arXiv:cond-mat/0505084v1 [cond-mat.stat-mech] 4 May 2005

Entropy flow in near-critical quantum circuits

Daniel Friedan*
Department of Physics and Astronomy, Rutgers, The State University of New Jersey, Piscataway, New Jersey 08854-8019, USA and The Science Institute, The University of Iceland, Reykjavik, Iceland

(Dated: May 3, 2005)

Near-critical quantum circuits are ideal physical systems for asymptotically large-scale quantum computers, because their low energy collective excitations evolve reversibly, effectively isolated from the environment. The design of reversible computers is constrained by the laws governing entropy flow within the computer. In near-critical quantum circuits, entropy flows as a locally conserved quantum current, obeying circuit laws analogous to the electric circuit laws. The quantum entropy current is just the energy current divided by the temperature. A quantum circuit made from a near-critical system (of conventional type) is described by a relativistic 1+1 dimensional relativistic quantum field theory on the circuit. The universal properties of the energy-momentum tensor constrain the entropy flow characteristics of the circuit components: the entropic conductivity of the quantum wires and the entropic admittance of the quantum circuit junctions. For example, near-critical quantum wires are always resistanceless inductors for entropy. A universal formula is derived for the entropic conductivity: $\sigma_S(\omega) = iv^2\mathcal{S}/\omega T$, where ω is the frequency, T the temperature, $\mathcal{S}$ the equilibrium entropy density and v the velocity of "light". The thermal conductivity is $\mathbf{Re}(T\sigma_S(\omega)) = \pi v^2\mathcal{S}\,\delta(\omega)$. The thermal Drude weight is, universally, $v^2\mathcal{S}$. This gives a way to measure the entropy density directly.

PACS numbers: 03.67.-a, 05.30.-d, 05.60Gg

I. INTRODUCTION

Asymptotically large-scale quantum computers will have to operate reversibly, close to equilibrium, effectively isolated from the environment.[1,2,3,4] A quantum computer has to maintain coherence as it evolves in time. It cannot afford to interact with the environment to discharge waste heat. On the other hand, there must be some method of external control, for programming and for input and output. External control requires contact with the environment. A fundamental problem is to reconcile external control with isolation and reversibility.

Near-critical quantum systems are ideal for the purpose. The low-energy collective excitations in a near-critical quantum system are governed by a locally conserved energy density operator. This means that they form an isolated quantum system which evolves reversibly in time. The low-energy excitations are effectively decoupled from microscopic environmental influences. External influences act only through a small number of relevant and marginal local couplings. Control is feasible, in principle, because only the relevant and marginal couplings need be tuned. It might even be arguable that near-critical quantum systems are the *only* physical systems that can operate reversibly and controllably at large scale, the only physical systems in which asymptotically large-scale quantum computation is practical.

This approach to the design of asymptotically large-scale quantum computers starts by singling out, on principle, the general class of useable physical systems. Left for later is the question of how, precisely, quantum bits are to be represented and manipulated in near-critical quantum systems, on the perhaps facile assumption that a Hilbert space is, after all, just a Hilbert space. When the low energy excitations are fermions, there is, of course, the obvious remark that the occupation number basis offers a quantum bit represention of the Hilbert space.

A quantum critical system is an extended system with a critical point at zero temperature. When the couplings of the system approach their critical values, collective excitations develop with energies and momenta that are very small on the characteristic microscopic scales. A "conventional" quantum critical system is one in which, as the system approaches criticality, the energies and momenta of the low-energy excitations scale in the same way (see, for example, Ref. 5,6). The energy-momentum dispersion relation then takes the relativistic form $E(p)^2 = E(0)^2 + v^2p^2$, where $E(0)$ is the energy gap, if any. The entire low-energy physics becomes relativistic near the critical point, the coefficient v being the speed of "light". The low energy physics is described by a relativistic quantum field theory, whose scale of energy and momentum is very much smaller than the microscopic scale. At low enough temperatures, such that kT is small on the microscopic energy scale, the physics of the near-critical system is entirely due to the low-energy excitations, and is described by the relativistic quantum field theory at temperature T. Near-critical quantum circuits are one-dimensional near-critical systems, described by relativistic quantum field theories in 1+1 dimensions: one space dimension and one time dimension. The "conventional" quantum critical systems are not the only kind of quantum critical system (see, for example, Ref. 6). It might be that some of the present considerations apply as well to "non-conventional" quantum critical systems, but only those described by relativistic quantum field theories will be considered here.

All near-critical quantum systems fall into a relatively small set of universality classes. These are the asymptotic limits of renormalization. The low-energy physics depends only on the universality class. The physically possible universality classes correspond to mathematically possible relativistic quantum field theories, and it might be conjectured that every mathematically possible relativistic quantum field theory can be realized as a physical system. Each relativistic quantum field theory is parametrized by a finite number of renormalized coupling constants, and by the speed of "light", v, which is the maximum speed of the low-energy excitations. All the low-energy properties of the near-critical quantum system are determined by the quantum field theory as universal functions of the renormalized coupling constants and v. Each universality class can be implemented in a variety of microscopically different physical systems. Nothing depends on the details of the physical implementation, except the value of v and the map that takes the microscopic coupling constants in the physical system to the renormalized coupling constants of the quantum field theory.

The relativistic quantum field theories are the universal "machine languages" for asymptotically large-scale quantum computers. Large-scale quantum computers can be designed theoretically, in the language of quantum field theory. If a particular universality class yields a theoretically useful design, the design can be implemented in any physical near-critical quantum system belonging to the universality class. Algorithms for asymptotically large-scale quantum computation are to be designed within the constrained vocabulary of relativistic quantum field theory, rather than the much more general vocabulary of quantum mechanics. The quantum field theory hamiltonians are very special among all possible quantum hamiltonians. Within this highly constrained set of "machine languages," it should be possible to make much more precise estimates of computational effectiveness than can be made for algorithms performed in general quantum mechanical systems. General estimates should be derivable from general properties of quantum field theory, and more specific estimates from the particular properties of each individual universality class.

Large-scale quantum computers are likely to be built as circuits, for much the same reasons as classical computers. The basic reason is that one-dimensional circuits can be packed in three-dimensional space. A more immediate reason is the expectation, or hope, that existing technologies for production of large-scale classical circuits can be adapted to production of large-scale quantum circuits. A near-critical quantum circuit is described by a 1+1 dimensional quantum field theory on the one-dimensional space of the circuit, a network of wires connected at junctions. The 1+1 dimensional relativistic quantum field theories are the universal "machine languages" for near-critical quantum circuit computers.

The 1+1 dimensional quantum field theory determines the possible characteristics of the elementary circuit components, the quantum wires and junctions. The challenge is to design circuits using these components that will perform useful large-scale quantum computations. Only a preliminary step is taken here. The laws governing entropy flow in near-critical quantum circuits are derived, and some basic, elementary calculations are done. Entropy is the currency of reversible computing. The laws governing the flow of entropy are the basic constraints on the movement of information within a reversible computer. It can be expected that the design of large-scale near-critical quantum circuit computers will have to take into account the laws governing entropy flow.

For control of a near-critical quantum circuit to be practical, the wires will have to be exactly critical in the bulk, with no relevant bulk couplings at all. The relevant couplings are the renormalizable couplings of positive scaling dimension. Their effects become large at low energies and momenta. Any relevant bulk coupling would have to be kept finely tuned everywhere along the length of the wires. This would be a daunting task, probably hopeless. Quantum wires with no relevant bulk couplings will automatically be critical in the bulk. No local perturbations will be able to make the bulk wire non-critical. There might still be marginal bulk couplings to control, but that will not require fine tuning. Such systems are gapless, and scale invariant in the bulk, and no local perturbation can produce a gap or disturb the scale invariance. The simplest example is the Berezinskii-Kosterlitz-Thouless line of critical points in the $U(1)$-invariant 1+1 dimensional gaussian model. A more exotic example is the $c = 24$ monster conformal field theory.[7,8,9] The latter is the tensor product of two chiral 1+1 dimensional conformal field theories, each of which has the largest finite simple group, the monster, as internal symmetry group. This quantum field theory has no renormalizable bulk couplings, neither relevant nor marginal. It is a fixed point of the renormalization group whose attracting basin is an open set in the space of physical systems. To realize the monster field theory physically, all that is needed is a physical system whose couplings lie somewhere with the attracting basin of the monster fixed point. No tuning or symmetry is needed at all. On the other hand, the monster field theory is constructed mathematically as a chiral orbifold. It is not clear how such a construction might be realized physically. If the monster field theory could be realized in physical quantum wires, it would have distinct advantages over the gaussian model: no need to screen the wires to maintain a bulk $U(1)$ symmetry, nor any need to control a marginal bulk coupling in the wires. The low energy properties of the bulk wires would be completely insensitive to local perturbations. The existence of such stable universality classes in 1+1 dimensions is another reason to prefer quantum circuits over quantum systems of higher dimension.

All the relevant couplings of the near-critical quantum circuit will now be in the circuit junctions. The junction couplings will represent the program and the input and output. Computation will be performed by the time evolution of the quantum state of the low energy collective excitations traveling along the bulk-critical wires, scattering in the circuit junctions. All excitations will travel at the speed of "light", v, because of the bulk scale invariance.

Controlling the entropy of the junctions will be a crucial function in a near-critical quantum circuit computer. Entropy will be moved in and out of the junctions during the evolution of the circuit. When the entropy of a junction is at its minimum, the junction is in a definite state, capable of supplying information, as program, or input, or output. When entropy flows into the junction, the state of the junction becomes uncertain, sensitive to the quantum excitations passing through. When entropy flows out again, the junction is left in a new definite state, which is a function of the original state and of the excitations that passed through the junction. The entropy flow characteristics of the circuit junctions will constrain the possible methods for input and output of information, and for control.

