The number g is an invariant of the universality class of the critical boundary condition. It was conjectured that g decreases from critical point to critical point under renormalization [1,2].

For a 1D quantum system that is critical in the bulk but is *not* critical at the boundary, the logarithm of the partition function at low temperature can be written in the form $\ln Z = (c\pi/6)(L/\beta) + \ln z_L$, and the boundary partition function z can be defined as $\lim_{L\to\infty} z_L$. That is, the partition function takes the universal form

$$\ln Z = (c\pi/6)(L/\beta) + \ln z \qquad (1)$$

up to corrections that vanish in the limit $L \to \infty$.

Given that the bulk system is critical and that $L = \infty$, the only dimensionful parameter is the temperature T. The logarithm of the boundary partition function is thus a function $\ln z(\mu\beta)$ that depends only on the temperature, in units of μ, where μ is a small temperature that sets the renormalization scale (or, equivalently, a small energy or inverse time or inverse distance). The total entropy then takes the universal form

$$S = (1 - \beta\partial/\partial\beta)\ln Z = (c\pi/3)(L/\beta) + s(\mu\beta), \qquad (2)$$

where $s(\mu\beta) = (1 - \beta\partial/\partial\beta)\ln z$ is the boundary entropy. At a critical point, s is equal to the constant $\ln g$.

We prove here a gradient formula

$$g_{ab}(\lambda)\beta^b(\lambda) = -\partial s/\partial\lambda^a, \qquad (3)$$

where the λ^a form a complete set of boundary coupling constants, $g_{ab}(\lambda)$ is a certain metric on the space of all boundary conditions, and $\beta^a(\lambda)$ is the boundary β function. It follows directly from the gradient formula that $\mu\partial s/\partial\mu = \beta^a\partial_a s = -g_{ab}\beta^a\beta^b$, so s decreases under the renormalization group except where $\beta^a = 0$ at the critical points. The gradient formula eliminates the possibility of esoteric asymptotic behavior under renormalization. Recurring trajectories such as limit cycles are excluded. The g conjecture for the renormalization group (RG) flows between critical points follows as a corollary of the gradient formula.

The gradient formula implies equally that the boundary entropy decreases with temperature, $\beta\partial s/\partial\beta = \mu\partial s/\partial\mu < 0$. The total entropy S obviously decreases with temperature because $\partial S/\partial\beta = -\beta\langle(H - \langle H\rangle)^2\rangle$. However, the decrease of the bulk contribution $(\pi/3)(L/\beta)c$ masks the change in s, so it is not obvious that the boundary entropy by itself must decrease with temperature. The gradient formula implies that it does. It follows that the thermodynamic boundary energy also decreases with temperature, $\mu\beta\partial u/\partial\beta = \partial s/\partial\beta < 0$.

Complete control over the possible behavior at asymptotically low temperature is still lacking because we do not prove that s is bounded below. If s is bounded below, then the system must go to a critical point at zero temperature. Of course, the total entropy S of any system is bounded below, as long as the system is of finite size. So, for any finite size L, $s_L = S - (c\pi/3)(L/\beta)$ is bounded below as $T \to 0$. However, the lower bound can descend without limit as $L \to \infty$, so $s = \lim_{L\to\infty} s_L$ is not necessarily bounded below as $T \to 0$. It still remains to be proved that $s(\mu\beta)$ is bounded below as $T \to 0$. If the boundary entropy is bounded below, then the boundary energy is also bounded below.

The gradient formula that we prove is mathematically equivalent to a gradient formula conjectured in string theory [3–7]. Evidence was given for the string theory conjecture [4,6,8], but the formula was never proved. It has been claimed that a proof was given in Ref. [7], but it was assumed there that the boundary β function $\beta^a(\lambda)$ is linear in the coupling constants λ^a. This is an invalid assumption. The β function cannot be linearized when there are marginally relevant couplings or, more generally, whenever resonance conditions occur (as discussed, for example, in Ref. [6]). Moreover, the conjectured string gradient formula is expressed in unphysical quantities, in terms of unnormalized correlation functions. Our contribution is to express the gradient formula in terms of normalized correlation functions and the boundary entropy, which are physical quantities, and to prove the formula using physical properties of the 1D quantum system. Some of the ideas used in the proof can be found in the string theory work [3,6]. The rewriting of the conjectured string gradient formula is based on an idea that is implicit in Ref. [7] and was mentioned explicitly to us [9]. Here, to avoid distracting from the physical meaning, we first prove the gradient formula in physical terms and only afterwards explain the connection to the string conjecture.

