From Gauge Theory To Integrability And Liouville Theory Via Coisotropic Branes

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(1) The connection with integrability explored by Nekrasov and Shatashvili;

(2) The connection to Liouville theory discovered by Alday, Gaiotto, and Tachikawa.

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(b) We map to a two-dimensional brane construction by looking at spacetime "torically."

(c) We use familiar facts about sigma models in a new way.

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The Ω deformation of an $N = 2$ supersymmetric gauge theory in four dimensions is made as follows. Such a theory contains a complex scalar field ϕ in the adjoint representation. Let σ be, for example, the real part of ϕ.

To make the Ω-deformation, we assume that we are given a vector field V that generates a $U(1)$ symmetry of spacetime. Then formally we make a change of variables

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$$\sigma \longrightarrow \sigma + \varepsilon V^{\mu} \frac{D}{Dx^{\mu}}.$$

This is not really an honest change of variables, because we are adding a derivative to a field.

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Still, it makes sense inside the $N = 2$ Lagrangian, because σ only appears in the form of $[\sigma, \sigma']$ (where σ' is another field) or $D_\alpha \sigma$. For example, the former transforms by

$$[\sigma, \sigma'] \to [\sigma + V^\mu D_\mu, \sigma']$$

$$= [\sigma, \sigma'] + \varepsilon V^\mu D_\mu \sigma'$$

which is just a function of fields, not a differential operator.

So the Ω-deformation gives a way to modify the Lagrangian, and since it does *not* come from an honest change of variables, the new Lagrangian is not equivalent to the old one. That is why the Ω-deformation can do something interesting.

However, the Ω-deformed theory is equivalent to the ordinary theory in different variables if $U(1)$ acts freely. In fact, suppose that spacetime is a product $M = Y \times S^1$, where the $U(1)$ symmetry acts by rotation of S^1. Let θ be an angular coordinate for this S^1 and let A_θ be the component of the gauge field in the θ direction.

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or equivalently

$$\frac{D}{D\theta} \longrightarrow \frac{D}{D\theta} - \varepsilon\sigma.$$

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which is invariant to first order in ε. Similarly for other terms

$$\sum_{\mu \neq \theta} \mathrm{Tr}\left([D_\mu, D_\theta]^2 + [D_\mu, \sigma]^2\right).$$

What is happening is this. Think of σ as the component of the gauge field in a hidden new direction, which I'll call s. Then our transformation

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is a rotation, to first order in ϵ. If we improve the formula so that it describes exactly a rotation in the $\theta - s$ plane, we can rotate away the Ω-deformation.

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is not an honest change of variables, but only a mnemonic, what we have here is not the discovery of a new symmetry of $N = 2$ super Yang-Mills. Instead, we have a way to show that the Ω-deformation can be rotated away away from the fixed points.

Now I'll tell you what we will apply this to, and here we will also get to idea (b), the idea of studying spacetime torically.

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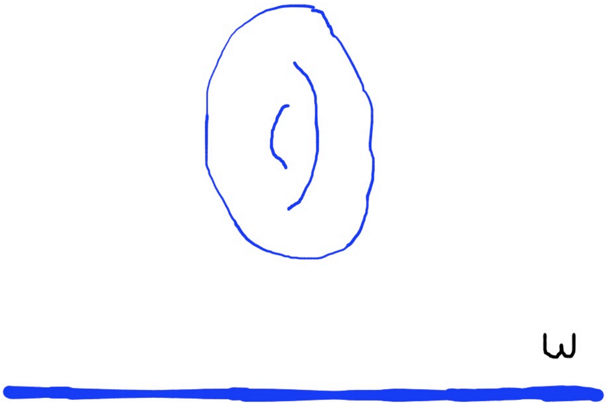
I'll actually consider first the case of making contact with the work of AGT on Liouville theory. For this, we are going to work in a Hamiltonian formalism and take spacetime to be, roughly, $\mathbb{R} \times S^3$, with a topological twist that preserves some supersymmetry.

