All possible symmetries of CFTs

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Introduction

Noether's theorem: symmetries constrain the dynamics.

How do the dynamics constrain possible symmetries?

In the context of relativistic QFT the question can be restated as follows:

What are the possible symmetries
of a theory with the non-trivial S matrix?

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Assumptions:

- the S matrix exists: a theory has a mass gap (theory is IR free);
- the S matrix is nontrivial: everything scatters into something;
- the Poincare group is part of the symmetry group;

Conclusion:

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There are many ways to evade the Coleman-Mandula theorem

- d=2
- SUSY
- CFT
- AdS

- d>2
- superalgebra (HLS 1975)
- the S matrix does not exist
- Vasiliev theory

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What are the possible symmetries

of a CFT with

the non-trivial spectrum and correlation functions?

Known symmetries of nontrivial CFTs

Before trying to prove that something is impossible let's summarize what we know is possible:

- symmetry can be infinite dimensional in $d = 2 \to d > 2$;
- SUSY $\to$ generators or currents of half-integer spin;
- internal symmetries are definitely allowed;
- no examples of non-trivial CFTs with conserved currents of spin higher than two.

Could we have more symmetries while having non-trivial correlation functions?

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Assumptions

Let's consider a set of QFTs for which the following is true:

- CFT: $\mathcal{H} = \bigoplus [\mathcal{O}_{\Delta,s}]$, OPE, j_2;
- the theory is unitary;
- the two-point function of stress tensors is finite;
- the theory contains conserved current j_s of spin higher than two $s > 2$.

Additional assumptions

- $d = 3$ (for $d > 3$ the same set of ideas is applicable);
- the stress tensor (or conserved current of spin two) is unique.

Conclusion

The theory contains an infinite number of currents j_s.
Their correlation functions

$$\langle j_{s_1}(x_1)\ldots j_{s_n}(x_n)\rangle$$

are fixed to be the free boson or the free fermion ones up to one real positive number.
This number is the coefficient in the two-point function of stress tensors.
Start with N bosons or fermions. Compute the correlation functions as the function of N. Continue them to arbitrary N.
N would be quantized if we put CFT on the nontrivial space?

Outline

- General idea
- Free fields
- Exploiting the twist gap
- Conserved currents sector
- Bi-primary fields
- Theories with higher spin symmetries broken at the $\frac{1}{N}$ level

General idea

We consider CFT on the plane.
We start from the extra conserved current j_s, build the extra symmetry charges

$$
\begin{aligned}
Q_s^\zeta &= \int_{\Sigma_{d-1}} *j_s^\zeta \\
j_s^\zeta &= j_{\mu\mu_1\dots\mu_{s-1}}\zeta^{\mu_1\dots\mu_{s-1}} \\
[Q_s^\zeta, \mathcal{O}(x)] &= \int_{S_{x+\epsilon}} *j_s^\zeta(x+\epsilon)\mathcal{O}(x)
\end{aligned}
$$

where ζ is the conformal Killing tensor. We study Ward identities which one gets by acting with these extra charges on the conserved currents $[Q_\zeta, j_s]$.

Free fields

Let's consider the free scalar field $\phi(x)$ and let's consider the charge built using constant CKT ζ. Then the action of this charge on the free field is

$$[Q^{\zeta}, \phi(x)] = \zeta^{\mu_1 \cdots \mu_{s-1}} \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \phi(x)$$

Consider now the WI that we get by acting on the correlation function

$$\langle [Q_s, \phi(x_1)\phi(x_2)\ldots\phi(x_n)] \rangle = 0$$

in momentum space it takes the form

$$\left(\sum_{i=1}^{n} k_i^{s-1}\right) \langle \phi(k_1)\phi(k_2)\ldots\phi(k_n) \rangle = 0$$

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Free Fields II

Notice that not only higher spin charges act in a very simple manner on the free field but also that the OPE

$$\phi(x)\phi(0) \sim \sum [j_s]$$

allows one to extract the correlation functions of conserved currents!

So, one can, in principle, hope to find the free field in the spectrum of CFT as soon as higher spin charge is present.

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Free Fields III

However, this is too naive... Let's consider N free bosons $\phi_i(x)$ and limit ourselves to a singlet sector.