The proof of the gradient formula applies to *all* local 1D quantum systems. It uses only the basic principles of quantum mechanics and locality. The gradient formula must therefore hold in every local 1D quantum mechanical model. The point of proving a result such as the gradient formula is to give reliable theoretical information about what is physically possible. For instance, when building devices out of low temperature 1D quantum systems joined at boundaries, it will be useful to know in advance, with certainty, what kinds of boundary behaviors are possible. It will be useful to know that the boundary must always behave as a thermodynamic system, except that it does not obey the third law. Proof also reveals what must be done to evade the theoretical limits. The gradient formula itself is not likely to be avoidable, since the proof depends only on the basic principles of quantum mechanics and renormalization, assuming only the existence of a local stress-energy tensor, which is assured by microscopic locality. Rather, attention is directed towards exotic systems, where the metric $g_{ab}(\lambda)$ degenerates, or where s is infinite [10,11], or even where s might not be bounded below, if this cannot be proved impossible. A lower bound on s would have to depend on the details of the bulk system. The bound could not be uniform, not a function of c alone. This can be seen in the critical $c = 1$ Gaussian model, where the values of g depend on the marginal coupling constant of the bulk model and can become arbitrarily close to zero [12].

The equilibrium observables of the system live on the cylindrical Euclidean space-time, periodic in Euclidean time with period β. The spacetime coordinates are $x^\mu = (x, \tau)$, where $0 \le x < L$ and $\tau \sim \tau + \beta$. The boundary is at $x = 0$. The stress-energy tensor $T_{\mu\nu}(x, \tau)$ expresses the response of the system to an infinitesimal local variation of the metric, $g_{\mu\nu} \to \delta_{\mu\nu} + \delta g_{\mu\nu}(x, \tau)$,

$$\delta \ln Z = (1/2) \iint \mu^2 d\tau dx \langle \delta g_{\mu\nu} T^{\mu\nu}(x, \tau)\rangle. \qquad (4)$$

To avoid potential confusion, we stress that the metric here is not dynamical. The metric describes the background geometry in which the 1D quantum system exists.

We specialize to $1 + 1$ dimensions the general analysis of the stress-energy tensor in space-times with boundary [13]. The stress-energy tensor can be written as a bulk part plus a boundary part:

$$T_{\mu\nu} = T^{\mathrm{bulk}}_{\mu\nu}(x, \tau) + \delta(\mu x) t_{\mu\nu}(\tau). \qquad (5)$$

There could also be a boundary operator proportional to $\delta'(x)\mathbf{1}$, but the identity operator makes no contribution to connected correlation functions, so we can ignore it.

The conservation equations follow from invariance of the physics under localized coordinate reparametrizations $\delta x^\mu = v^\mu(x, \tau)$ where the vector field v^μ is tangent to the boundary, i.e., $v^x(0, \tau) = 0$. A nearly critical system is insensitive to the detailed coordinate labeling of its local degrees of freedom (e.g., the coordinate labeling of lattice sites in a lattice model). The coordinate reparametrization is equivalent to a change in the metric tensor $\delta g_{\mu\nu} = \partial_\mu v_\nu + \partial_\nu v_\mu$. Plugging this into the formula for $\delta \ln Z$ and setting the variation to zero, we obtain, after integration by parts, the bulk conservation equation $\partial^\mu T^{\mathrm{bulk}}_{\mu\nu} = 0$ and also $\int \mu d\tau (\mu T^{\mathrm{bulk}}_{x\tau} v^\tau - t_{\mu\nu}\partial^\mu v^\nu) = 0$ at the boundary, which is equivalent to the boundary conservation equations $t_{xx} = t_{x\tau} = t_{\tau x} = 0$ and

$$\mu T^{\mathrm{bulk}}_{x\tau}(0, \tau) + \partial_\tau \theta = 0, \qquad (6)$$

where $\theta(\tau) \equiv t_{\tau\tau}(\tau)$. The boundary operator θ was described in Ref. [14]. The trace of the stress-tensor is

$$T^\mu_\mu = \Theta(x, \tau) = \Theta_{\mathrm{bulk}}(x, \tau) + \delta(\mu x)\theta(\tau). \qquad (7)$$

The system is critical in the bulk, so $\Theta_{\mathrm{bulk}}(x, \tau) = 0$ up to contact terms. The full trace is $\Theta = \delta(\mu x)\theta(\tau)$, entirely a boundary operator.