The junction entropy, s, was first described at critical points in the space of junction couplings, in the form $g = \exp(s_{crit})$, the "noninteger ground state degeneracy".[10] The junction is connected to long wires. The junction entropy is what remains when the bulk entropy of the wires is subtracted from the total entropy of junction plus wires (with a subtlety: boundary conditions are needed for the far ends of the wires, whose entropies must also be subtracted). When the junction is critical, along with the bulk wires, the whole system is scale invariant, so the junction entropy is independent of temperature, and can be attributed to the ground state. It was conjectured in Ref. 10, and supported by considerable evidence, that the value of g should always be larger at the ultraviolet critical point than at the infrared critical point, that it should decrease under the renormalization group flow, from fixed point to fixed point. This is the *g-theorem*.

The g-theorem was proved by establishing a gradient formula, $\partial s/\partial\lambda^a = -g_{ab}(\lambda)\beta^b$, which expresses the variation of the junction entropy with respect to the junction coupling constants in terms of the junction beta-function, $\beta^a(\lambda)$, and a certain positive metric, $g_{ab}(\lambda)$, on the space of junction couplings.[11] The gradient formula holds for all junctions, critical or not. The only scale in the junction is set by the temperature, T, so the renormalization group equation for the junction entropy is $T\mathrm{d}s/\mathrm{d}T = -\beta^a\partial s/\partial\lambda^a = \beta^a g_{ab}\beta^b$. The junction entropy thus decreases with decreasing temperature, which is to say that it decreases under the renormalization group. The junction contains minimum entropy at $T = 0$, at its IR fixed point and maximum entropy at $T = \infty$, at its critical point, its UV fixed point. The decrease of the junction entropy with temperature is not merely an obvious consequence of thermodynamics. The junction is a bounded sub-system, but as part of a near-critical system it cannot be treated as finite.

It may well be useful that a junction is close to its critical point exactly when its entropy is close to its minimum. Minimal entropy means that the junction is as sensitive as possible to the excitations passing through it. Close to critical means that the characteristic response time of the junction is long. The combination seems ideal: the junction is then able to process over long times the effects of the excitations passing through it.

The gradient formula can be regarded as explaining how to control the junction entropy: how the junction entropy changes in response to changes in the junction couplings. By relating the change in entropy to the junction beta-function, the gradient formula suggests that control might be simplest to achieve in supersymmetric universality classes, since supersymmetry typically simplifies the renormalization group flow. Supersymmetry is found at a special value of the marginal coupling constant in the $U(1)$-invariant gaussian model.[12,13,14] Supersymmetry is also found in the $c = 24$ monster conformal field theory.[15]

The junction entropy, $s(T)$, is not a useful quantity. Only *changes*, $\Delta s(T)$, are physically significant, describing the movement of information in and out of the junction. Of particular interest is the maximum change, $s(\infty) - s(0)$, which could be called the information capacity of the junction. To find the junction entropy itself requires global measurement, but changes in the junction entropy can be determined locally, near the junction, by studying how entropy flows into and out from the junction.

II. SUMMARY

It is an elementary observation that entropy flows in near-critical quantum circuits as a locally conserved quantum current. Every 1+1 dimensional quantum field theory has a conserved energy-momentum tensor, $T^\mu_\nu(x,t)$. The energy density is $T^t_t(x,t)$, the energy current is $T^x_t(x,t)$. The entropy density operator, $\rho_S(x,t)$, is simply the change in the energy density from its equilibrium value, divided by the temperature:

$$\rho_S(x,t) = k\beta T^t_t(x,t) - k\beta\langle T^t_t(x,t)\rangle_{eq}\ . \tag{1}$$

This is the local, quantum mechanical expression of the formula of Clausius, $\Delta S = \Delta Q/T$. Local conservation of energy then implies local conservation of entropy. The entropy current is simply the energy current divided by the temperature, $j_S(x,t) = k\beta T^x_t(x,t)$. The idea that entropy moves locally within a compound system goes back at least to Gibbs.[16] What is perhaps slightly novel here is the construction of the entropy density as a *quantum* operator, whose expectation value in equilibrium is the thermodynamic quantity.

The flow of entropy can now be treated in analogy with the flow of electric charge in electric circuits. Entropic circuit laws can be written, in analogy with the laws of electric circuits. The entropic potential, $\Phi_S(x,t)$, defined in analogy with the electric potential, is the temperature drop, $-\Delta T(x,t)$. The entropic field, analogous to the electric field, is the temperature gradient: $E_S(x,t) = -\partial_x \Phi_S(x,t)$. The entropic circuit laws determine the entropy flow in a complex circuit from the entropy characteristics of its components. A complex circuit is regarded as a collection of junctions connected by quantum wires. Each junction can stand for a complex sub-circuit, or can be elementary, without substructure. The flow of entropy in a wire is characterized by its entropic conductivity. The flow of entropy through a junction is characterized by its entropic admittance. The entropic conductivity of a wire, $\sigma_S(\omega)$, is the linear response coefficient which gives the entropy current, $\Delta I_S(t) = \sigma_S(\omega)\Delta E_S(t)$, that flows in response to an infinitesimal uniform entropic field alternating at frequency ω. The junction entropic admittance gives the entropy current, $\Delta I_S(t)_A = \sum_{B=1}^{N} Y_S(\omega)_{AB}\, \Delta V_S(t)^B$, that flows out each of the wires connected to the junction, $A = 1 \dots N$, in response to small alternating changes, $\Delta V_S(t)^B$, in the entropic potentials on the wires.

The laws governing entropy flow are more stringent than the laws for electricity, because the energy current is part of the conserved, symmetric energy-momentum tensor, $T^{\mu\nu} = T^{\nu\mu}$. The symmetry of the energy-momentum tensor relates the energy current to the momentum density: $T^x_t(x,t) = -v^2 T^t_x(x,t)$. This significantly constrains the entropy characteristics of a quantum circuit.

Universal equal-time commutation relations are derived for the energy density and current operators. The formula for the equal-time commutator $[T^t_t(x',t), T^t_t(x,t)]$ implies the continuity equation for entropy:

$$\partial_t \langle \rho_S(x,t)\rangle + \partial_x \langle j_S(x,t)\rangle = k\beta E_S(x,t)\langle j_S(x,t)\rangle - k\beta \partial_x \left[\Phi_S(x,t)\langle j_S(x,t)\rangle\right] . \tag{2}$$

The formula for $[T^t_t(x',t), T^x_t(x,t)]$ implies an equation for entropy conduction in bulk wires:

$$\begin{aligned}\partial_t \langle j_S(x,t)\rangle \;&=\; -k^2\beta^2 v^2 \frac{c_{UV}}{6}\frac{\hbar v}{2\pi}\partial_x^2 E_S(x,t) + k^2\beta^2 v^2 \langle T^t_t(x,t) - T^x_x(x,t)\rangle E_S(x,t) \\ &\quad + \left[1 + k\beta\Phi_S(x,t)\right] k\beta v^2 \partial_x \langle T^x_x(x,t)\rangle \, .\end{aligned} \tag{3}$$

where c_{UV} is the bulk conformal central charge in the short-distance limit (for discussion of 1+1 dimensional conformal field theory, see, for example, Refs. 5 and 17). The continuity and conduction equations are exact. No linear response approximations have been made. The conduction equation is useful in the limit of uniform flow, when the spatial derivatives can be neglected, and in the bulk-critical limit, where conformal invariance allows replacing $T^x_x(x,t)$ with $-T^t_t(x,t)$. The conduction equation, in the linear response approximation and the uniform limit, implies a general formula for the entropic conductivity of near-critical quantum wire:

$$\sigma_S(\omega) = \frac{iv^2\mathcal{S}}{\omega T} \tag{4}$$

where ω is the frequency, T is the temperature, $\mathcal{S}$ is the equilibrium entropy density, and v the velocity of "light". Near-critical quantum wires, as circuit elements, are resistanceless inductors for entropy. A direct derivation of Eq. (4), using the Kubo formula, is given in Appendix B. The entropic conductivity is just the complex thermal conductivity divided by temperature, so the thermal conductivity of near-critical quantum wire is

$$\kappa(\omega, T) = \mathbf{Re}(T\sigma_S(\omega)) = \pi v^2 \mathcal{S}\, \delta(\omega) \, . \tag{5}$$

The coefficient $v^2\mathcal{S}$ is the universal thermal Drude weight for a near-critical one-dimensional quantum system.

The universal formula for the entropic conductivity of quantum wire that is near-critical but not critical is not directly relevant to the design of quantum circuit computers, given the argument that the quantum wires in such circuits should be exactly critical in the bulk. On the other hand, the universal formula for the thermal conductivity, Eq. (5), could be useful in other contexts, as it gives a way to determine directly, by experiment, the entropy density of the low energy collective excitations in near-critical one-dimensional and quasi-one-dimensional quantum systems.

When the quantum wires are critical in the bulk, the entropic conductivity for non-uniform flow is calculated from the continuity and conduction equations, in the linear response approximation, using bulk conformal invariance:

$$\sigma_S(q,\omega) = \frac{c}{12}\frac{2\pi k^2 v}{\hbar}\left[1 + \left(\frac{\hbar v \beta q}{2\pi}\right)^2\right]\left(\frac{i}{\omega + i\epsilon + vq} + \frac{i}{\omega + i\epsilon - vq}\right) \tag{6}$$

where c is the bulk conformal central charge and q is the wave-number. The low-energy excitations all travel at the speed of "light". The entropy current splits into independent right-moving and left-moving chiral currents,

$j_R(x,t) = j_R(x - vt)$ and $j_L(x,t) = j_L(x + vt)$. This can be read out from Eq. (6), from the poles in $\sigma_S(q,\omega)$ at $\omega = \pm vq$. The chiral entropy currents are just the chiral energy currents divided by the temperature. The chirality of the entropy currents — the uniform speed of the excitations and the absence of bulk interactions between the left-moving and right-moving entropy currents — might be a useful feature of bulk-critical quantum wires.

This paper defines the entropy density and current operators, notes that circuit laws for entropy follow by formal analogy to the electric circuit laws, derives the entropy continuity and conduction equations, and the universal formula for the entropic conductivity. All the derivations and calculations are completely elementary. A subsequent paper[18] treats the flow of entropy through near-critical quantum circuit junctions.