This is consistent CFT with higher spin symmetries but there is no free field in the spectrum! Thus, we should look for something else...

The next easiest object is the bi-local operator

$$B(x, y) = \phi_i(x)\phi_i(y)$$

- is always present;
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Twist gap

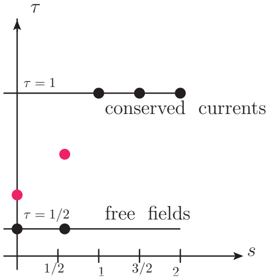

Figure: Spectrum of the unitary CFT in $d = 3$

The unitary constrains the possible dimensions of operators as follows

$$\begin{aligned} \Delta &\geq s + 1/2, \quad s = 0, 1/2 \\ \Delta &\geq s + 1, \quad s \geq 1 \end{aligned}$$

Thus, if we introduce the twist $\tau = \Delta - s$ then the operators with the twist

$$1/2 \leq \tau < 1$$

could have only 0 or $1/2$ spin.

Twist gap II

Let's imagine that we have some CFT such that there is a scalar operator ϕ_Δ with the twist lying inside the twist gap

$$\Delta = \frac{1}{2} + \gamma, \quad \gamma < 1/2.$$

Suppose also that it is charged under the $j_4 \to Q_4$.
Let's build the charge Q_{---} using CKT along the minus direction.

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Notice that Q_{---} has dimension $\Delta = 3$ spin $s = 3$ and twist 0. Then from the unitarity and conformal symmetry of the OPE it follows that

$$[Q_{---}, \phi_\Delta] = \sum_i c_i \partial_-^3 \phi_{i,\Delta}$$

Diagonalizing the action of the charge and writing WI for the four point function in momentum space one gets

$$(\sum k_{-,i}^3)\langle \phi_\Delta(k_1)\phi_\Delta(k_2)\phi_\Delta(k_3)\phi_\Delta(k_4)\rangle = 0$$

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$$\langle \phi_\Delta(x_1)\phi_\Delta(x_2)\phi_\Delta(x_3)\phi_\Delta(x_4)\rangle = \langle \phi_\Delta(x_1)\phi_\Delta(x_2)\rangle \ \langle \phi_\Delta(x_3)\phi_\Delta(x_4)\rangle + \ldots$$

Due to the fact that all operators couple to stress tensor $\langle \phi_\Delta \phi_\Delta T\rangle \neq 0$ and the fact that $\langle TT\rangle$ is finite, the stress tensor should be present in the OPE.

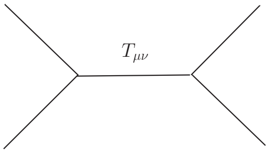

Figure: Stress tensor should be present in the OPE

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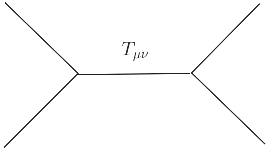

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Twist gap IV

However, looking at the OPE of the factorized four point function one can check that for $\Delta \neq \frac{1}{2}$ such operator is absent...
Thus, our assumption about existence of such an operator was wrong...
This excludes all charged under higher spin symmetries operators that have the twist inside the twist gap $1/2 < \tau < 1$.
Now let's switch to conserved currents...

Conserved currents: basic properties

To attack the sector of conserved currents let's recall the basic properties of three point functions of conserved currents (Giombi-Prakash-Yin, Costa-Penedones-Poland-Rychkov).

$$\langle j_{s_1} j_{s_2} j_{s_3} \rangle = \langle \text{boson} \rangle + \langle \text{fermion} \rangle + \langle \text{odd} \rangle$$

where

$$\mathcal{F}_{even} = e^{\frac{1}{2}(Q_1+Q_2+Q_3)} e^{P_1+P_2} (b \cosh P_3 + f \sinh P_3)$$

and the odd piece is given by

$$\langle j_{s_1}(\vec{x}_1, \lambda_1) j_{s_2}(\vec{x}_2, \lambda_2) j_{s_3}(\vec{x}_3, \lambda_3) \rangle_{odd} \sim \int dt d^3 \vec{x}_0 t^{s_1+s_2+s_3-1}$$

$$\frac{(\lambda_1 \not{x}_{10} \not{x}_{02} \lambda_2)^{(s_1+s_2-s3)} (\lambda_1 \not{x}_{10} \not{x}_{03} \lambda_3)^{(s_1+s_3-s2)} (\lambda_2 \not{x}_{20} \not{x}_{03} \lambda_3)^{(s_2+s_3-s1)}}{(x_{10}^2)^{2s_1+1} (x_{20}^2)^{2s_2+1} (x_{30}^2)^{2s_3+1}}$$