The space of boundary conditions is parametrized by the coupling constants λ^a, which couple to the renormalized local boundary fields ϕ_a, $\partial_a \ln Z = \int \mu d\tau \langle \phi_a(\tau)\rangle$. The boundary trace θ can be decomposed into a linear combination of the boundary fields and the identity operator

$$\theta = \beta^a(\lambda)\phi_a + h(\lambda)\mathbf{1}, \qquad (8)$$

where $\beta^a(\lambda)$ is the boundary β function. We will not have to worry about the term $h(\lambda)\mathbf{1}$ because θ will only appear within connected correlation functions.

The foregoing are operator statements. In correlation functions, the stress-energy tensor will also have contact terms. The generator of dilatations $\delta g_{\mu\nu} = 2(\delta\mu/\mu)\delta_{\mu\nu}$ is $\iint \mu^2 d\tau dx \Theta(x, \tau)$, so the renormalization group equation for $\ln Z$ is

$$(\mu\partial/\partial\mu)\ln Z = \iint \mu^2 d\tau dx \langle\Theta(x,\tau)\rangle = \beta^a\partial_a \ln Z + \mu\beta h(\lambda), \quad (9)$$

where Θ is given by Eq. (7). For the one-point functions,

$$(\mu\partial/\partial\mu)\langle\phi_b(\tau_1)\rangle = \iint \mu^2 d\tau dx \langle\Theta(x,\tau)\phi_b(\tau_1)\rangle_c = (\beta^a\partial_a\delta_b^c + \gamma_b^c - \delta_b^c)\langle\phi_c(\tau_1)\rangle, \quad (10)$$

where the coefficients $\gamma_b^c - \delta_b^c$ come from contact terms of $\Theta_{\rm bulk}$ and θ with ϕ_b. Because of the contact terms, $\Theta_{\rm bulk}$ cannot be omitted. The identity $\gamma_b^a = \partial_b\beta^a$ follows from $[\mu\partial/\partial\mu, \partial_a] = 0$, which in turn follows from the definition of the λ^a as the coupling constants renormalized at scale μ. We will need one last property of the stress-energy tensor: that $T_{\mu\nu}^{\rm bulk}(x,\tau)$ decays as $\exp(-4\pi x/\beta)$ in connected correlation functions far from the boundary. This follows from the fact that when x is far from the boundary, $T_{\mu\nu}^{\rm bulk}(x,\tau)$ behaves as in the bulk theory without boundary.

We prove the gradient formula, Eq. (3), with the metric on the space of boundary conditions given by

$$g_{ab}(\lambda) = \int \mu d\tau_1 \int \mu d\tau \langle\phi_a(\tau_1)\phi_b(\tau)\rangle_c f(\tau - \tau_1), \quad (11)$$

where $f(\tau) = 1 - \cos(2\pi\tau/\beta)$. This is essentially the metric proposed in Ref. [3], except that Ref. [3] used the unnormalized, full two-point function, while we use the normalized, connected two-point function. Because we are using the connected two-point function, we can write

$$g_{ab}\beta^b = \int \mu d\tau_1 \int \mu d\tau \langle\phi_a(\tau_1)\theta(\tau)\rangle_c f(\tau - \tau_1). \quad (12)$$

The identity component of θ makes no contribution to the connected two-point function. Let us deal with the term containing the cosine:

$$A_a(\tau_1) \equiv \int \mu d\tau \langle\phi_a(\tau_1)\theta(\tau)\rangle_c(-\cos[2\pi(\tau-\tau_1)/\beta]) = \int \mu d\tau \langle\phi_a(\tau_1)\partial_\tau\theta(\tau)\rangle_c 2v^\tau(0,\tau), \quad (13)$$

where we integrate by parts on the boundary and define $v^\tau(0,\tau) \equiv (\beta/4\pi)\sin[2\pi(\tau-\tau_1)/\beta]$ as a tangent vector field on the boundary. The correlation functions are distributions, so integration by parts is justified. By the boundary conservation law Eq. (6),

$$A_a(\tau_1) = \int \mu d\tau \langle\phi_a(\tau_1)\mu T_{x\tau}^{\rm bulk}(\tau)\rangle_c(-2)v^\tau(0,\tau). \quad (14)$$

Next, we extend the boundary vector field $v^\tau(0,\tau)$ to a conformal Killing vector field $v^\mu(x,\tau)$ in the bulk. That is, $v^x(0,\tau) = 0$ and $\partial_\mu v_\nu + \partial_\nu v_\mu = g_{\mu\nu}\partial_\sigma v^\sigma$. Such a vector field is most easily found as an analytic vector field $v^w = (2\pi/\beta)(v^x + iv^\tau)$ in the complex coordinate $w = 2\pi(x + i\tau)/\beta$, $v^w = [\exp(w - w_1) - \exp(-w + w_1)]/4$. Then $\partial_\sigma v^\sigma = \cos[2\pi(\tau-\tau_1)/\beta]\cosh(2\pi x/\beta)$. Now we integrate by parts in the bulk, using the bulk conservation equation, to obtain

$$A_a(\tau_1) = \iint \mu^2 d\tau dx \langle\phi_a(\tau_1)T_{\mu\nu}^{\rm bulk}(x,\tau)\rangle_c 2\partial^\mu v^\nu. \quad (15)$$