Much of what is done here is related to other works. Quantum wires and junctions have been much studied (see, for example, Ref. 19 and references therein). The idea of using quantum critical systems for quantum computers is far from new (see, for example, Ref. 20). But the goal and the reasoning are perhaps different here. The goal is to design asymptotically large-scale quantum computers, requiring large quantum systems that operate reversibly and are controllable. The reason for proposing that asymptotically large-scale quantum computers should be built from near-critical quantum circuits is the effective isolation provided by renormalization. Renormalizaton, by decoupling the low energy excitations from the microscopic physics, solves the hard part of the control problem and makes reversible operation practical. The program that follows from this reasoning is to base the design strategy on the laws of entropy flow.

It is quite standard to express thermal transport properties of quantum systems in terms of the local energy density and current operators.[21] The entropy density and current operators defined here can be considered a trivial repackaging of the energy density and current. Entropy is the quantity of interest in reversible computing, but bookkeeping entropy is equivalent to bookkeeping energy in reversible processes. On the other hand, it is the entropy current and density operators that are appropriate for the formulation of circuit laws, because the entropic potential, $\Phi_S(x,t)$, which couples to the local entropy density in analogy with the electric potential, has a thermodynamic interpretation as the local drop in temperature, $\Phi_S(x,t) = -\Delta T(x,t)$. No such thermodynamic quantity couples to the energy density.

Formulas for the thermal conductivity equivalent to Eq. (5) were previously derived for the special cases of free massive fermions and bosons and for the general bulk-critical case,[22] but it was not noted for any of these cases that the thermal Drude weight takes the form $v^2\mathcal{S}$.

III. ENTROPY FLOW

Every relativistic quantum field theory has a locally conserved, symmetric energy-momentum tensor, $T^\mu_\nu(x,t)$, which represents the response of the system to local deformations of the space-time geometry. In 1+1 dimensions, the space-time metric, $g_{\mu\nu}$, has components $g_{tt} = -v^2$, $g_{xx} = 1$, $g_{xt} = g_{tx} = 0$. Any local deformation, $\delta g_{\mu\nu}(x,t)$, can be represented as a local variation of the couplings in the hamiltonian, so is equivalent to a perturbation of the hamiltonian by a local quantum field, the energy-momentum tensor. The expectation values in the quantum field theory change by

$$\delta\langle\cdots\rangle = \int\int \mathrm{d}t\,\mathrm{d}x \,\left(\frac{-1}{2}\right)\delta g_{\mu\nu}(x,t)\,\langle\frac{i}{\hbar}T^{\mu\nu}(x,t)\cdots\rangle \tag{7}$$

where $T^{\mu\nu'}g_{\nu'\nu} = T^\mu_\nu$. The energy-momentum tensor is symmetric, $T^{\mu\nu} = T^{\nu\mu}$, because the space-time metric is symmetric. Symmetry of $T^{\mu\nu}$ is simply the indentity $T^x_t = -v^2 T^t_x$.

The individual components of the energy-momentum tensor are the energy density, $T^t_t(x,t)$, the energy current, $T^x_t(x,t)$, the momentum density, $T^t_x(x,t)$, and the momentum current, $T^x_x(x,t)$. Energy and momentum are each locally conserved,

$$\partial_\mu T^\mu_\nu(x,t) = 0\,. \tag{8}$$

The hamiltonian is

$$H_0 = \int \mathrm{d}x\ T^t_t(x,t)\,. \tag{9}$$

Local conservation of energy,

$$\partial_t T^t_t(x,t) + \partial_x T^x_t(x,t) = 0\,, \tag{10}$$

expresses the effective isolation of the near-critical quantum system. The degrees of freedom of the quantum field theory are the low energy collective degrees of freedom of the near-critical quantum system. They form a closed system, neither gaining nor losing energy, effectively decoupled from microscopic fluctuations.

The formula of Clausius,

$$\Delta S = \frac{\Delta Q}{T} \,, \tag{11}$$

expresses the change of entropy in a reversible process as the change in heat divided by the temperature. The constant of proportionality, $1/T$, is also written $k\beta$, where k is Boltzmann's constant, the fundamental unit of entropy. In a near-critical system, the change in heat within a local region, $\mathcal{R}$, is

$$\Delta Q_{\mathcal{R}} = \int_{\mathcal{R}} \mathrm{d}x \; \Delta\langle T^t_t(x,t)\rangle \tag{12}$$

because all available forms of energy are included in $T^t_t(x,t)$. The entropy within region $\mathcal{R}$ changes by

$$\Delta S_{\mathcal{R}} = k\beta \int_{\mathcal{R}} \mathrm{d}x \; \Delta\langle T^t_t(x,t)\rangle \,. \tag{13}$$

This local version of the Clausius formula can be derived formally by calculating the change of entropy when the hamiltonian of the local region is perturbed infinitesimally from H_0 to $H = H_0 + \Delta H$, at constant temperature. The equilibrium expectation values of observables, $\langle\mathcal{O}\rangle_{eq}$, are perturbed to $\langle\mathcal{O}\rangle = \langle\mathcal{O}\rangle_{eq} + \Delta\langle\mathcal{O}\rangle$. The partition function is

$$Z = \mathrm{Tr}(\mathrm{e}^{-\beta H}) \tag{14}$$

and the entropy is

$$S = k\left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln Z = k\ln Z + k\beta\langle H\rangle \,. \tag{15}$$

The infinitesimal change of entropy is

$$\begin{aligned} \Delta S &= k\Delta\ln Z + k\beta\Delta\langle H_0\rangle + k\beta\langle\Delta H\rangle_{eq} \\ &= k\beta\langle-\Delta H\rangle_{eq} + +k\beta\Delta\langle H_0\rangle + k\beta\langle\Delta H\rangle_{eq} \\ &= k\beta\Delta\langle H_0\rangle \\ &= k\beta\Delta\langle\int \mathrm{d}x \; T^t_t(x,t)\rangle \,. \end{aligned} \tag{16}$$

The Clausius formula for finite perturbations follows by integrating over infinitesimal perturbations.

Write the local change of entropy as

$$\Delta S_{\mathcal{R}} = \Delta\int_{\mathcal{R}} \mathrm{d}x \; \langle\rho_S(x,t)\rangle \tag{17}$$

where

$$\rho_S(x,t) = k\beta T^t_t(x,t) - \langle k\beta T^t_t(x,t)\rangle_{eq} \,. \tag{18}$$

The operator $\rho_S(x,t)$ can be interpreted as the variation of the entropy density away from its equilibrium value, which is the natural baseline against which to measure the flow of entropy. The time derivative of the entropy density operator is

$$\partial_t\rho_S(x,t) = k\beta\partial_t T^t_t(x,t) = -k\beta\partial_x T^x_t(x,t) \tag{19}$$

so the local entropy current is

$$j_S(x,t) = k\beta T^x_t(x,t) \,. \tag{20}$$

The entropy current, $j_S(x,t)$, is the entropy per unit time flowing to the right through the point x. The local entropy current and the local entropy density, $\rho_S(x,t)$, are *quantum* observables. Entropy flows as a locally conserved *quantum* current:

$$\partial_t\rho_S(x,t) + \partial_x j_S(x,t) = 0 \,. \tag{21}$$

IV. QUANTUM CIRCUITS AND ENTROPIC CIRCUIT LAWS

A near-critical quantum circuit is described by a 1+1 dimensional relativistic quantum field theory on the one-dimensional space of the circuit. That space consists of a set of line segments, the wires, and a set of points, the junctions. Each wire boundary is identified with one of the junctions. A junction to which only a single wire is connected is simply an end of the wire.

The existence of the locally conserved quantum entropy current implies that the flow of entropy is governed by entropic circuit laws derived by formal analogy with the laws of electric circuits, which are expressed for example in Kirchoff's laws. The circuit laws determine the performance of the whole circuit given the characteristics of its parts, the entropic conductivity of the wires and the entropic admittance of the junctions.

The characteristics of the wires and junctions are determined by their linear responses to small external perturbations by an entropic potential, $\Phi_S(x,t)$, which is the external source that couples to the entropic charge density, in analogy with the electric potential. The perturbed hamiltonian is

$$H = H_0 + \Delta H(t) = H_0 + \int \mathrm{d}x\ \rho_S(x,t)\Phi_S(x,t)\,. \tag{22}$$

$\Phi_S(x,t)$ has dimensions of temperature. The entropic field

$$E_S(x,t) = -\partial_x \Phi_S(x,t) \tag{23}$$

has dimensions of temperature/distance.

Compare this perturbation to a change of temperature from T to $T+\Delta T$. The equilibrium density matrix of the unperturbed system is

$$\rho_{eq} = \frac{1}{Z_0}\mathrm{e}^{-\beta H_0}\,. \tag{24}$$

Under an infinitesimal change of temperature, the equilibrium density matrix changes by

$$\Delta\rho_{eq} = \rho_{eq}(k\beta^2\Delta T)\ (H_0 - \langle H_0\rangle_{eq})\,. \tag{25}$$

The same effect can be obtained at constant temperature by adding an infinitesimal perturbation to the hamiltonian,

$$\Delta H = (H_0 - \langle H_0\rangle_{eq})(-k\beta\Delta T) = \int \mathrm{d}x\ \rho_S(x,t)(-\Delta T)\,. \tag{26}$$

Therefore, imposing an infinitesimal static entropic potential, $\Delta\Phi_S(x,t) = -\Delta T(x)$, is equivalent to making an infinitesimal local variation of the temperature, $\Delta T(x)$, in the limit where both become constant in space. Integrating infinitesimal perturbations gives $\Delta\Phi_S(x,t) = -\Delta T$ for finite changes of temperature. Increasing the entropic potential means *decreasing* the temperature. The entropic potential is the local *drop* in temperature. The entropic field $E_S(x,t) = -\partial_x\Phi_S(x,t)$ is the local temperature gradient, in the limit where both are uniform in space.

When the external perturbation is turned on, entropy flows along the entropic field from higher entropic potential to lower, from regions of *lower* temperature to regions of *higher* temperature. The entropic potential acts like the temperature dial on the thermostat of a heating system. When a negative entropic potential is introduced in a local region, the temperature dial there is turned *up*. The couplings in the hamiltonian are changed locally, so that the system behaves as if at a higher temperature, locally. The system responds by evolving towards local equilibrium at the new temperature. Initially, there is too little entropy in the perturbed region for that region to be in equilibrium at the new, higher temperature, so entropy flows *into* the perturbed region from regions of higher entropic potential elsewhere in the system.

