Lower Bound on the Entropy of Boundaries and Junctions in (1 + 1)-Dimensional Quantum Critical Systems

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A lower bound is derived for the boundary entropy $s = \ln g$ of a $(1 + 1)$-dimensional quantum critical system with boundary under the conditions $c \geq 1$ on the bulk conformal central charge and $\Delta_1 > (c - 1)/12$ on the most relevant bulk scaling dimension. This is the first general restriction on the possible values of g for bulk critical systems with $c \geq 1$.

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A $(1 + 1)$-dimensional quantum critical system is described by a 2D conformal field theory (the bulk CFT). A critical boundary is described by a conformally invariant boundary condition on the bulk CFT. The combination—a bulk CFT with a conformally invariant boundary condition—is a boundary CFT [1]. Critical junctions in critical quantum circuits are described by boundary CFTs. For an N-wire junction, the bulk CFT is the N-fold product of the CFTs describing the individual wires. The critical junction is described by a conformal boundary condition for the product CFT. In string theory, branes in spacetime are described by conformal boundary conditions on the string world sheet.

Affleck and Ludwig [2] defined a number g for each boundary CFT—the universal noninteger ground state degeneracy. The entropy localized in the boundary—the boundary entropy—is $\ln g$. It is defined as the total entropy of the system minus the bulk entropy $\pi c L/6\beta$, which is proportional to the length L in the limit of large L. The coefficient of L is determined by conformal invariance, in terms of the inverse temperature β and the conformal central charge c of the bulk CFT.

For $c < 1$, there is a complete classification of all possible boundary CFTs [3]. There is also a complete classification of conformal boundary conditions for the $c = 1$ Gaussian model [4–6]. Until now, no limitations have been known on the possible values that g can take for any other $c \geq 1$ bulk systems.

For noncritical boundary conditions in a bulk CFT, the boundary entropy s is defined in the same way by subtracting the universal bulk entropy from the total entropy. Now s depends on the temperature. Under a change of the thermal length scale β the effective boundary condition evolves along the boundary renormalization group flow (the boundary RG flow). The bulk system, being scale invariant, stays the same. A fixed point of the boundary RG flow is a boundary CFT. At a fixed point $s = \ln g$. It is not obvious that s decreases with decreasing temperature—that the second law of thermodynamics applies to the boundary—because of the subtraction of bulk entropy in the definition of s. In fact, the boundary entropy s does decrease along the boundary RG flow, so it decreases with temperature [7]. The result is actually stronger: the boundary RG beta function is the gradient of the function s on the space of boundary conditions. All that is missing to control the asymptotic low temperature behavior is a lower bound on s. Such a lower bound would be an analogue of the third law of thermodynamics. Again, the existence of a lower bound on s is nonobvious because of the subtraction of the bulk entropy. Unsuccessful attempts have been made to prove that s is bounded below [8]. Without a lower bound, we cannot exclude the possibility that s might decrease to $-\infty$ as the temperature drops to zero.

Here we prove a lower bound $g > g_B(c, \Delta_1)$ that applies to any $c \geq 1$ bulk system that has $\Delta_1 > (c - 1)/12$, where Δ_1 is the most relevant bulk scaling dimension. The proof assumes nothing about the boundary condition besides criticality and unitarity. The bound does not imply a boundary third law of thermodynamics, since it applies only to critical boundary conditions. It does imply that a noncritical boundary with entropy s below the bound cannot flow to a critical boundary condition at zero temperature. If such a system exists, its boundary entropy must necessarily decrease without limit towards $s = -\infty$ at zero temperature.

One of us has argued that critical quantum circuits are natural physical systems for asymptotically large scale quantum computers [9]. The quantum wires should be critical in the bulk, so that the low-energy excitations are protected against microscopic fluctuations by universality (the RG) and travel at uniform speed. The processing elements are to be the circuit junctions. A junction can be considered as a boundary condition on the CFT describing the independent wires entering it. A lower bound on $\ln g$ leads to an upper bound on the information capacity of the junction, giving a general constraint on the design of critical quantum circuits.

In string theory, g is the brane tension. The lower bound on the brane tension might be useful once it is extended to superconformal boundary CFTs and if the condition $\Delta_1 > (c-1)/12$ can be relaxed.