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Other useful properties to remember are

$$\langle \mathcal{O}\mathcal{O}T \rangle \neq 0$$

and also

$$\langle j_s j_s j_{s'} \rangle = 0$$

when s' is odd.

We will be again interested in all-minus charges $Q_s = Q_{-\dots-}$.
So let's introduce the following notations

$$\langle j_{s_1 -\dots-}(x_1) j_{s_2 -\dots-}(x_2) j_{s_3 -\dots-}(x_3) \rangle = \langle s_1 s_2 s_3 \rangle$$

Rule of thumb: whenever you think that there should be a tensor index — it is the minus index.

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Conserved currents: action of the higher spin charges

As the next step consider the action of minus-charge on the all-minus component of conserved currents. Again, unitarity fixes it up to several constants

$$[Q_s, k] = \sum_{i=-s}^{s} c_i \partial^{(s-i)}(k+i).$$

We can do a little bit better

$$[Q_s, 2] \sim \partial(s-1).$$

This term must be there! And this opens the flow...

$$[Q_s, X] \sim Y$$

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Conserved currents: example

Consider the theory that contains spin four current j_4. From it we build minus charge Q_4. When acting on the stress tensor

$$[Q_4, 2] \sim \partial 4$$

Let's consider the $\langle [Q_4, 224] \rangle = 0$ WI. On the general grounds there will be

$$\partial_{x_1} \langle 4(x_1) 2(x_2) 4(x_3) \rangle \neq 0$$

thus, we get algebraic equation

$$c_{422} \partial_{x_1} \langle 4(x_1) 2(x_2) 4(x_3) \rangle + c_{222} \partial^3_{x_1} \langle 2(x_1) 2(x_2) 4(x_3) \rangle + c_{022} \partial^3_{x_1} \langle 0(x_1) 2(x_2) 4(x_3) \rangle + \ldots = 0$$

on the coefficients...

Conserved currents: example

Consider the theory that contains spin four current j_4. From it we build minus charge Q_4. When acting on the stress tensor

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Conserved currents: results

From this simple exercise we can learn that

- there are three families of solutions (boson, fermion, odd);
  Is there interacting HS CFT with odd parts?
- if 4 is present, 6 is necessary present;
- for boson and odd the scalar 0 is necessary present.

Repeating a similar exercise for the scalar $\langle 022\rangle$ one can show that uniqueness of the stress tensor restricts

$$\begin{aligned}\langle 222\rangle &= \langle \text{boson}\rangle + \langle \text{odd}\rangle\\ \langle 222\rangle &= \langle \text{fermion}\rangle\end{aligned}$$

This is one of the many examples when boson and fermion solutions are separates.

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Lemma

By playing a little bit more one can be convinced that the following is true:

A theory that contains any conserved current of spin $s > 2$ contains an infinite number of currents and spin four $s = 4$ conserved current is necessary present in the theory.

However, let's come back to the three point function of stress tensors to appreciate it a little bit more...

Energy one point function

Knowing something about three point functions of stress tensors allows one to compute the so-called one point energy correlator .

$$\langle \mathcal{O}[\Psi] | \hat{\mathcal{E}}(\vec{n}) | \mathcal{O}[\Psi] \rangle$$

$$\hat{\mathcal{E}}(\vec{n}) = \lim_{r \to \infty} r^2 \int_{-\infty}^{\infty} dt n^i T_i^0(t, r\vec{n})$$

Some intuition about this object:

- measure the energy flow at infinity;
- small coupling - jets and showering;
- strong coupling - uniform flow (Hofman-Maldacena).