In an operating circuit, the only external perturbations will be in the junctions that serve as external controls. The entropic potential everywhere else in the circuit is an auxiliary variable, determined by a subset of the circuit equations as a function of the entropic currents and charges. The remaining circuit laws and the external entropic potentials in the control junctions then determine the flow of entropy within the circuit.

V. ENTROPIC CONDUCTIVITY AND ADMITTANCE

The entropic conductivity of a quantum wire is defined by analogy with the electrical conductivity. When a wire is perturbed by a small entropic field, $\Delta E_S(x,t) = \mathrm{e}^{iqx-i\omega t}\Delta E_S(0,0)$, an entropy current is induced to flow. The entropy current is given, to first order in the perturbation, by a linear response formula

$$\Delta\langle j_S(x,t)\rangle = \sigma_S(q,\omega)\Delta E_S(x,t)\,. \tag{27}$$

The linear response coefficient $\sigma_S(q,\omega)$ is the entropic conductivity. The entropic conductivity for uniform flow is

$$\sigma_S(\omega) = \lim_{q\to 0} \sigma_S(q,\omega)\,. \tag{28}$$

The entropic conductivity is just the complex thermal conductivity divided by the temperature, $T\sigma_S = \sigma_{thermal}$, since ΔE_S is the temperature gradient and $\Delta\langle j_S\rangle$ is the energy current divided by temperature.

The circuit laws for uniform entropy conduction through wires are analogous to those for electrical conduction. Let A,A' label the two ends of a wire. Let $\Delta I_S(t)_A$ be the entropy current entering the wire at end A, and $\Delta I_S(t)_{A'}$ the entropy current entering at end A'. Let $\Delta V_S(t)^A$ be the entropic potential at end A, and $\Delta V_S(t)^{A'}$ the entropic potential at end A'. Let $\Delta E_S(t)$ be the entropic field in the wire. Let l be the length of the wire. The conduction equations for uniform entropy flow in the wire are

$$l\,\Delta E_S(t) \;=\; \Delta V_S(t)^A - \Delta V_S(t)^{A'} \tag{29}$$

$$\Delta I_S(t)_A \;=\; \sigma_S(\omega)\Delta E_S(t) = -\Delta I_S(t)_{A'}\,. \tag{30}$$

The entropic admittance of a junction is defined by analogy with the electrical admittance. Label the N wire-ends attached to the junction by indices $A,B = 1\ldots N$. Let $\Delta V_S(t)^B$ be an infinitesimal change of entropic potential at the end of wire B, where it is attached to the junction, and let $\Delta I_S(t)_A$ be the resulting change in the entropic current flowing out of the junction through wire A. For alternating potentials

$$\Delta V_S(t)^B = \mathrm{e}^{-i\omega t}\,\Delta V_S(0)^B \tag{31}$$

the junction admittance equation is

$$\Delta I_S(t)_A = \sum_{B=1}^{N} Y_S(\omega)_{AB}\,\Delta V_S(t)^B \tag{32}$$

where the matrix $Y_S(\omega)_{AB}$ is the entropic admittance of the junction.

The entropic conductivity, $\sigma_S(\omega)$, of the wire and the entropic admittance matrices, $Y_S(\omega)_{AB}$, of the elementary junctions are to be calculated in the 1+1 dimensional quantum field theory. The conduction equations for the wires and the admittance equations for the junctions then determine the entropy flow properties of the circuit.

VI. THE CONTINUITY EQUATION FOR ENTROPY

Observables in the unperturbed system evolve in time by

$$\partial_t \mathcal{O}(t) = \frac{i}{\hbar}[H_0, \mathcal{O}(t)]\,. \tag{33}$$

The hamiltonian of the perturbed system is $H = H_0 + \Delta H(t)$. The perturbation, $\Delta H(t)$, vanishes at early times. The perturbed system starts at early time unperturbed and in equilibrium. The density matrix, $\rho(t)$, starts at early time equal to the unperturbed equilibrium density matrix, ρ_{eq}. It evolves in time by

$$\partial_t \rho(t) = -\frac{i}{\hbar}[H, \rho(t)] \tag{34}$$

The expectation values in the perturbed system, $\langle \mathcal{O}(t)\rangle = \mathrm{Tr}[\rho(t)\mathcal{O}(t)]$, evolve in time by

$$\partial_t \langle \mathcal{O}(t)\rangle = \langle \partial_t \mathcal{O}(t)\rangle + \langle \frac{i}{\hbar}[\Delta H(t), \mathcal{O}(t)]\rangle\,. \tag{35}$$

This time evolution formula is especially useful when the equal-time commutator can be evaluated explicitly.

The entropy density evolves in time by

$$\begin{aligned}\partial_t\langle\rho_S(x,t)\rangle \;&=\; \langle\partial_t\rho_S(x,t)\rangle + \langle\frac{i}{\hbar}[\Delta H(t),\rho_S(x,t)]\rangle \\ &=\; \langle -\partial_x j_S(x,t)\rangle + \langle\frac{i}{\hbar}[\int \mathrm{d}x'\;\rho_S(x',t)\Phi_S(x',t),\rho_S(x,t)]\rangle\end{aligned} \tag{36}$$

or

$$\partial_t \langle \rho_S(x,t)\rangle + \partial_x \langle j_S(x,t)\rangle = \int \mathrm{d}x'\ \Phi_S(x',t)\langle \frac{i}{\hbar}[\rho_S(x',t), \rho_S(x,t)]\rangle\ . \tag{37}$$

First, integrate over x to find the rate of change of the total entropy:

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= \int \mathrm{d}x'\ \Phi_S(x',t)\langle \frac{i}{\hbar}[\rho_S(x',t), k\beta H_0]\rangle \\ &= k\beta \int \mathrm{d}x'\ \Phi_S(x',t)\langle -\partial_t \rho_S(x',t)\rangle \\ &= k\beta \int \mathrm{d}x'\ \Phi_S(x',t)\langle \partial_x j_S(x',t)\rangle \\ &= k\beta \int \mathrm{d}x'\ E_S(x',t)\langle j_S(x',t)\rangle\end{aligned} \tag{38}$$

which can be written

$$\frac{\mathrm{d}S}{\mathrm{d}t} = k\beta \frac{\mathrm{d}W}{\mathrm{d}t} \tag{39}$$

where $W(t)$ is the work done on the system by the external entropic potential, given by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}\langle H_0\rangle}{\mathrm{d}t} = \int \mathrm{d}x\ E_S(x,t)\langle j_S(x,t)\rangle\ . \tag{40}$$

Eq. (37), unintegrated, describes the local production of entropy:

$$\partial_t \langle \rho_S(x,t)\rangle + \partial_x \langle j_S(x,t)\rangle = k\beta\, p(x,t) \tag{41}$$

where

$$p(x,t) = \int \mathrm{d}x'\ \Phi_S(x',t)\langle \frac{i}{\hbar}[\rho_S(x',t), T^t_t(x,t)]\rangle \tag{42}$$

is the density of power delivered to the system by the imposed entropic field. The power delivered to the system by the external field is a source of entropy. This does not have an analogue in the continuity equation for electric charge.

The equal-time commutator appearing in Eqs. (37) and (42) is calculated in Appendix A:

$$\frac{i}{\hbar}[T^t_t(x',t), T^t_t(x,t)] = \partial_{x'}\delta(x'-x)T^x_t(x,t) - \partial_x \delta(x'-x)T^x_t(x',t)\ . \tag{43}$$

Substituting in Eq. (37) gives the continuity equation for entropy:

$$\partial_t \langle \rho_S(x,t)\rangle + \partial_x \langle j_S(x,t)\rangle = k\beta E_S(x,t)\langle j_S(x,t)\rangle - k\beta \partial_x \left[\Phi_S(x,t)\langle j_S(x,t)\rangle\right]\ . \tag{44}$$

The form of this equation suggests that it might be worthwhile to redefine the entropy current as $j_S(x,t) + k\beta\Phi_S(x,t)j_S(x,t)$ away from equilibrium, so that the power will be proportional to the current.

The continuity equation for entropy is exact. In the linear response approximation, it becomes

$$\partial_t \Delta\langle \rho_S(x,t)\rangle + \partial_x \Delta\langle j_S(x,t)\rangle = k\beta \Delta E_S(x,t)\langle j_S(x,t)\rangle_{eq} - \partial_x \left[k\beta\Delta\Phi_S(x,t)\langle j_S(x,t)\rangle_{eq}\right]\ . \tag{45}$$

where, for any operator $\mathcal{O}$,

$$\Delta\langle \mathcal{O}\rangle = \langle \mathcal{O}\rangle - \langle \mathcal{O}\rangle_{eq}\ . \tag{46}$$

Entropy is locally conserved to first order in the perturbation, unless there is a persistent equilibrium entropy current, $\langle j_S(x,t)\rangle_{eq}$. The distribution of entropy is stationary in equilibrium, $\partial_t\langle \rho_S(x,t)\rangle_{eq} = 0$, so $\partial_x\langle j_S(x,t)\rangle_{eq} = 0$. Therefore, if there is an equilibrium entropy current, it must flow in closed loops within the circuit and must flow uniformly within each wire. Entropy current could be stored in such persistent loops. Since $\partial_x\langle j_S(x,t)\rangle_{eq} = 0$, the linearized continuity equation can be written

$$\partial_t \Delta\langle \rho_S(x,t)\rangle + \partial_x \Delta\langle j_S(x,t)\rangle = 2k\beta \Delta E_S(x,t)\langle j_S(x,t)\rangle_{eq}\ . \tag{47}$$

In the static limit, this becomes

$$\partial_x \Delta\langle j_S(x,t)\rangle = \partial_x \left[-2k\beta\Delta\Phi_S(x,t)\langle j_S(x,t)\rangle_{eq}\right] \ . \tag{48}$$

Assuming that a local change in temperature has only a local effect on the equilibrium current (possibly a dubious assumption), then

$$\Delta\langle j_S(x,t)\rangle = -2k\beta\Delta\Phi_S(x,t)\langle j_S(x,t)\rangle_{eq} \ . \tag{49}$$

Take the uniform limit, recalling that $\Delta\Phi_S = -\Delta T$, to obtain

$$\Delta\langle j_S(x,t)\rangle_{eq} = 2\frac{\Delta T}{T}\langle j_S(x,t)\rangle_{eq} \ . \tag{50}$$

The temperature dependence of the persistent entropy current is now completely determined:

$$\langle j_S(x,t)\rangle_{eq} = T^2 f_S(x) \tag{51}$$

where $f_S(x)$ is independent of temperature, and $\partial_x f_S(x) = 0$. This argument is not entirely solid, but if it holds, then it will imply that a strong persistent entropy current can be generated in a loop by first generating a weak entropy current in the loop at low temperature, then raising the temperature of the loop.