Actually, S^3 has $SO(4)$ symmetry, and a maximal torus in $SO(4)$ is $SO(2) \times SO(2) = U(1) \times U(1)$. We make an Ω-deformation for both $U(1)$'s with parameters $\varepsilon_1, \varepsilon_2$ and we symbolically refer to the resulting theory as $N = 2$ on $\mathbb{R} \times S^3_{\varepsilon_1,\varepsilon_2}$.

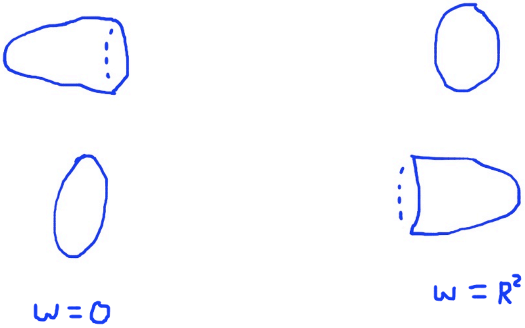
Thinking of S^3 as the locus $\sum_{i=1}^4 y_i^2 = R^2$, one $U(1)$ rotates y_1, y_2 and the other rotates y_3, y_4. The only $U(1) \times U(1)$-invariant is $w = y_1^2 + y_2^2$; it ranges from 0 to R^2.

The picture looks a bit like this except near the ends:



w parametrizes a line-interval I. What we have here except near the endpoints of w is a T^2 compactification to $\mathbb{R} \times I$. The low energy theory is an effective two-dimensional theory with supersymmetric branes at the two ends. The only thing that is unusual is that the branes come purely from geometry.

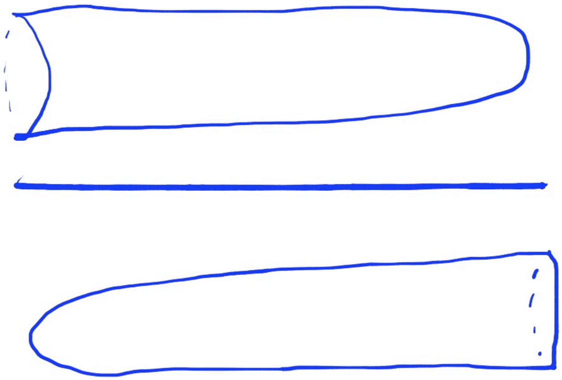
Near each end, one circle shrinks – the one in the $y_1 - y_2$ plane or the one in the $y_3 - y_4$ plane. The other circle does nothing. Schematically:



At each end, the geometry of S^3 looks like $D \times S^1$ or $S^1 \times D$, where D is a "cigar-like" geometry:

More exactly, to specify the Ω-deformation, the geometry looks like $D_{\varepsilon_1} \times S^1_{\varepsilon_2}$ at one end and $S^1_{\varepsilon_1} \times D_{\varepsilon_2}$ at the other.

Here is another picture:



The cigar geometry preserves half of the supersymmetry, so in the effective two-dimensional description, it produces a half-BPS brane. In the case of compactification on S^3 or $S^3_{\varepsilon_1,\varepsilon_2}$, the two ends together preserve only $1/4$ of the supersymmetry.

The same cigar geometry appears in our other problem – the link between gauge theory and integrability explored by Nekrasov and Shatashvili. Here the geometry considered is $\mathbb{R} \times S^1 \times D_\epsilon$ where D_ϵ is the cigar. The effective two-dimensional geometry is $\mathbb{R} \times I$, where I is the "base" of the cigar, regarded as a circle fibration.

The brane at the left end of the cigar comes from geometry, just like in our S^3 compactification. But the brane at the other end comes from a choice of a supersymmetric boundary condition in 4d gauge theory:

We can pick the boundary condition at the far end so that the whole picture is half-BPS. (This contrasts with S^3 or $S^3_{\varepsilon_1,\varepsilon_2}$ compactification, which is only $1/4$ BPS.)