Energy one point function: no showering, no odd piece

Consider the case when three point function of stress tensors is boson+odd. Then one can show that

$$\langle T_{11} - T_{22} | \mathcal{E}(\vec{n}) | T_{11} - T_{22} \rangle = \frac{q^0}{2\pi}(1 + \cos 4\theta + d_{\mathrm{odd}} \sin 4\theta)$$

expanding near $\theta = \frac{\pi}{4}$ we find that for any $d \neq 0$

$$\langle \mathcal{E}(\theta) \rangle < 0$$

our theory is secretly non-unitary. So $\langle 222 \rangle$ is either a purely free boson or free fermion. So that

$$\langle \mathcal{E}(\theta) \rangle = \frac{q^0}{2\pi}(1 \pm \cos 4\theta)$$

Notice that for some angles the energy flow is zero.
This does not happen in the theory where the showering occurs.

Four point functions

One can also consider the four point functions of four scalars 0. After showing that

$$[Q_4, 0] = \partial^3 0 + \partial 2$$

we get the differential equation

$$\partial^3 \langle 0000 \rangle + \partial \langle 2000 \rangle + ... = 0.$$

This time we need to solve genuine differential equations for the functions of cross ratios...
The result is that the solution is fixed up to one constant

$$\langle 0000 \rangle = \langle \text{disconnected} \rangle + \frac{1}{c} \langle \text{connected} \rangle$$

Four point functions: conclusion

- using only one additional charge we can fix two correlation functions $\langle 0000 \rangle$ and $\langle 2000 \rangle$;
- $\langle 2000 \rangle_{odd}$ must be set to zero to obey WI, by the OPE it sets to zero odd piece of three point functions;
- the free fermion story is very similar.

We got this using only one additional charge in a very explicit way...
But we have an infinite number of them. So there should be a shorter way to the answer that uses all the symmetries!

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Introduction to bi-primaries

Let's consider the scalar bilinear operator built from conserved even spin current j_s

$$b_s(x,y) = \sum c_{i,n}(x-y)^i \partial^n j_s(\frac{x+y}{2})$$

defined by the equality

$$\langle b_s(x,y) j_s(z)\rangle = \langle \phi(x)\phi(y) j_s(z)\rangle$$

The properties of $b_s(x,y)$

- transforms as the bi-primary;
- obeys the Laplace equation in x and in y.

This is not a free field! For any theory there exists $b_2(x,y)$.

Taking the light-cone limit

As the next step we consider the light-cone limit of two conserved currents $j_s(x)j_{s'}(0)$. There are three types of limits that project to three different parts of three point functions (boson, fermion, odd).

For simplicity we present here the bosonic limit

$$\underline{j_s(x)j_{s'}(0)}_b = \left(\lim_{y_{12}\to 0^+} + \lim_{y_{12}\to 0^-}\right)|y_{12}| \lim_{x_{12}^+\to 0} j_s(x_1)j_{s'}(x_2)$$

here we assumed that the only operator of a twist less than one is the identity. All operators from the twist gap are absent due to the argument presented before.

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Simplification of the three point functions

The crucial simplification that occurs in the limit is the following

$$\langle \underline{j_s j_{s'}}_b j_{s''}(x_3)\rangle = \partial_1^s \partial_s^{s'} \langle b_{s''}(\underline{x_1, x_2}) j_{s''}(x_3)\rangle$$

where

$$\langle b_{s''}(\underline{x_1, x_2}) j_{s''}(x_3)\rangle = \frac{1}{\sqrt{x_{13}x_{23}}}\left(\frac{x_{12}}{x_{13}x_{23}}\right)^{s''}$$

and all indices are minuses as usual.
This allows us to analyze infinitely many WIs in a very simple manner.

Looking for the bi-primary

To find the bi-primary that consists of two free fields we take the light-cone limit of two stress tensors

$$\underline{j_2(x)j_2(y)}_b = \partial_1^2\partial_2^2 B(\underline{x,y}) , \qquad B(x,y) = \sum_{s\,\text{even}} c_s b_s(x,y)$$

if $\langle 222\rangle_b \neq 0$ then, at least, $c_2 \neq 0$.

Then we would like to prove that in the theory with higher spin symmetries $B(\underline{x,y})$ behaves as the normal ordered product of two bosonic free fields.

This will be done first by showing that

$$[Q_s, B(\underline{x_1,x_2})] = (\partial_1^{s-1} + \partial_2^{s-1}) B(\underline{x_1,x_2})$$

and then showing that this implies that correlators have the free field form.