VII. THE ENTROPY CONDUCTION EQUATION

The time evolution of the entropy current is

$$\partial_t\langle j_S(x,t)\rangle = \langle \partial_t j_S(x,t)\rangle + \int \mathrm{d}x' \ \Phi_S(x',t)\langle \frac{i}{\hbar}[\rho_S(x',t), j_S(x,t)]\rangle \ . \tag{52}$$

Symmetry of the energy-momentum tensor implies

$$j_S(x,t) = k\beta T^x_t(x,t) = -k\beta v^2 T^t_x(x,t) \ . \tag{53}$$

Local conservation of momentum implies

$$\partial_t j_S(x,t) = -k\beta v^2 \partial_t T^t_x(x,t) = k\beta v^2 \partial_x T^x_x(x,t) \ . \tag{54}$$

Eq. (52) becomes

$$\partial_t\langle j_S(x,t)\rangle = k\beta v^2 \partial_x \langle T^x_x(x,t)\rangle + \int \mathrm{d}x' \ \Phi_S(x',t)\langle \frac{i}{\hbar}[\rho_S(x',t), j_S(x,t)]\rangle \ . \tag{55}$$

The equal-time commutator appearing in Eq. (55) is calculated in Appendix A:

$$\begin{aligned}\frac{i}{\hbar}[T^t_t(x',t), v^{-2}T^x_t(x,t)] \ &= \ \frac{c_{UV}}{6}\frac{\hbar v}{2\pi}\partial_x^3\delta(x'-x) + \partial_x\delta(x'-x)[T^x_x(x,t) - T^t_t(x,t)] \\ &\quad + \delta(x'-x)\partial_x T^x_x(x,t)\end{aligned} \tag{56}$$

where c_{UV} is the bulk conformal central charge in the short-distance limit. Substituting in Eq. (55) gives the conduction equation:

$$\begin{aligned}\partial_t\langle j_S(x,t)\rangle \ &= \ -k^2\beta^2 v^2 \frac{c_{UV}}{6}\frac{\hbar v}{2\pi}\partial_x^2 E_S(x,t) + k^2\beta^2 v^2 \langle T^t_t(x,t) - T^x_x(x,t)\rangle E_S(x,t) \\ &\quad + \left[1 + k\beta\Phi_S(x,t)\right] k\beta v^2 \partial_x \langle T^x_x(x,t)\rangle \ .\end{aligned} \tag{57}$$

The conduction equation is especially useful in two situations: when the entropy flow in the wire is uniform, so $\partial_x\langle T^x_x(x,t)\rangle = 0$, or when the wire is conformally invariant in the bulk, so $T^x_x = -T^t_t$.

VIII. UNIFORM ENTROPY CONDUCTION

When the entropy flow is uniform, the spatial derivatives in Eq. (57) are negligible, so the conduction equation becomes

$$\partial_t \langle j_S(x,t)\rangle = k^2\beta^2 v^2 \langle T_t^t(x,t) - T_x^x(x,t)\rangle E_S(x,t)\ . \tag{58}$$

In the linear response approximation, this becomes

$$\partial_t \Delta\langle j_S(x,t)\rangle = k^2\beta^2 v^2 \langle T_t^t(x,t) - T_x^x(x,t)\rangle_{eq} \Delta E_S(x,t)\ . \tag{59}$$

The quantity $k\beta\langle T_t^t(x,t) - T_x^x(x,t)\rangle_{eq}$ is the equilibrium entropy density, $\mathcal{S}$, by the following argument.[23] The equilibrium energy density in a long wire of length l is

$$\mathcal{E} = \langle T_t^t(x,t)\rangle_{eq} = -\frac{\partial}{\partial\beta}\frac{\ln Z_0}{l}\ . \tag{60}$$

The free energy density,

$$\mathcal{F} = -\frac{1}{\beta}\frac{\ln Z_0}{l}\ , \tag{61}$$

is independent of l in the limit of large l, so

$$\mathcal{F} = (1 + l\frac{\partial}{\partial l})\mathcal{F} = -\frac{1}{\beta}\frac{\partial}{\partial l}\ln Z_0 = \langle T_x^x(x,t)\rangle_{eq}\ . \tag{62}$$

The equilibrium entropy density is

$$\mathcal{S} = k\left(1 - \beta\frac{\partial}{\partial\beta}\right)\frac{\ln Z_0}{l} = k\beta(\mathcal{E} - \mathcal{F}) = k\beta^2\frac{\partial}{\partial\beta}\mathcal{F}\ . \tag{63}$$

It can now be written in two equivalent ways:

$$\mathcal{S} = k\beta\langle T_t^t(x,t) - T_x^x(x,t)\rangle_{eq} = k\beta^2\frac{\partial}{\partial\beta}\langle T_x^x(x,t)\rangle_{eq}\ . \tag{64}$$

Substituting in the formula for uniform conduction, Eq. (59), gives

$$\partial_t \Delta\langle j_S(x,t)\rangle = k\beta v^2 \mathcal{S}\Delta E_S(x,t)\ . \tag{65}$$

The entropic conductivity for uniform entropy flow is therefore

$$\sigma_S(\omega) = \frac{ik\beta v^2\mathcal{S}}{\omega} = \frac{iv^2\mathcal{S}}{\omega T}\ . \tag{66}$$

The complex thermal conductivity is

$$\sigma_{thermal}(\omega) = T\sigma_S(\omega) = \frac{iv^2\mathcal{S}}{\omega} \tag{67}$$

and the thermal conductivity is

$$\kappa(\omega,T) = \mathbf{Re}(T\sigma_S(\omega)) = \pi v^2\mathcal{S}\,\delta(\omega)\ . \tag{68}$$

An alternative derivation of Eq. (66) is given in Appendix B, using the Kubo formula for the conductivity instead of the linearized conduction equation.

As a consistency check, take the static limit of the conduction formula, Eq. (57), make the linear response approximation, then take the uniform limit, letting the entropic potential become constant in space, $\Delta\Phi_S(x,t) = -\Delta T$. The perturbed system will be in equilibrium at temperature $T + \Delta T$. The conduction equation becomes

$$0 = k\beta v^2\Delta\langle T_x^x(x,t)\rangle + k^2\beta^2 v^2\langle T_t^t(x,t) - T_x^x(x,t)\rangle_{eq}\Delta T \tag{69}$$

or

$$0 = \Delta\mathcal{F} + \mathcal{S}\Delta T \tag{70}$$

which is just the thermodynamic relation between free energy and entropy.

IX. BULK-CRITICAL WIRE

When the quantum wire is critical in the bulk, the 1+1 dimensional quantum field theory is conformally invariant in the bulk:

$$\Theta(x,t) = -T^{\mu}_{\mu}(x,t) = -T^{t}_{t}(x,t) - T^{x}_{x}(x,t) = 0\,. \tag{71}$$

Then $T^x_x(x,t)$ can be replaced by $-T^t_t(x,t)$ in the conduction equation, Eq. (57), giving

$$\begin{aligned}\partial_t\langle j_S(x,t)\rangle = &-k\beta v^2\partial_x\langle T^t_t(x,t)\rangle - k^2\beta^2 v^2\frac{c}{6}\frac{\hbar v}{2\pi}\partial_x^2 E_S(x,t)\\ &+ k^2\beta^2 v^2 E_S(x,t)2\langle T^t_t(x,t)\rangle - k^2\beta^2 v^2\Phi_S(x,t)\partial_x\langle T^t_t(x,t)\rangle\,.\end{aligned} \tag{72}$$

In the linear response approximation, this becomes

$$\partial_t\Delta\langle j_S(x,t)\rangle + v^2\partial_x\Delta\langle\rho_S(x,t)\rangle = k^2v^2\beta^2\left[2\langle T^t_t(x,t)\rangle_{eq} - \frac{c}{6}\frac{\hbar v}{2\pi}\partial_x^2\right]\Delta E_S(x,t)\,. \tag{73}$$

For bulk-critical wire, the equilibrium energy density is[24,25]

$$\mathcal{E}_{crit} = \langle T^t_t(x,t)\rangle_{eq} = \frac{c}{12}\frac{2\pi}{\hbar v}\frac{1}{\beta^2} \tag{74}$$

so the linearized conduction equation is

$$\partial_t\Delta\langle j_S(x,t)\rangle + v^2\partial_x\Delta\langle\rho_S(x,t)\rangle = k^2\frac{c}{6}\frac{2\pi v}{\hbar}\left[1 - \left(\frac{\hbar v}{2\pi}\right)^2\beta^2\partial_x^2\right]\Delta E_S(x,t)\,. \tag{75}$$

This equation and the linearized continuity equation, Eq. (47), together determine the entropy flow in the wire. For $\Delta E_S(x,t) = \mathrm{e}^{iqx-i\omega t+\epsilon t}\Delta E_S(0,0)$, the entropy current is

$$\Delta\langle j_S(x,t)\rangle = \sigma_S(q,\omega)\Delta E_S(x,t)\,. \tag{76}$$

with

$$\begin{aligned}\sigma_S(q,\omega) = k^2\frac{2\pi v}{\hbar}\frac{c}{12}&\left[1+\left(\frac{\hbar v\beta q}{2\pi}\right)^2\right]\left(\frac{i}{\omega+i\epsilon-vq}+\frac{i}{\omega+i\epsilon+vq}\right)\\ &+ vk\beta\langle j_S\rangle_{eq}\left(\frac{i}{\omega+i\epsilon-vq}-\frac{i}{\omega+i\epsilon+vq}\right)\,.\end{aligned} \tag{77}$$

