A New Look At The Gopakumar-Vafa and Ooguri-Vafa Formulas

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(This talk is based on a paper with M. Dedushenko.)

We introduce a set of coupling constants $\vec{t}$ that is dual to the homology class $\vec{d}$, and define the genus g topological string partition free energy

$$\mathcal{F}_g(\vec{t}) = \sum_{\vec{d} \in H_2(Y,\mathbb{Z})} a_{g,\vec{d}} \exp(2\pi i \vec{d} \cdot \vec{t}). \tag{1}$$

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The full topological string partition function free energy is

$$\mathcal{F}(g_{\rm st}, \vec{t}) = \sum_{g=0}^{\infty} g_{\rm st}^{2g-2} \mathcal{F}_g(\vec{t}). \tag{2}$$

(The partition function including disconnected contributions is $\mathcal{Z} = e^{\mathcal{F}}$.)

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$$C = \sum_{I=1}^{b_2} A^I(x) \cdot \omega_I(y)$$

where $\omega_I(y)$ are harmonic two-forms on Y, normalized to give a basis of $H^2(Y, \mathbb{Z})$ mod torsion, and $A^I(x)$ are $U(1)$ gauge fields on $\mathbb{R}^4$.

Type IIA on a Calabi-Yau manifold Y has eight unbroken supersymmetries and the effective action can be described in a superspace with four bosonic coordinates x^μ, four fermionic coordinates θ^{Ai} of negative chirality, and four more $\bar{\theta}^{\dot{A}i}$ of positive chirality. (Here all indices $A, \dot{A}, i$ take the values 1,2. A and $\dot{A}$ are spinor indices of respectively negative and positive chirality in four dimensions, and the index i is there because Type IIA on a Calabi-Yau has $\mathcal{N} = 2$ supersymmetry in four dimensions.)

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Of the $b_2 + 1$ $U(1)$ gauge fields, b_2 linear combinations are in vector multiplets, which for $\mathcal{N} = 2$ supersymmetry in four dimensions are described by chiral superfields

$$\mathcal{X}^\Lambda(x, \theta) = X^\Lambda + \theta \Psi^\Lambda + \theta^2 F^\Lambda + \ldots$$

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where F^Λ is a $U(1)$ gauge field strength. The remaining $U(1)$ gauge field is the "graviphoton," which is part of the supergravity multiplet. It also is part of a chiral superfield, but this is not a chiral superfield of spin zero. Rather, the anti-selfdual part of the graviphoton field strength, which I will write as a bispinor $W_{AB}(x)$ $(= W_{BA}(x) = \sigma^{\mu\nu}_{AB} W_{\mu\nu}(x))$, partly because this is natural in string perturbation theory, is the bottom component of a chiral superfield that itself is a bispinor $\mathcal{W}_{AB}(x, \theta) = W_{AB}(x) + \dots$.

In general, in a supersymmetric theory, a term in the effective action might be a "D-term," meaning that it can be written as an integral over all of superspace, in our case $\int \mathrm{d}^4x \mathrm{d}^4\theta \mathrm{d}^4\bar{\theta}(\dots)$, or an "F-term," meaning that it cannot be so written.

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In the case at hand, for every $g \geq 0$, there is a possible F-term

$$\mathcal{I}_g = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{F}_g(\mathcal{X}^0, \ldots, \mathcal{X}^{b_2})(\mathcal{W}_{AB}\mathcal{W}^{AB})^g.$$

These particular interactions have the remarkable property that, from the standpoint of Type IIA superstring perturbation theory, $\mathcal{I}_g$ is generated only in genus g. The proof of this just follows from the low energy supergravity. One transforms the interaction $\mathcal{I}_g$ to the string frame and finds (to simplify slightly) that it is proportional to g_{st}^{2g-2}.

The insight of BCOV is that the function $\mathcal{F}_g(\mathcal{X}^\Lambda)$ is the topological string partition function $\mathcal{F}_g(\vec{t})$ if one sets $t^I = \mathcal{X}^I/\mathcal{X}^0$, $I = 1, \ldots, b_2$. (Technically, here $\mathcal{X}^0$ is the multiplet that corresponds to the RR 1-form in ten dimensions and $\mathcal{X}^I$, $I = 1, \ldots, b_2$ correspond to the vector fields that arise from the C-field).