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Proof of the simple transformation law

We would like to compute $[Q_s, B(\underline{x_1, x_2})]$.
We can compute

$$[Q_s, \underline{j_2, j_2}_b] = \underline{[Q, j_2] j_2}_b + \underline{j_2 [Q, j_2]}_b .$$

The action of Q commutes with the limit and we can write $[Q, j_2]$ in terms of currents and derivatives (with indices and derivatives all along the minus directions).
Thus, in the end we can write

$$[Q_s, B(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1})\tilde{B}(\underline{x_1, x_2}) + (\partial_1^{s-1} - \partial_2^{s-1})B'(\underline{x_1, x_2})$$

To prove that $\tilde{B} = B$ and that B' is absent boils down to considering the set of WIs of the form $\langle [Q_s, \hat{B}s'] \rangle$.

Proof that $\tilde{B} = B$

Consider $\langle [Q_s, B \;\; 2] \rangle = 0$.
One sees then that $\langle \tilde{B} \; 2 \rangle \neq 0$.
Consider $B - \tilde{B}$ where we normalize $\tilde{B}$ in such a way that the difference does not contain b_2.
To show that all the other currents will be also absent consider

$$\langle [Q_{s'}, (B - \tilde{B}) 2] \rangle = 0$$

Again the chain nature of WIs and the structure of correlation functions are extremely restrictive.

Higher spin symmetries broken at $\frac{1}{N}$ order

It is not complicated to generalize our consideration to the cases when the higher spin symmetries are broken at the $\frac{1}{N}$ order in large N limit.

In this case the dimensions of conserved currents will

$$\Delta_s = s + 1 + O(\frac{1}{N})$$

and the divergence of the currents will take the form

$$\partial_\mu j^\mu = \frac{1}{\sqrt{N}} \sum \mathcal{O}_i$$

where we assumed vector-like large N expansion and also the fact that we know the spectrum of operators at $N = \infty$.

The operators in the RHS should have the right quantum numbers.

In a completely analogous way to the exact symmetries we can analyze all possible structures in the RHS.

Now what will happen with the WIs?

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Shadow Ward Identities

In the case of fermions we would get

$$\partial_\mu j^\mu{}_{---} = \frac{1}{\sqrt{N}}[a_1 \partial_- \tilde{j}_0 j_{--} + a_2\ \tilde{j}_0 \partial_- j_{--}].$$

If we consider $\langle 222 \rangle$ WI we would get terms like

$$\int_V (\partial \langle j_2(x) j_2 \rangle) \langle \tilde{j}_0(x) j_2 j_2 \rangle$$

using the fact that all indices are minus this can be rewritten as

$$\partial^5 \int_V \langle j_0(x) j_0 \rangle \langle \tilde{j}_0(x) j_2 j_2 \rangle$$

Now notice that the integral has the all properties of $\langle j_0 j_2 j_2 \rangle$. This is the mechanism of the appearance of the twist one scalar in the WI story. The operator $\tilde{j}_0$ of dimension 2 is substituted by the scalar of dimension $d - 2$ which is what sometimes is called "shadow" field.

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Conclusions

- we analyzed the problem of possible symmetries of CFTs in $d > 2$;
- in $d = 3$ using unitarity, conformal symmetry and uniqueness of stress tensor we showed that addition of conserved currents of spin $s > 2$ makes the theory trivial;
- the case of multiple stress tensors looks very similar and hopefully will be addressed in the near future;
- our analysis heavily relied on the structure of three point functions of conserved currents in $d = 3$. The analysis for $d > 3$ is completely analogous and should be easy especially using the simplicity of the light-cone limit;
- using the same approach we can analyze the cases when higher spin symmetries are broken at $\frac{1}{N}$ order;

Conclusions

- all gravitational higher spin symmetric theories in AdS that preserve symmetry at the quantum level are described by free fields at the boundary;
- there is more freedom to have a theory which has higher spin symmetry at the classical level only;
- a theory that is dual to free fermion/scalar at the leading N order necessarily generates odd structures at one loop (this should be true in parity violating versions of Vasiliev theory considered recently. Our analysis supports duality with CS theory).

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