The change in the entropy density is

$$\begin{aligned}\Delta\langle\rho_S(x,t)\rangle = k^2\frac{2\pi}{\hbar}\frac{c}{12}&\left[1+\left(\frac{\hbar v\beta q}{2\pi}\right)^2\right]\left(\frac{i}{\omega+i\epsilon-vq}-\frac{i}{\omega+i\epsilon+vq}\right)\Delta E_S(x,t)\\ &+ k\beta\langle j_S\rangle_{eq}\left(\frac{i}{\omega+i\epsilon-vq}+\frac{i}{\omega+i\epsilon+vq}\right)\Delta E_S(x,t)\,.\end{aligned} \tag{78}$$

In the limit of uniform flow,

$$\sigma_S(\omega) = \lim_{q\to 0}\sigma_S(q,\omega) = \frac{c}{6}\frac{2\pi k^2 v}{\hbar}\frac{i}{\omega} \tag{79}$$

which agrees with the general formula for the entropic conductivity, Eq. (66), since the equilibrium entropy density of bulk-critical wire is

$$\mathcal{S}_{crit} = \frac{c}{6}\frac{2\pi}{\hbar v}\frac{k}{\beta}\,. \tag{80}$$

Eq. (79) is equivalent to the formula for the thermal conductivity of bulk-critical quantum wire, $\kappa(\omega,T) = (k^2\pi^2 Tvc/3\hbar)\delta(\omega)$, which was previously derived in Ref. 22.

As another check, take the static limit, $\Delta E_S(x,t) = -\partial_x \Delta\Phi_S(x)$. Eq. (78) gives the change in the entropy density:

$$\Delta\langle\rho_S(x,t)\rangle = -k^2 \frac{2\pi}{\hbar v}\frac{c}{6}\left[1 - \left(\frac{\hbar v\beta}{2\pi}\right)^2 \partial_x^2\right]\Delta\Phi_S(x)\,. \tag{81}$$

Let the entropic potential become uniform, $\Delta\Phi_S(x) = -\Delta T$. The system should respond to the perturbation by going to equilibrium at temperature $T + \Delta T$. The induced change in entropy density will be, according to Eq. (81),

$$\Delta\langle\rho_S(x,t)\rangle = k^2 \frac{2\pi}{\hbar v}\frac{c}{6}\Delta T \tag{82}$$

which agrees with the temperature derivative of the equilibrium entropy density, Eq. (80).

X. CHIRAL ENERGY AND ENTROPY CURRENTS

The energy-momentum tensor has only two independent components when the quantum wire is critical in the bulk. They can be written as two currents

$$T_R(x,t) = \frac{1}{2}(vT_t^t(x,t) + T_t^x(x,t)) \tag{83}$$

$$T_L(x,t) = \frac{1}{2}(vT_t^t(x,t) - T_t^x(x,t))\,. \tag{84}$$

The conservation laws, $\partial_\mu T_\nu^\mu(x,t) = 0$, become

$$(\partial_t + v\partial_x)T_R(x,t) = (\partial_t - v\partial_x)T_L(x,t) = 0\,, \tag{85}$$

so each is a chiral current, depending on a single coordinate:

$$T_R(x,t) = T_R(x - vt) \tag{86}$$

$$T_L(x,t) = T_L(x + vt)\,. \tag{87}$$

The entropy current is a sum of chiral entropy currents:

$$j_S(x,t) = j_R(x,t) - j_L(x,t) \tag{88}$$

$$\rho_S(x,t) = \frac{1}{v}j_R(x,t) + \frac{1}{v}j_L(x,t) \tag{89}$$

where

$$j_R(x,t) = j_R(x - vt) = k\beta T_R(x,t) - \frac{1}{2}k\beta v\langle T_t^t(x,t)\rangle_{eq} \tag{90}$$

$$j_L(x,t) = j_L(x + vt) = k\beta T_L(x,t) - \frac{1}{2}k\beta v\langle T_t^t(x,t)\rangle_{eq}\,. \tag{91}$$

$j_R(x,t)$ flows to the right, $j_R(x,t) = j_R(x + v\delta t, t + \delta t)$, and $j_L(x,t)$ flows to the left, $j_L(x,t) = j_L(x - v\delta t, t + \delta t)$, both at the speed of "light", v.

The chiral energy currents are, up to normalization, the usual chiral components, $T(z)$ and $\bar{T}(\bar{z})$, of the euclidean energy-momentum tensor: $T_R(z) = -\hbar v^2 T(z)/2\pi$, $T_L(\bar{z}) = -\hbar v^2 \bar{T}(\bar{z})/2\pi$, where $z = x + iv\tau$, $\bar{z} = x - iv\tau$, $\tau = it$. The usual analytic operator product expansions of $T(z)$ and $\bar{T}(\bar{z})$ are equivalent to equal-time commutation relations (see Appendix B of Ref. 18 for details). These are equivalent to the general equal-time commutation relations derived in Appendix A, specialized to the bulk-critical case.

A naive calculation shows the condition for uniform entropy flow in bulk-critical wire. Consider a wire of length l. Let $I(t) = I(0)\cos(i\omega t)$ be the entropy current flowing into the wire from the left, at $x = -l/2$. The same entropy current flows out of the wire to the right, at $x = +l/2$. The boundary conditions completely determine the expectation values of the chiral currents,

$$\langle j_S(-\frac{l}{2},t)\rangle = \langle j_L(-\frac{l}{2} - vt)\rangle - \langle j_R(-\frac{l}{2} + vt)\rangle = I(0)\cos(i\omega t) \tag{92}$$

$$\langle j_S(+\frac{l}{2},t)\rangle = \langle j_L(+\frac{l}{2} - vt)\rangle - \langle j_R(+\frac{l}{2} + vt)\rangle = I(0)\cos(i\omega t)\,, \tag{93}$$

so the entropy current inside the wire is

$$\langle j_S(x,t)\rangle = I(t)\frac{\cos(\omega x/v)}{\cos(\omega l/2v)}\,. \tag{94}$$

The condition for uniform flow is therefore $\omega l \ll v$.

XI. CONCLUSION

An argument has been presented that near-critical quantum circuits are, in principle, ideal physical systems for large-scale quantum computers, because they are effectively isolated and controllable. The relativistic quantum field theories in 1+1 dimensions are then universal "machine languages" for large-scale quantum circuit computers. It was remarked that laws governing the flow of entropy are basic constraints on the design of reversible quantum computers, and that entropy flows in near-critical quantum circuits as a conserved quantum current, so circuit laws can be written for entropy flow in analogy with the electric circuit laws.

It was argued that the quantum wires should be stably bulk-critical, with no relevant bulk couplings, to avoid intractable control problems in the bulk wires. It was pointed out that bulk-critical quantum wires have some possibly useful features: all excitations move along the wires at a uniform speed, v, and the entropy current separates into left and right moving chiral currents which do not interact with each other in the bulk.

The continuity equation for entropy, Eq. (2), and the equation for the conduction of entropy in wires, Eq. (3), were derived. The entropy continuity and conduction equations are exact. Neither depends on a linear response approximation. They are universal equations, because they follow from universal equal-time commutation relations of the energy density and current.

The conduction equation, in the limit of uniform flow, was shown to imply a formula for the entropic conductivity of near-critical quantum wire, $\sigma_S(\omega) = iv^2\mathcal{S}/\omega T$, equivalent to a formula for the thermal conductivity, $\kappa(\omega,T) = \pi v^2\mathcal{S}\delta(\omega)$. This formula provides a way to measure directly, by experiment, the entropy density, $\mathcal{S}$, of the low-energy excitations in one-dimensional and quasi-one-dimensional near-critical quantum systems.

It is hoped that these will be useful preliminary steps towards the design of near-critical quantum circuits that can perform large-scale quantum computations.

Acknowledgments

I thank A. Konechny for many discussions. I thank the members of an informal Rutgers seminar — S. Ashok, A. Ayyer, D. Belov, E. Dell'Aquila, B. Doyon, and R. Essig — for listening to a preliminary version of this work, and for their comments and questions. I thank M. Douglas and G. Moore for reminding me that the monster conformal field theory is an example of a completely stable renormalization group fixed point in 1+1 dimensions, and G. Moore for pointing out Ref. 15. I thank S. Lukyanov for pointing towards some of the condensed matter literature, leading in particular to Refs. 21,22. I thank N. Andrei for helpful comments on the manuscript and for explaining to me that there are quantum critical phenomena which are not described by relativistic quantum field theories.

This work was supported by the Rutgers New High Energy Theory Center.

APPENDIX A: EQUAL-TIME COMMUTATORS OF $T^t_t(x,t)$ AND $T^t_x(x,t)$

The universal equal-time commutators of the energy and momentum densities

$$\frac{i}{\hbar}[T^t_t(x',t),\, T^t_t(x,t)] \;=\; -\partial_x\delta(x'-x)2T^x_t(x,t) - \delta(x'-x)\partial_x T^x_t(x,t) \tag{A1}$$

$$\frac{i}{\hbar}[T^t_x(x',t),\, T^t_x(x,t)] \;=\; \partial_x\delta(x'-x)2T^t_x(x,t) + \delta(x'-x)\partial_x T^t_x(x,t) \tag{A2}$$

$$\begin{aligned}\frac{i}{\hbar}[T^t_t(x',t),\, T^t_x(x,t)] \;=\; & -\frac{c_{UV}}{6}\frac{\hbar v}{2\pi}\partial_x^3\delta(x'-x) + \partial_x\delta(x'-x)[T^t_t(x,t) - T^x_x(x,t)] \\ & - \delta(x'-x)\partial_x T^x_x(x,t)\end{aligned} \tag{A3}$$

are derived here from the Ward identities for the operator product of two energy-momentum tensors. The number c_{UV} is the bulk conformal central charge at short-distance.