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In the case of the cigar brane

one scalar – the one coming from the holonomy around the shrinking circle – will have to vanish at the boundary. The other three scalars obey Neumann boundary conditions and do not vanish.

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What is happening is that in reducing from four to two dimensions, a vector multiplet turns into a linear multiplet, not a hypermultiplet. To turn it into a hypermultiplet, we need to do T-duality on one of the two fields b_1, b_2 that come from the holonomies of A around T^2. We choose to do T-duality on the holonomy around the shrinking circle.

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The brane $\mathcal{B}$ thus preserves a whole family of topological supercharges. But if we want to make contact with either the work of Nekrasov & Shatashvili or that of Alday, Gaiotto & Tachikawa, the important structure is that it is a B-brane of complex structure I. This is the complex structure in which the Seiberg-Witten or Hitchin fibration is holomorphic.

Now what happens when we turn on the Ω-deformation and replace the cigar D by its deformed cousin D_ε? Away from the tip of the cigar, the theory is unchanged, modulo a "rotation" that acts non-trivially on the supersymmetries. The brane $\mathcal{B}$ is no longer a B-brane of complex structure I.

The supersymmetries that are preserved by the Ω-deformed brane $\mathcal{B}_\varepsilon$ are different from the supersymmetries that are preserved at $\epsilon = 0$.

What can happen?

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In 2d, in bulk, there are four left-moving SUSY's Q_L^a and four right-moving ones Q_R^a. The unbroken SUSY's in the presence of a boundary are always linear combinations of left- and right-moving SUSY's.

At $\epsilon = 0$, the unbroken combinations are $Q_L^a + Q_R^a$, $a = 1, \ldots, 4$, where the a index transforms in the fundamental representation of the $SO(4) \cong SU(2) \times SU(2)$ R-symmetry group.

At $\varepsilon \neq 0$, the unbroken SUSY's are $Q_L^a + \sum_b \mathcal{R}_b^a Q_R^b$, where $\mathcal{R}_b^a$ is a certain $SO(4)$ rotation matrix, which depends on ε.

Concretely, what happens is that the modes traveling down the string, when they reach the boundary, are reflected back with an $SO(4)$ rotation by the matrix $\mathcal{R}^a_b$.

The way that this comes about is familiar in string theory. From the point of view of a $1+1$-dimensional sigma-model, there is a B-field at the end of the string.

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A B-field has another consequence that is also familiar: noncommutativity.

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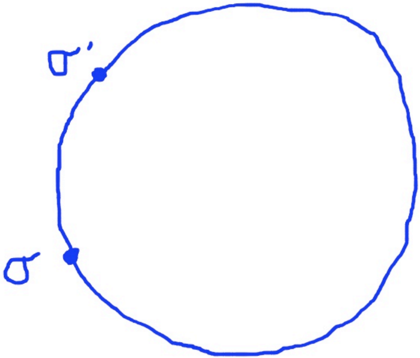
At $\epsilon \neq 0$, additively, the space of $(\mathcal{B}_\varepsilon, \mathcal{B}_\varepsilon)$ strings is the space of holomorphic functions in complex structure $\mathcal{I}$, where $\mathcal{I}$ may or may not coincide with I, depending on the choice of ε.

At $\epsilon \neq 0$, additively, the space of $(\mathcal{B}_\varepsilon, \mathcal{B}_\varepsilon)$ strings is the space of holomorphic functions in complex structure $\mathcal{I}$, where $\mathcal{I}$ may or may not coincide with I, depending on the choice of ε. For the Nekrasov-Shatashvili problem, $\mathcal{I} = I$, but for AGT, $\mathcal{I}$ is more generic and is equivalent to J.

Even more remarkable is the multiplication of $(\mathcal{B}_\varepsilon, \mathcal{B}_\varepsilon)$ strings. It is given by the familiar sort of noncommutative deformation. An important fact is that the B-field is of type $(2,0) \oplus (0,2)$ with respect to $\mathcal{I}$.

One approach to this statement is via the theory of coisotropic A-branes, of Kapustin and Orlov. This approach is explained in the paper, but I will largely skirt this language today.