There is no boundary term at large x because of the decay condition $T_{\mu\nu}^{\rm bulk}(x,\tau) \sim \exp(-4\pi x/\beta)$. Then we use the fact that v^μ is a conformal Killing vector to write

$$A_a(\tau_1) = \iint \mu^2 d\tau dx \langle\phi_a(\tau_1)\Theta_{\rm bulk}(x,\tau)\rangle_c\partial_\sigma v^\sigma. \quad (16)$$

Finally, we can approximate $\partial_\sigma v^\sigma \sim 1$ because $\Theta_{\rm bulk} = 0$, except for contact terms. The boundary operator $\phi_a(\tau_1)$ is renormalizable, and $\Theta_{\rm bulk}$ has dimension 2, so the most singular contact terms in the two-point function are of the form $\delta(x)\delta'(\tau-\tau_1)$ and $\delta'(x)\delta(\tau-\tau_1)$. But $\partial_\sigma v^\sigma(x,\tau) - 1$ vanishes to second order at $x = 0$, $\tau = \tau_1$, so there is no error. Thus

$$A_a(\tau_1) = \iint \mu^2 d\tau dx \langle\phi_a(\tau_1)\Theta_{\rm bulk}(x,\tau)\rangle_c. \quad (17)$$

Using Eqs. (17) and (7) in Eq. (12), we arrive at

$$g_{ab}\beta^b = \int \mu d\tau_1 \iint \mu^2 d\tau dx \langle\phi_a(\tau_1)[\Theta_{\rm bulk}(x,\tau) + \delta(\mu x)\theta(\tau)]\rangle_c = \int \mu d\tau_1 \iint \mu^2 d\tau dx \langle\phi_a(\tau_1)\Theta(x,\tau)\rangle_c = \int \mu d\tau_1(\mu\partial/\partial\mu)\langle\phi_a(\tau_1)\rangle = (\mu\partial/\partial\mu - 1)\partial_a \ln Z = -\partial_a s, \quad (18)$$

which is the gradient formula.

Each element of the gradient formula is covariant under renormalization. The boundary entropy s is covariant, $\mu\partial s/\partial\mu = \beta^a\partial_a s$, even though the partition function is not [see Eq. (9)]. Using Eq. (9),

$$(\mu\partial/\partial\mu - \beta^a\partial_a)s = (\mu\partial/\partial\mu - \beta^a\partial_a)(1 - \mu\partial/\partial\mu)\ln Z = (1 - \mu\partial/\partial\mu)(\mu\beta h) = 0. \quad (19)$$

That is, the entropy is not sensitive to a shift of the ground state energy. The covariance of β^a is just its μ independence. The metric g_{ab} is covariant under renormalization because it is defined in terms of normalized, connected correlation functions, in Eq. (11).

To show that the metric g_{ab} is positively definite, we need only remark that $g_{ab}\delta\lambda^a\delta\lambda^b$ is given in Eq. (11) as a positive two-point function of $\phi = \delta\lambda^a \phi_a$, integrated against a positive function.

The cosine term in the metric plays a twofold role. On the one hand, it provides the $\Theta_{\rm bulk}$ term in the correlation function of Θ with the boundary operator. On the other hand, the cosine term renders the metric independent of contact terms in the two-point functions of the boundary operators. Such terms could spoil the positivity of the metric. The metric, as defined by Eq. (11), is independent of contact terms. During the proof of the gradient formula, we split it into two parts, each of which does depend on the contact terms. At that point, the two-point functions have to be treated as distributions. In the end, when the two terms are joined together, the result is independent of the contact terms. The technical roles of the cosine term are evident, but we do not see a deeper meaning. The cosine first appeared in the string theory metric proposed in Ref. [3]. But the proposal was not natural in string theory, as it involved integrating dimension zero fields. So we still do not see a natural interpretation of the cosine term.