Make an infinitesimal local variation of the space-time metric, $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}(x,t)$, combined with an infinitesimal space-time transformation, $x^\mu \to x^\mu + \delta x^\mu(x,t)$. The combined change in the metric is

$$g_{\mu\nu} \to g_{\mu\nu} + \partial_\mu(\delta x_\nu) + \partial_\nu(\delta x_\mu) + \delta g_{\mu\nu} + \delta x^\alpha \partial_\alpha(\delta g_{\mu\nu}) + \partial_\mu(\delta x^\alpha)\delta g_{\alpha\nu} + \partial_\nu(\delta x^\alpha)\delta g_{\mu\alpha} \,. \tag{A4}$$

Vary $\ln Z$, keeping terms that are first order in δx^μ and in $\delta g_{\mu\nu}$, to obtain the Ward identity on the time-ordered product of two energy-momentum tensors:

$$\begin{aligned} \frac{i}{\hbar}\,\partial'_{\mu'}\, t\{T^{\mu'}_{\nu'}(x',t')\,T^\mu_\nu(x,t)\} &= \partial_\alpha\left[\delta(x'-x)\delta(t'-t)\right]\left(\delta^\alpha_{\nu'}T^\mu_\nu - g_{\nu'\nu}g^{\alpha\beta}T^\mu_\beta - \delta^\mu_{\nu'}T^\alpha_\nu\right)(x,t) \\ &\quad + \delta(x'-x)\delta(t'-t)\partial_{\nu'}T^\mu_\nu(x,t)\,. \end{aligned} \tag{A5}$$

Integrate both sides of the Ward identity over t' from $t-\epsilon$ to $t+\epsilon$ to obtain:

$$\begin{aligned} &\frac{i}{\hbar}[T^t_{\nu'}(x',t),\, T^\mu_\nu(x,t)] + \partial_{x'}\int_{t-\epsilon}^{t+\epsilon} \mathrm{d}t' \;\frac{i}{\hbar}\, t\{T^x_{\nu'}(x',t')\,T^\mu_\nu(x,t)\} = \\ &\int_{t-\epsilon}^{t+\epsilon}\mathrm{d}t'\;\Big\{\partial_\alpha\left[\delta(x'-x)\delta(t'-t)\right]\left(\delta^\alpha_{\nu'}T^\mu_\nu - g_{\nu'\nu}g^{\alpha\beta}T^\mu_\beta - \delta^\mu_{\nu'}T^\alpha_\nu\right)(x,t) \\ &\qquad + \delta(x'-x)\delta(t'-t)\partial_{\nu'}T^\mu_\nu(x,t)\Big\}\,. \end{aligned} \tag{A6}$$

The time integral on the lhs picks out the contact terms in the time-ordered operator product. The energy-momentum tensor has scaling dimension 2, so the contribution of the contact terms has the form:

$$\begin{aligned} &\int_{t-\epsilon}^{t+\epsilon}\mathrm{d}t'\;\frac{i}{\hbar}\, t\{T^{\mu'}_{\nu'}(x',t')\,T^\mu_\nu(x,t)\} = \\ &\int_{t-\epsilon}^{t+\epsilon}\mathrm{d}t'\;\left[C^{\mu'\mu\alpha\beta}_{\nu'\nu}\partial_\alpha\partial_\beta + B^{\mu'\mu\alpha}_{\nu'\nu}(x,t)\partial_\alpha + A^{\mu'\mu}_{\nu'\nu}(x,t)\right]\left[\delta(x'-x)\delta(t'-t)\right] \end{aligned} \tag{A7}$$

for some operator-valued tensors A, B, C. The operators $C^{\mu'\mu\alpha\beta}_{\nu'\nu}$ have scaling dimension 0, so are multiples of the identity.

By Eqs. (A6) and (A7), the equal-time commutators of the energy and momentum densities are:

$$\begin{aligned} \frac{i}{\hbar}[T^t_{\nu'}(x',t),\, T^t_\nu(x,t)] &= c_{\nu'\nu}\partial^3_x\delta(x'-x) + b_{\nu'\nu}(x,t)\partial^2_x\delta(x'-x) \\ &\quad + \left(\delta^x_{\nu'}T^t_\nu - \delta^t_{\nu'}T^x_\nu - g_{\nu'\nu}T^t_x + a_{\nu'\nu}\right)(x,t)\partial_x\delta(x'-x) \\ &\quad + \partial_{\nu'}T^t_\nu(x,t)\delta(x'-x) \end{aligned} \tag{A8}$$

where $a_{\nu'\nu} = A^{tt}_{\nu'\nu}$, $b_{\nu'\nu} = B^{ttxx}_{\nu'\nu}$, and $c_{\nu'\nu} = C^{ttxx}_{\nu'\nu}$.

The antisymmetry of the commutators is equivalent to:

$$0 = c_{\nu'\nu} - c_{\nu\nu'} \tag{A9}$$
$$0 = b_{\nu'\nu} + b_{\nu\nu'} \tag{A10}$$
$$0 = \partial_x(a_{xx} - 2T^t_x) \tag{A11}$$
$$0 = \partial_x a_{tt} \tag{A12}$$
$$0 = \partial_x(a_{xt} + a_{tx} + T^x_x - T^t_t) \tag{A13}$$
$$0 = a_{tx} - a_{xt} - 2\partial_x b_{tx} - T^t_t - T^x_x\,. \tag{A14}$$

Therefore $b_{xx} = b_{tt} = 0$, $b_{xt} = -b_{tx}$, and, up to multiples of the identity operator,

$$a_{xx} = 2T^t_x \tag{A15}$$
$$a_{tt} = 0 \tag{A16}$$
$$a_{xt} = \partial_x b_{tx} - T^x_x \tag{A17}$$
$$a_{tx} = \partial_x b_{tx} + T^t_t\,. \tag{A18}$$

Ignoring multiples of the identity operator for the time being, the only unknown is the operator $b_{tx}(x,t)$. The equal-time commutators are, up to multiples of the identity,

$$\frac{i}{\hbar}[T^t_t(x',t),\, T^t_t(x,t)] \;=\; -\partial_x\delta(x'-x)2T^x_t(x,t) - \delta(x'-x)\partial_x T^x_t(x,t) \tag{A19}$$

$$\frac{i}{\hbar}[T^t_x(x',t),\, T^t_x(x,t)] \;=\; \partial_x\delta(x'-x)2T^t_x(x,t) + \delta(x'-x)\partial_x T^t_x(x,t) \tag{A20}$$

$$\begin{aligned}\frac{i}{\hbar}[T^t_t(x',t),\, T^t_x(x,t)] \;&=\; +\partial_x\delta(x'-x)(T^t_t - T^x_x)(x,t) - \delta(x'-x)\partial_x T^x_x(x,t) \\ &\quad + \partial_x\left[\partial_x\delta(x'-x)b_{tx}(x,t)\right] .\end{aligned} \tag{A21}$$

Take the time derivative of both sides of Eq. (A19). In the time derivative of Eq. (A19), use Eq. (A21) to evaluate the commutators. The equation that results is:

$$0 = 2\partial_x^3\delta(x'-x)b_{tx}(x,t) + 3\partial_x^2\delta(x'-x)\partial_x b_{tx}(x,t) + \partial_x\delta(x'-x)\partial_x^2 b_{tx}(x,t)\,. \tag{A22}$$

So $b_{tx}(x,t) = 0$.

Eqs. (A19–A21), with $b_{tx}(x,t) = 0$, give the equal-time commutators up to multiples of the identity. These are exactly Eqs. (A1–A3), up to multiples of the identity. So all that remains is to determine the multiples of the identity operator that appear in the equal-time commutators.

The terms proportional to the identity operator in Eqs. (A1–A3) are determined by evaluating the expectation values of the equal-time commutators in the ground-state. The spectral representation of the ground-state two-point function of the energy-momentum tensor is:[26]

$$\langle 0|\,\frac{i}{\hbar}\,t\{T^{\mu'}_{\nu'}(x',t')\,T^{\mu}_{\nu}(x,t)\}\,|0\rangle = \int_0^\infty \mathrm{d}(m^2)\,\rho_c(m^2)\;G^{\mu'\mu}_{\nu'\nu}(x'-x,t'-t;m^2) \tag{A23}$$

$$G^{\mu'\mu}_{\nu'\nu}(x,t;\mu) = \frac{1}{(2\pi)^2}\int\!\!\int \mathrm{d}p_x\mathrm{d}p_t\;\mathrm{e}^{ip_x x + ip_t t}\,\frac{(p_{\nu'}p^{\mu'} - \delta^{\mu'}_{\nu'}p^2)(p_\nu p^\mu - \delta^\mu_\nu p^2)}{p_\mu p^\mu + m^2 + i\epsilon}\,. \tag{A24}$$

The conformal central charge in the short distance limit, c_{UV}, is given by

$$\int \mathrm{d}(m^2)\,\rho_c(m^2) = \frac{c_{UV}}{6}\,\frac{\hbar v}{2\pi}\,. \tag{A25}$$

Extract the equal-time commutator from Eq. (A23) by evaluating at $t' = t + \epsilon$ and at $t' = t - \epsilon$ and taking the difference:

$$\begin{aligned}\langle 0|\,\frac{i}{\hbar}\,[T^{\mu'}_{\nu'}(x',t)\,T^{\mu}_{\nu}(x,t)]\,|0\rangle &= \int_0^\infty \mathrm{d}(m^2)\,\rho_c(m^2)\;\frac{1}{2\pi}\int \mathrm{d}p_x\;\mathrm{e}^{ip_x(x'-x)} \\ &\quad \frac{1}{2\pi}\int \mathrm{d}p_t\;\frac{\mathrm{e}^{ip_t\epsilon} - \mathrm{e}^{-ip_t\epsilon}}{p_\mu p^\mu + m^2 + i\epsilon}\left[(p_{\nu'}p^{\mu'} - \delta^{\mu'}_{\nu'}p^2)(p_\nu p^\mu - \delta^\mu_\nu p^2)\right]\,.\end{aligned} \tag{A26}$$

In particular,

$$\langle 0|\,\frac{i}{\hbar}[T^t_t(x',t)\,T^t_t(x,t)]\,|0\rangle \;=\; 0 \tag{A27}$$

$$\langle 0|\,\frac{i}{\hbar}[T^t_x(x',t)\,T^t_x(x,t)]\,|0\rangle \;=\; 0 \tag{A28}$$

$$\langle 0|\,\frac{i}{\hbar}[T^t_t(x',t)\,T^t_x(x,t)]\,|0\rangle \;=\; -\partial_x^3\delta(x'-x)\frac{c_{UV}}{6}\frac{\hbar v}{2\pi}\,. \tag{A29}$$

This fixes the terms proportional to the identity operator in Eqs. (A1–A3), finishing their derivation.