The modular duality formula for a boundary CFT is [10]

$$\mathrm{tr}\,\exp(-\beta H_{\mathrm{bdry}}) = \langle B|\exp(-2\pi H_{\mathrm{bulk}}/\beta)|B\rangle.$$

On the left is the thermodynamic partition function $Z_L(\beta)$ at inverse temperature β for a finite segment of the system of length $L=1$. The boundary conditions at the two ends of the segment are the same. The Hamiltonian is H_{bdry}. The Hilbert space is called the *boundary sector* (in string theory, the *open string sector*). In the Euclidean spacetime interpretation, $Z_L(\beta)$ is the partition function of a finite 2D cylinder with length L and Euclidean time periodic with period β. The right-hand side is obtained by reinterpreting L as Euclidean time and β as the length of a circle or, by scale invariance, Euclidean time $2\pi L/\beta$ and a spatial circle of length 2π. The Hamiltonian for the circle is H_{bulk}. The Hilbert space of the bulk system on the circle is called the *bulk sector* (the *closed string sector*). The boundary condition on each end of the cylinder is described by a bulk state $|B\rangle$. The modular duality formula states that the partition function depends only on the 2D geometry, so the two quantum mechanical interpretations give the same result.

Conformal invariance implies that each side of the duality formula can be expressed as a sum over the characters of the irreducible unitary representations of the Virasoro algebra. For $c>1$ (we consider the case $c=1$ separately below) the duality formula becomes

$$\chi_0(i\beta) + \sum_j \chi_{h_j}(i\beta) = g^2\chi_0(i/\beta) + \sum_k b_k^2 \chi_{\Delta_k/2}(i/\beta),$$

where the characters $\chi_h(i\beta)$ are given by

$$\chi_h(i\beta) = \frac{f_h(\beta)}{\eta(i\beta)}, \qquad f_h(\beta) = \begin{cases} q^{-\gamma}(1-q), & h=0 \\ q^{-\gamma+h}, & h>0, \end{cases}$$

$$q = e^{-2\pi\beta}, \quad \eta(i\beta) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad \gamma = \frac{c-1}{24}.$$

The character $\chi_0(i\beta)$ is the contribution to the partition function from the boundary sector representation that contains the ground state, whose energy is $-2\pi c/24$. The characters $\chi_{h_j}(i\beta)$ are the contributions from the representations with lowest energies $2\pi(h_j - c/24)$. Unitarity and uniqueness of the ground state imply all $h_j > 0$. The boundary scaling fields are in one-to-one correspondence with the energy eigenstates in the boundary sector, via radial quantization. A *primary* boundary field of scaling dimension h_j corresponds to the lowest energy state in the representation labeled by j. The bulk scaling fields are in one-to-one correspondence with the energy eigenstates in the bulk sector. The terms on the right side of the duality formula come from the closed sector representations whose lowest energy states correspond to the spin-zero primary scaling fields whose scaling dimensions are $0 < \Delta_1 \le \Delta_2 \le \cdots$. The numbers g and b_k characterize and completely determine the conformally invariant boundary state $|B\rangle$.

Rattazzi *et al.* [11] developed the linear functional method for deriving bounds on the low-lying scaling dimensions of conformal field theories from crossing formulas for correlation functions. Hellerman [12] showed that the same method could be applied to the modular duality formula for the bulk partition function of a 2D CFT to obtain an upper bound on the dimension of the lowest nontrivial primary field, and, with one of us, to obtain bounds on state degeneracies [13]. Here we apply the linear functional method to the modular duality formula for boundary CFT to derive a lower bound on g.

We want a bound on g that depends only on properties of the bulk system so it will apply to all possible critical boundary conditions for a given bulk critical system. The derivation should use only universal facts about the boundary condition: the uniqueness of the boundary sector ground state and the positivity of the scaling dimensions h_j, which follows from unitarity. Otherwise, nothing should be assumed about the numbers h_j or b_k.

We start by multiplying both sides of the duality formula by $\eta(i\beta) = \beta^{-1/2}\eta(i/\beta)$ to get

$$f_0 + \sum_j f_{h_j} = g^2\tilde f_0 + \sum_k b_k^2 \tilde f_{\Delta_k,} \tag{1}$$

where

$$\tilde f_\Delta = \begin{cases} \beta^{-1/2}\tilde q^{-\gamma+\Delta/2}(1-\tilde q), & \Delta = 0 \\ \beta^{-1/2}\tilde q^{-\gamma+\Delta/2}, & \Delta > 0, \end{cases} \quad \tilde q = e^{-2\pi/\beta}.$$

Then we apply a linear functional—a distribution $\rho(\beta)$—to both sides of Eq. (1), giving

$$(\rho, f_0) + \sum_j (\rho, f_{h_j}) = g^2(\rho, \tilde f_0) + \sum_k b_k^2(\rho, \tilde f_{\Delta_k}),$$

where $(\rho, F) = \int_0^\infty d\beta \rho(\beta) F(\beta)$. If we can choose $\rho(\beta)$ so that

$$(\rho, f_h) \ge 0, \qquad \forall h > 0 \tag{2}$$

$$(\rho, \tilde f_\Delta) \le 0, \qquad \forall \Delta \ge \Delta_1 \tag{3}$$

then we get an inequality

$$g^2(\rho, \tilde f_0) \ge (\rho, f_0). \tag{4}$$

Next, using the identity

$$\beta^{-1/2}\tilde{q}^{-\gamma+\Delta/2} = \int_{-\infty}^{\infty} dy e^{-\pi\beta y^2+2\pi i y\sqrt{\Delta-2\gamma}} \quad (5)$$