The conjectured string theory gradient formula involves an additional boundary coupling constant λ^0 which couples to the identity operator $\phi_0 = \mathbf{1}$. The string partition function is $Z_s = \exp(\mu\beta\lambda^0) z(\mu\beta)$, where $z(\mu\beta)$ is the boundary partition function, from Eq. (1). The string β function, β_s^a, is the ordinary β^a for the ordinary coupling constants; plus, from Eq. (9), $\beta_s^0 = \lambda^0 + h(\lambda)$. The conjectured string theory gradient formula is $G^s_{ab}\beta_s^b = -\partial_a g_s$, where $g_s = (1 - \mu\partial/\partial\mu) Z_s$ and the string metric is

$$G^s_{ab}(\lambda) = \int \mu d\tau_1 \int \mu d\tau Z_s \langle \phi_a(\tau_1)\phi_b(\tau)\rangle f(\tau - \tau_1). \tag{20}$$

These string formulas are unphysical when applied to 1D quantum systems. No physical probe couples to the identity operator $\phi_0 = \mathbf{1}$, so λ^0 is not a physical coupling constant. Unnormalized correlation functions are not measurable. Changes in g_s are not locally measurable because g_s is constructed from z, not $\ln Z$. On the other hand, all of the elements of the physical gradient formula, Eq. (3), can by measured by local operations at the boundary of the 1D system. The string gradient formula *is* formally sensible from the string theory perspective. The λ^a are the wave modes of space-time fields; λ^0 is the zero mode of the tachyon field. The equation $\beta_s^a = 0$ has the form of a space-time equation of motion. The function $g_s(\lambda)$ has the form of a space-time action.

The unphysical parameter λ^0 can be eliminated by extremizing g_s [7,9]. We carry out this idea. We calculate that $\partial g_s/\partial\lambda^0 = 0$ at $\lambda^0 = a_0$, $\mu\beta a_0 = -\mu\partial \ln z/\partial\mu$. We calculate that, at $\lambda^0 = a_0$, $g_s = Z_s = z\exp(\mu\beta a_0)$, which is the physical quantity $\exp(s)$. It now becomes straightforward to show the equivalence between the string gradient formula and the physical gradient formula. The string gradient formula is trivial in the direction of λ^0 and is precisely the physical formula on the subspace $\lambda^0 = a_0$. To be explicit, the components of the string metric are $G^s_{00} = (\mu\beta)^2 Z_s$, $G^s_{0b} = \mu\beta Z_s \partial_a \ln Z_s$, and $G^s_{ab} = Z_s(g_{ab} + \partial_a \ln Z_s \partial_b \ln Z_s)$, where the indices a, b now range only over the physical coupling constants. The string gradient formula splits into two equations: $G^s_{00}\beta_s^0 + G^s_{0b}\beta^b = -\partial_0 g_s$ and $G^s_{a0}\beta_s^0 + G^s_{ab}\beta^b = -\partial_a g_s$. The first is satisfied identically; it is just the RG equation for Z_s, $\mu\partial Z_s/\partial\mu = (\beta^a\partial_a + \mu\beta\beta_s^0) Z_s$. The second equation, after substituting and then using the RG equation for Z_s, becomes $Z_s g_{ab}\beta^b = Z_s(-1 + \mu\partial/\partial\mu)\partial_a \ln Z_s$, which is exactly the physical gradient formula, since $\partial_a \ln Z_s = \partial_a \ln Z$. So, by proving the physical gradient formula, we have also proven the string gradient formula.

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[1] I. Affleck and A.W. Ludwig, Phys. Rev. Lett. **67**, 161 (1991).
[2] I. Affleck and A.W. Ludwig, Phys. Rev. B **48**, 7297 (1993).
[3] E. Witten, Phys. Rev. D **46**, 5467 (1992).
[4] E. Witten, Phys. Rev. D **47**, 3405 (1993).
[5] S. Shatashvili, Phys. Lett. B **311**, 83 (1993).
[6] S. Shatashvili, Alg. Anal. **6**, 215 (1994).
[7] D. Kutasov, M. Marino, and G. Moore, J. High Energy Phys. 10 (2000) 45.
[8] A. Konechny, hep-th/0310258 [IJMP (to be published)].
[9] G. Moore (private communication).
[10] See two notes on conformal boundary conditions for the $c = 1$ Gaussian model by D. Friedan, http://www.physics.rutgers.edu/pages/friedan.
[11] S.-L. Tseng, J. High Energy Phys. 04 (2002) 51.
[12] S. Elitzur, E. Rabinovici, and G. Sarkissian, Nucl. Phys. B **541**, 246 (1999).
[13] D. M. McAvity and H. Osborn, Nucl. Phys. B **406**, 655 (1993).
[14] S. Ghoshal and A. Zamolodchikov, Int. J. Mod. Phys. A **9**, 3841 (1994); *ibid.* **9**, 4353(E) (1994).