APPENDIX B: $\sigma_S(\omega) = iv^2\mathcal{S}/\omega T$ FROM THE KUBO FORMULA

The Kubo formula for the entropy current induced in a wire by an entropic potential $\Delta V_S(x,t)$ is

$$\begin{aligned}\Delta\langle j_S(x_2,t_2)\rangle \;&=\; \int_{-\infty}^{t_2}\mathrm{d}t_1\;\langle\frac{i}{\hbar}[\Delta H(t_1), j_S(x_2,t_2)]\rangle_{eq} \\ &=\; \int_{-\infty}^{t_2}\mathrm{d}t_1\;\langle\frac{i}{\hbar}[\int \mathrm{d}x_1\,\Delta V_S(x_1,t_1)\rho_S(x_1,t_1), j_S(x_2,t_2)]\rangle_{eq}\,.\end{aligned} \tag{B1}$$

The Kubo formula is the solution of the time evolution equation, Eq. (35), in the linear response approximation.

For an alternating entropic potential, $\Delta V_S(x,t) = \mathrm{e}^{iqx-i\omega t}\Delta V_S(0,0)$, the induced current is

$$\Delta\langle j_S(x,t)\rangle = \sigma_S(q,\omega)\Delta E_S(x,t) \tag{B2}$$

where $\Delta E_S(x,t) = -iq\Delta V_S(x,t)$. The Kubo formula for the entropic conductivity is

$$\begin{aligned}\sigma_S(q,\omega) &= \frac{i}{q}\int \mathrm{d}x_1 \int_{-\infty}^{t_2} \mathrm{d}t_1\ \mathrm{e}^{i\omega(t_2-t_1)-iq(x_2-x_1)}\langle\frac{i}{\hbar}[\rho_S(x_1,t_1), j_S(x_2,t_2)]\rangle_{eq} \\ &= k^2\beta^2\frac{i}{q}\int \mathrm{d}x_1 \int_{-\infty}^{t_2} \mathrm{d}t_1\ \mathrm{e}^{i\omega(t_2-t_1)-iq(x_2-x_1)}\langle\frac{i}{\hbar}[T^t_t(x_1,t_1), T^x_t(x_2,t_2)]\rangle_{eq}\ .\end{aligned} \tag{B3}$$

Introduce the Fourier transform of the energy-momentum tensor:

$$\tilde{T}^\mu_\nu(p,\eta) = \int \mathrm{d}x \int \mathrm{d}t\ \mathrm{e}^{i(\eta t - px)} T^\mu_\nu(x,t)\ . \tag{B4}$$

Write its two-point functions:

$$\langle \tilde{T}^{\mu'}_{\nu'}(p',\eta')\,\tilde{T}^\mu_\nu(p,\eta)\rangle_{eq} = (2\pi)^2\delta(p'+p)\delta(\eta'+\eta)G^{\mu'\mu}_{\nu'\nu}(p,\eta)\ . \tag{B5}$$

The equilibrium expectation values of the commutators are given by

$$\langle\frac{i}{\hbar}[\tilde{T}^{\mu'}_{\nu'}(p',\eta')\,\tilde{T}^\mu_\nu(p,\eta)]\rangle_{eq} = (2\pi)^2\delta(p'+p)\delta(\eta'+\eta)\frac{i}{\hbar}\left(1-\mathrm{e}^{\beta\hbar\eta}\right)G^{\mu'\mu}_{\nu'\nu}(p,\eta)\ . \tag{B6}$$

The Kubo formula becomes

$$\sigma_S(q,\omega) = \frac{k^2\beta^2}{\hbar}\int \mathrm{d}\eta\ \frac{1}{\omega+i\epsilon-\eta}\left(1-\mathrm{e}^{\beta\hbar\eta}\right)\frac{1}{iq}G^{tx}_{tt}(q,\eta)\ . \tag{B7}$$

Conservation and symmetry of the energy-momentum tensor imply

$$\eta\tilde{T}^x_t(q,\eta) = -v^2 q\tilde{T}^x_x(q,\eta) \tag{B8}$$

so

$$\frac{1}{q}G^{tx}_{tt}(q,\eta) = -\frac{v^2}{\eta}G^{tx}_{tx}(q,\eta) \tag{B9}$$

so

$$\sigma_S(q,\omega) = k^2v^2\beta^3\int \mathrm{d}\eta\ \frac{i}{\omega+i\epsilon-\eta}\left(1-\mathrm{e}^{\beta\hbar\eta}\right)\frac{1}{\beta\hbar\eta}G^{tx}_{tx}(q,\eta)\ . \tag{B10}$$

In the uniform limit, $q\to 0$,

$$\lim_{q\to 0} G^{tx}_{tx}(q,\eta) = \delta(\eta)\langle H_0\, T^x_x(x,t)\rangle_{eq} = -\delta(\eta)\frac{\partial}{\partial\beta}\langle T^x_x(x,t)\rangle_{eq} \tag{B11}$$

so

$$\sigma_S(\omega) = \lim_{q\to 0}\sigma_S(q,\omega) = k^2v^2\beta^3\frac{i}{\omega+i\epsilon}\frac{\partial}{\partial\beta}\langle T^x_x(x,t)\rangle_{eq} \tag{B12}$$

The equilibrium entropy density is (see Eq. (64)):

$$\mathcal{S} = k\beta^2\frac{\partial}{\partial\beta}\langle T^x_x(x,t)\rangle_{eq} \tag{B13}$$

so

$$\sigma_S(\omega) = \frac{ik\beta v^2\mathcal{S}}{\omega}\ . \tag{B14}$$

The thermal conductivity is

$$\kappa(\omega, T) = \mathbf{Re}(T\sigma_S(\omega)) = \pi v^2 \mathcal{S}\delta(\omega) \, . \tag{B15}$$

---

* Electronic address: friedan@physics.rutgers.edu

1 R. Landauer, IBM J. Research and Development **3**, 183 (1961).
2 C. Bennett, IBM J. Research and Development **17**, 525 (1973).
3 P. Benioff, J. Stat. Phys. **29**, 515 (1982).
4 P. Benioff, Phys. Rev. Lett. **48**, 1581 (1982).
5 J. Cardy, *Scaling and Renormalization in Statistical Physics* (Cambridge University Press, Cambridge, 1996).
6 S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).
7 I. B. Frenkel, J. Lepowsky, and A. Meurman, in *Vertex Operators in Mathematics and Physics - Proceedings of a Conference November 10-17, 1983*, edited by J. Lepowsky, S. Mandelstam, and I. Singer (Springer, New York, 1985), no. 3 in Publications of the Mathematical Sciences Research Institute, pp. 231–273.
8 I. B. Frenkel, J. Lepowsky, and A. Meurman, Proc. Nat. Acad. Sci. USA **81**, 3256 (1984).
9 I. Frenkel, J. Lepowsky, and A. Meurman, *Vertex Operator Algebras and the Monster* (Pure and Applied Mathematics Volume 134, Academic Press, San Diego, 1988).
10 I. Affleck and A. W. Ludwig, Phys. Rev. Lett. **67**, 161 (1991).
11 D. Friedan and A. Konechny, Phys. Rev. Lett. **93**, 030402 (2004), hep-th/0312197.
12 D. Friedan, Z. Qiu, and S. Shenker, in *Vertex Operators in Mathematics and Physics - Proceedings of a Conference November 10-17, 1983*, edited by J. Lepowsky, S. Mandelstam, and I. Singer (Springer, New York, 1985), no. 3 in Publications of the Mathematical Sciences Research Institute, pp. 419–449.
13 D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. **52**, 1575 (1984).
14 D. Friedan and S. Shenker (1986), *Supersymmetric critical phenomena and the two dimensional gaussian model,* preprint, Enrico Fermi Institute, reprinted in *Conformal Invariance and Applications to Statistical Mechanics*, eds. C. Itzykson, H. Saleur, and J.B. Zuber (World Scientific, Singapore, 1988), pp. 578–579 .
15 L. Dixon, P. Ginsparg, and J. Harvey, Comm. Math. Phys. **119**, 221 (1988).
16 J. W. Gibbs (1888), *Letter to the Secretary of the Electrolysis Committee of the British Association for the Advancement of Science*, Report Brit. Asso. Adv. Sci. (1888), pp. 343–346, reprinted in *The Collected Works of J. Willard Gibbs,* Yale University Press (New Haven, 1928, 1948), vol. 1, pp. 408-412.
17 P. DiFrancesco, P. Mathieu, and D. Senechal, *Conformal Field Theory* (Springer-Verlag, New York, 1997).
18 D. Friedan (2005), cond-mat/0505085.
19 C. Chamon, M. Oshikawa, and I. Affleck, Phys. Rev. Lett. **91**, 206403 (2003).
20 S. Sachdev, in *The New Physics For the Twenty-First Century*, edited by G. Fraser (Cambridge University Press, Cambridge, 2005), 2nd ed.
21 J. M. Luttinger, Physical Review **135**, A1505A1514 (1964).
22 E. Orignac, R. Chitra, and R. Citro, Phys. Rev. **B67**, 134426 (2003).
23 A. B. Zamolodchikov, Nucl. Phys. **B342**, 695 (1990).
24 H. Blöte, J. Cardy, and M. Nightingale, Phys. Rev. Lett. **56**, 742 (1986).
25 I. Affleck, Phys. Rev. Lett. **56**, 746 (1986).
26 A. Cappelli, D. Friedan, and J. I. Latorre, Nucl. Phys. **B352**, 616 (1991).