Noncommutativity has the usual open string origin:



Interactions of open strings are always noncommutative, but usually this does not reduce to something as simple as deformation quantization. For this to happen, the antisymmetric contraction has to dominate the symmetric one:

$$\langle X^I(\sigma)X^J(\sigma'\rangle \sim G^{IJ}\ln(\sigma-\sigma')^2 \\ +\theta^{IJ}\epsilon(\sigma-\sigma').$$

The usual way to ensure that the antisymmetric contraction dominates is to make $F + B$ large, relying on the fact that $G^{IJ} \sim 1/(F+B)^2$ while $\theta^{IJ} \sim 1/(F+B)$.

Here we get another route to the same goal: the symmetric contraction vanishes because one only considers holomorphic functions and G^{IJ} is of type $(1,1)$ in complex structure $\mathcal{I}$.

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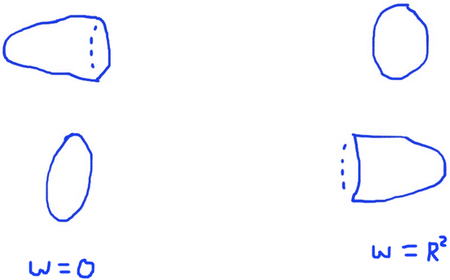
For definiteness in what follows, we specialize to a large class of $N = 2$ theories considered by Gaiotto. They arise by compactification of the six-dimensional $(0, 2)$ theory on a Riemann surface C. And X is a moduli space of Higgs bundles on C.

This is a useful description in complex structure I (where the Hitchin fibration is holomorphic). In complex structure J, X is a moduli space of flat connections on C (with structure group the complexification of the usual gauge group G).

Let $\mathcal{R}$ be the algebra of $(\mathcal{B}_\varepsilon, \mathcal{B}_\varepsilon)$ strings. If $\mathcal{I} = J$ (relevant to AGT), then the holomorphic functions on X in complex structure $\mathcal{I}$ are the traces of holonomies of the flat connection along arbitrary paths in the Riemann surface C.

$\mathcal{R}$ has a noncommutative deformation that arises in three-dimensional Chern-Simons gauge theory and also in two-dimensional conformal field theory.

In the case of AGT – which recall has to do with studying the $N = 2$ gauge theory on $\mathbb{R} \times S^3_{\varepsilon_1,\varepsilon_2}$ – both branes come from geometry.



Thus there are two similar branes, say $\mathcal{B}_{\varepsilon_1}$ at one end and $\mathcal{B}_{\varepsilon_2}$ at the other end. On the space of $(\mathcal{B}_{\varepsilon_1}, \mathcal{B}_{\varepsilon_2})$ strings, the algebra $\mathcal{R}$ acts at the left end and a dual algebra $\tilde{\mathcal{R}}$ acts at the right end.

It turns out that the deformation parameters of the two algebras are $b = \varepsilon_1/\varepsilon_2$ and $b^{-1} = \varepsilon_2/\varepsilon_1$, as one might expect from Liouville duality and the work of AGT.

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The other problem – of NS – is a little different. The algebra $\mathcal{R}$ arises by deformation quantization of the ring of holomorphic functions on X in complex structure $\mathcal{I}$. But we want actual quantization of something, not just deformation quantization.

To get the picture of NS, the second brane does not come from geometry.

It comes from a boundary condition that we place at the far end of the cigar:

Simple supersymmetric boundary conditions in the gauge theory lead in the two-dimensional sigma-model to an ordinary Lagrangian A-brane $\mathcal{B}_L$ with support a Lagrangian submanifold L. By definition, L is Lagrangian for the symplectic structure ω of the A-model. But generically, it is symplectic with respect to $\tilde{\omega} = F + B$.

When this happens, the space of $(\mathcal{B}_\varepsilon, \mathcal{B}_L)$ strings is a quantization of L in symplectic structure $\tilde{\omega}$.

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(Aldi and Zaslow; Gukov and EW)

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