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What did Gopakumar and Vafa (1998) add to this story? They suggested that one should view Type IIA on $\mathbb{R}^4 \times Y$ in terms of M-theory on $\mathbb{R}^4 \times S^1 \times Y$, where S^1 is sometimes called the M-theory circle. In this correspondence, the string coupling constant is determined by the radius of the M-theory circle. Since $\mathcal{I}_g$ for given g has a known dependence on $g_{\rm st}$, its dependence on the radius of the S^1 is essentially also known and we can calculate in the region where the S^1 is large and the M-theory description is useful.

In string perturbation theory, $\mathcal{F}_g(\mathcal{X}^\Lambda)$ is computed by counting superstring worldsheets wrapped on a complex submanifold $\Sigma \subset Y$. This corresponds in M-theory to an M2-brane wrapped on $S^1 \times \Sigma$. Such an M2-brane can be studied in a Hamiltonian formalism in terms of states propagating in the S^1 direction. So it should be possible to compute the F-terms $\mathcal{I}_g$ by summing over contributions of wrapped M2-brane states.

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Moreover, they suggested that this sum should be viewed as the superspace effective action evaluated in a background in which a constant anti-selfdual graviphoton field is turned on. This superspace effective action is supposed to be computed by a one-loop calculation, summing over one-loop diagrams with BPS states running around the loop.

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Before explaining the computation that leads to the GV formula, I want to first address some general questions about it. Perhaps the first question is whether contributions to the interactions $\mathcal{I}_g$ can arise by dimensional reduction from a supersymmetric action in five dimensions. To the extent that the $\mathcal{I}_g$ can arise that way, we can only determine them by knowing the 5d effective action for M-theory on $\mathbb{R}^5 \times Y$; an interaction that is already present in five dimensions cannot be determined by a Schwinger computation that involves further compactification from $\mathbb{R}^5$ to $\mathbb{R}^4 \times S^1$.

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So we can only learn about F-terms by expanding about a supersymmetric background. In field theory, one can possibly use a background that is supersymmetric but not a classical solution, but it is doubtful that we would understand how to do that in string/M-theory. But at a minimum we need a supersymmetric background.

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If one turns on an arbitrary linear combination of anti-selfdual $U(1)$ gauge fields F^{Λ}_{AB} on $\mathbb{R}^4$, one will get no gravitational backreaction – since the energy momentum tensor of a selfdual or anti-selfdual Maxwell field vanishes. But generically one will get scalar backreaction. It turns out that scalar backreaction is avoided and moreover one gets a supersymmetric background precisely if one turns on only the graviphoton and not any of the gauge fields that are in vector multiplets.

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So a supersymmetric classical solution with the necessary properties actually exists. It is most simply described as a solution in a 5d spacetime with Lorentz signature. The solution was called the supersymmetric Gödel solution by Gauntlett, Gutowski, Hull, Pakis and Reall (GGHPR, 2003):

$$\mathrm{d}s^2 = -(\mathrm{d}t + V_\mu \mathrm{d}x^\mu)^2 + \sum_{\mu=1}^{4} (\mathrm{d}x^\mu)^2, \quad V_\mu = \frac{1}{8} T^-_{\mu\nu} x^\nu,$$

where $T^-_{\mu\nu}$ is the constant anti-selfdual graviphoton.

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where $T^-_{\mu\nu}$ is the constant anti-selfdual graviphoton. To use this in M-theory compactification to Type IIA, we need to compactify the t direction but also we want the metric on the circle to be Euclidean. So we rotate $t \to iy$ where y is a periodic variable.

It is not possible to make both the graviphoton $T^{-}_{\mu\nu}$ and the metric $\mathrm{d}s^2$ real. The lesser evil is to take the graviphoton imaginary so that the metric is real. The graviphoton being imaginary in the background is really not a problem since Schwinger's original calculation worked fine in either an electric or a magnetic field; an imaginary magnetic field in Euclidean space is somewhat like a real electric field in Lorentz signature.