we see that condition (2) implies $(\rho, \tilde{f}_0) > 0$ so we have a lower bound on g,

$$g^2 \geq g_B^2[\rho] = \frac{(\rho, f_0)}{(\rho, \tilde{f}_0)}. \quad (6)$$

Maximizing over all distributions $\rho(\beta)$ satisfying conditions (2) and (3), we obtain the optimal bound

$$g^2 \geq g_B^2(c, \Delta_1) = \max_\rho g_B^2[\rho]. \quad (7)$$

It is not obvious that there exists *any* distribution $\rho(\beta)$ satisfying both conditions (2) and (3). Using identity (5), condition (3) requires

$$\int_{-\infty}^{\infty} dy(\rho, f_{\gamma+y^2/2})\cos(2\pi y\sqrt{\Delta_1 - 2\gamma}) \leq 0. \quad (8)$$

If $\Delta_1 \leq 2\gamma$ this is incompatible with condition (2). So the linear functional method can give a bound only if $\Delta_1 > (c-1)/12$.

The next step is to approximate the space of distributions by distributions of the form $(\rho, F) = \mathcal{D}F(\beta)$, where $\mathcal{D}$ is an Nth order differential operator in β. A bound $g_B^2(c, \Delta_1, N, \beta)$ is obtained by taking the maximum in Eq. (7) over the differential operators of order N. The bound can only improve as N increases. The partition function is real analytic in β so we can expect the limit $N \to \infty$ to exhaust the space of linear functionals for any choice of β, giving the optimal bound $g_B^2(c, \Delta_1) = \lim_{N\to\infty} g_B^2(c, \Delta_1, N, \beta)$. We stop here at $N = 1$, contenting ourselves with finding any bound at all. Elsewhere we will use the numerical techniques of Ref. [14] (semi-definite programming) to approximate the optimal bound from the linear functional method.

For $N = 1$, we write the general first order operator

$$\mathcal{D} = a_0 + a_1\left(-\frac{1}{2\pi}\frac{\partial}{\partial\beta} + \gamma\right).$$

Condition (2) is $a_0, a_1 \geq 0$. There is no bound if $a_1 = 0$, and the bound does not change if we scale $\mathcal{D}$, so we might as well set $a_1 = 1$. Condition (3) then becomes

$$a_0 \leq A_1(\beta) = \frac{\Delta_1 - 2\gamma}{2\beta^2} - \frac{1}{4\pi\beta} - \gamma.$$

These conditions require $A_1(\beta) \geq 0$ which cannot be satisfied for any value of β if $\Delta_1 - 2\gamma \leq 0$, so to get a bound we have to assume $\Delta_1 > 2\gamma$, the necessity of which we have already seen from the general analysis. The bound (7) is

$$g_B^2[\rho] = A_2(\beta)\frac{a_0 - A_3(\beta)}{a_0 + A_4(\beta)}, \quad (9)$$

where

$$A_2(\beta) = \beta^{1/2}q^{-\gamma}\tilde{q}^{\gamma}\frac{1-q}{1-\tilde{q}}, \qquad A_3(\beta) = \frac{q}{1-q},$$

$$A_4(\beta) = \gamma + \frac{1}{4\pi\beta} + \frac{\gamma}{\beta^2} + \frac{1}{\beta^2}\frac{\tilde{q}}{1-\tilde{q}}.$$

Since $A_{2,3,4}(\beta) > 0$, the highest bound is obtained when a_0 takes its maximum value $A_1(\beta)$, so

$$g_B^2(c, \Delta_1, 1, \beta) = A_2(\beta)\frac{A_1(\beta) - A_3(\beta)}{A_1(\beta) + A_4(\beta)}. \quad (10)$$

The bound is empty unless $A_1(\beta) - A_3(\beta) > 0$, which is stronger than $A_1(\beta) \geq 0$, so

$$A_1(\beta) - A_3(\beta) > 0 \quad (11)$$

is the only condition we need to impose to get a bound.