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There are a few more questions we should ask and here is another one: Quantum mechanics has wave-particle duality but the range of validity of a calculation based on waves can be quite different from that of a calculation based on particles. Is the Schwinger calculation that leads to the GV formula supposed to be a calculation based on particles or on fields?

To answer this question, observe that – to the extent that we do understand M-theory – we understand it when the length scales involved are much greater than the Planck length.

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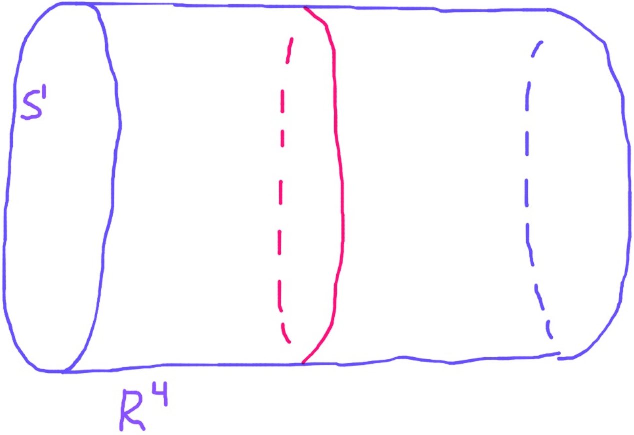
To do the particle calculation, one starts by thinking of the M2-brane wrapped on $p \times S^1 \times \Sigma \subset \mathbb{R}^4 \times S^1 \times Y$ (where p is a point in $\mathbb{R}^4$ and $\Sigma \subset Y$) as a sort of instanton:



In drawing the picture, I actually ignored Σ and just depicted the M2-brane worldvolume $p \times S^1 \times \Sigma$ as a particle worldline $p \times S^1 \subset \mathbb{R}^4 \times S^1$, with a point $p \in \mathbb{R}^4$.

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Let us see instead what happens for a half-BPS particle that is invariant under the four supersymmetries $Q^{\dot{A}i}$. Such a BPS particle comes from an M2-brane wrapped on a complex Riemann surface $\Sigma \subset Y$.

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The BPS particles that have no moduli except the ones forced by the symmetries are massive hypermultiplets; they come from an M2-brane wrapped on an isolated genus 0 curve $\Sigma \subset Y$. Let us analyze the contribution of such a particle. First we need to understand its mass. The Kahler form of Y is

$$\omega = \sum_I h^I \omega_I$$

where ω_I, $I = 1, \ldots, b_2$ are a basis of $H^2(Y, \mathbb{Z})$ and h^I are Kahler moduli. (I will take some minor shortcuts to avoid too many details.)

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$$\zeta(\vec{q}) = \sum_I q_I h^I$$

and (in units in which the M2-brane tension is 1) the mass of the BPS state is

$$M(\vec{q}) = \zeta(\vec{q}).$$

(We don't take the absolute value since a holomorphically wrapped M2-brane has $\zeta > 0$.)

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The fact that the effective magnetic field seen by the BPS state depends only on its mass is very important in getting the GV formula.

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$$I_0 = \int \mathrm{d}t \left(-M + \frac{M}{2}\sum_{\mu}(\dot{x}^{\mu})^2 + \frac{iM}{2}\psi_{Ai}\frac{\mathrm{d}}{\mathrm{d}t}\psi^{Ai} \right).$$

It turns out that this is enough since any higher order terms are "irrelevant" when the radius R of the M-theory circle is large and don't contribute to the F-terms.

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$$P^\mu = p^\mu = M\dot{x}^\mu$$

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It is straightforward to verify (the appropriate nonrelativistic limit of) the supersymmetry algebra.

Now let us turn on the graviphoton field. There is one obvious term that we need to add to the action: a coupling $\int \mathrm{d}t\, A_\mu(\vec{q})\dot{x}^\mu$ to the background gauge field $A(\vec{q}) = \sum_I q_I A^I$.

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$$I_1 = \frac{M}{2}\int \mathrm{d}t T^-_{\mu\nu} x^\mu \dot{x}^\nu.$$

The minimal possible action is thus

$$I = I_0 + I_1 = M\int \mathrm{d}t \left(-1 + \frac{1}{2}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\right)^2 + \frac{i}{2}\psi_{Ai}\dot{\psi}^{Ai} + \frac{1}{2}T^-_{\mu\nu}x^\mu \dot{x}^\nu \right).$$

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It turns out that this is the complete supersymmetric action, modulo irrelevant terms of higher dimension.

Once one turns on the graviphoton, one has to modify the translation generators P^μ, which no longer coincide with the canonical momenta p^μ

$$P^\mu = p^\mu - \frac{M}{2} T^{-\mu\nu} x_\nu$$

and no longer commute

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This is not surprising for a charged particle in a constant magnetic field. We can still use the old formulas for the supercharges

$$Q^{Ai} = M\psi^{Ai}, \quad Q^{\dot{A}i} = \gamma_\mu^{A\dot{A}} P_\mu \psi^i_A,$$

and they are obviously still conserved. But since the definition of P_μ has changed, the algebra they generate is deformed – in precisely the way seen in supergravity.

Now let us evaluate the path integral for fluctuations around the particle trajectory $p \times S^1$.

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$$\int \mathrm{d}^4 x \mathrm{d}^4 \theta \, \exp(-I_{\mathrm{cl}}) \cdot \frac{\sqrt{\det \mathcal{D}_F}}{\sqrt{\mathcal{D}_B}},$$

where I_{cl} is the classical action and $\det \mathcal{D}'_F$, $\det \mathcal{D}'_B$ are fermionic and bosonic one-loop determinants with zero-modes removed.

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where I_{cl} is the classical action and $\det \mathcal{D}'_F$, $\det \mathcal{D}'_B$ are fermionic and bosonic one-loop determinants with zero-modes removed. As is usual in instanton physics, we remove the zero-modes from the determinants and replace them with an integral over the corresponding moduli or collective coordinates x^μ and θ^{Ai}.

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So our answer is going to be

$$\int \mathrm{d}^4x\mathrm{d}^4\theta\, \exp(-2\pi i q_I \mathcal{Z}^I)\, \frac{\sqrt{\det \mathcal{D}_F}}{\sqrt{\mathcal{D}_B}}. \quad (3)$$

We still have to evaluate the determinants, but they are more or less the simplest functional determinants one will see. The fermionic determinant is completely trivial, since the fermions are free ($\mathcal{D}_F = i\mathrm{d}/\mathrm{d}t$), and the bosonic determinant is a classical example since $\mathcal{D}_B = \mathcal{D}_1\mathcal{D}_2$ where $\mathcal{D}_1 = -\mathrm{d}/\mathrm{d}t$ and $\mathcal{D}_2 = \mathrm{d}/\mathrm{d}t + T^-$, where T^- is a constant matrix.

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$$\int \mathrm{d}^4x\mathrm{d}^4\theta \, \exp(-2\pi i q_I \mathcal{Z}^I)\frac{T^2}{\sinh^2(\pi R T)}$$

where $T = \sqrt{T^-_{\mu\nu}T^{-\mu\nu}}$.

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$$\int \mathrm{d}^4x\mathrm{d}^4\theta \sum_{k=1}^{\infty} \frac{1}{k} \exp(-2\pi i k q_I \mathcal{Z}^I) \frac{T^2}{\sinh^2(\pi k R T)}.$$

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$$\int \mathrm{d}^4x\mathrm{d}^4\theta \sum_{k=1}^{\infty} \frac{1}{k} \exp(-2\pi i k q_I \mathcal{Z}^I) \frac{T^2}{\sinh^2(\pi kRT)}.$$

(It takes some discussion to justify ignoring the self-interactions of the BPS particles for $k > 1$, since after all the interactions between M2-branes are strong.)

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To get the GV formula as usually written, one just has to take the formulas I have described and re-express them in terms of the usual variables of 4d supergravity, as opposed to the 5d variables that we used in the derivation.

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The Ooguri-Vafa formula is obtained in the same way, by carrying out a similar analysis for a BPS superparticle that propagates in 2+1 rather than 4+1 dimensions.