At this point there is no reason to stick to one particular value of β. The dependence on β will disappear as $N \to \infty$ but for finite N we can sample a larger subspace of distributions if we vary β. The best bound that can be obtained with a first-order $\mathcal{D}$ is

$$g_B^2(c, \Delta_1, 1) = \max_\beta g_B^2(c, \Delta_1, 1, \beta), \quad (12)$$

where the maximum is taken over all β satisfying condition (11). There is a unique positive solution β_1 of $A_1(\beta_1) - A_3(\beta_1) = 0$ and condition (11) is equivalent to $0 < \beta < \beta_1$. So for $\Delta_1 > 2\gamma$ there is a lower bound

$$g^2 \geq g_B^2(c, \Delta_1, 1)$$

with

$$g_B^2(c, \Delta_1, 1) = \max_{0<\beta<\beta_1} A_2(\beta)\frac{A_1(\beta) - A_3(\beta)}{A_1(\beta) + A_4(\beta)}. \quad (13)$$

There is no analytic expression for the $N = 1$ bound, but it can be calculated numerically for any given value of c and Δ_1. In general, the detailed form of the $N = 1$ bound as a function of c and Δ_1 is not particularly interesting since it is not even the optimal linear functional bound. At this stage, we are only interested in the fact that there is any lower bound on g.

For $c = 1$, there are degenerate Virasoro representations that do not occur for $c > 1$. Equation (1) holds with the modification that, for integers $n \geq 1$, $f_{n^2} = q^{-\gamma+n^2}(1 - q^{2n+1})$, $\tilde{f}_{2n^2} = \beta^{-1/2}\tilde{q}^{-\gamma+n^2}(1 - \tilde{q}^{2n+1})$. As before, we apply a first order differential operator with appropriate positivity conditions to get a lower bound on g that depends on β, then we maximize over β. We omit the calculations. The result is shown in Fig. 1. Included for comparison is the smallest value of g^2 for the $c = 1$ Gaussian model. Note that for the Gaussian model the lowest bulk scaling dimension is $\Delta_1 = \min(R^2/2, 1/2R^2)$ so that $\Delta_1 \leq 1/2$. However, in the derivation of the bound, we could as well have taken Δ_1 to be the lowest dimension that actually occurs in the boundary state and so contributes to the right-hand side of the duality formula. (This

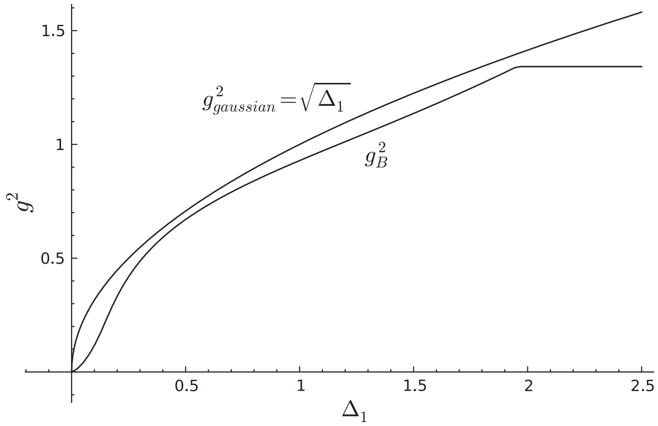

FIG. 1. The $N = 1$ bound for $c = 1$ compared to the minimum value of g^2 for the $c = 1$ Gaussian model.

does not lead to a substantial generalization of our result because in general we have no information on which bulk states actually occur in the boundary state.) For example, if we consider Gaussian model boundary conditions that are invariant under one of the $U(1)$ symmetries, then Δ_1 in this sense is $R^2/2$. Thus we can continue the comparison past the maximal value $\Delta_1 = 1/2$ as done on Fig. 1. The $N = 1$ bound is moderately good except when $\Delta_1 \approx 0$.

Several future directions are more or less obvious. We can explore how much the bound can be improved by numerically maximizing over differential operators of degree $N > 1$. We can apply the linear functional method to supersymmetric CFTs to get bounds on brane tensions in superstring theory. We can try to find linear functional bounds for specific bulk CFTs by exploiting knowledge of the bulk spectrum. For example, the most interesting bulk universality class for critical quantum circuits is the Monster CFT [15], which has $c = 24$ and $\Delta_1 = 4$. It is interesting because it has no relevant or marginal bulk perturbations. Our $N = 1$ lower bound is $g_B^2(24, 4, 1) = 0.0273$. The known conformal boundary conditions [16] have $g^2 = 1$. Numerical calculation of the $N = 83$ bound, making use of the fact that all the bulk scaling dimensions Δ_k are even integers ≥ 4, gives a bound $g^2 > 1$–6.03×10^{-19}, strikingly close to 1.

The most pressing problem is to overcome the restriction $\Delta_1 > (c - 1)/12$. We expect—from consideration of the effective low energy field theory of string theory in the presence of branes—that there should be a lower bound on g for all Δ_1 which goes to zero as Δ_1 goes to zero. We have shown that in consequence of (8) our present method cannot be extended straightforwardly. Some new ideas will be needed. The linear functional method applied to the boundary partition function is a practical compromise, well short of the exact lower bound that would follow from a complete solution of the conformal bootstrap for boundary CFT. We do not know in what direction to improve the linear functional method to get past the restriction $\Delta_1 > (c - 1)/12$.

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