More S-dualities from Outer-automorphism Twists

Yuji Tachikawa (IAS)

based on [arXiv:1009.0339]
and on numerous previous works by various people

October 2010

Introduction

$\mathcal{N} = 4$ super Yang-Mills with group G,

or, in general, $\mathcal{N} = 2$ gauge theory with zero one-loop beta function:

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \text{ doesn't run, and tunable.}$$

What happens when it becomes very strong? $\longrightarrow$ Dual descriptions.

$\mathcal{N} = 4$: Montonen-Olive duality

$$G \text{ at } \tau \text{ is equivalent to } {}^LG \text{ at } \tau' = -\frac{1}{n_G \tau}$$

- $n_G = 1$: $A_{N-1} = \mathbf{SU}(N)$, $\quad D_N = \mathbf{SO}(2N)$, E_N
- $n_G = 2$: $B_N = \mathbf{SO}(2N+1)$, $C_N = \mathbf{USp}(2N)$, F_4
- $n_G = 3$: G_2

$$B_N \leftrightarrow C_N, \text{ otherwise } G = {}^LG$$

(as far as the Lie algebra is concerned, that is.)

What's ${}^{L}G$?

- Called the Goddard-Nuyts-Olive dual, or the Langlands dual of G.
- $\mathcal{N} = 4$ SYM has six adjoint scalars.
- Give one a vev Φ in the Cartan of g and break G to $U(1)^r$.
- Each root α gives a W-boson with mass $\alpha \cdot \Phi$.
- Each root corresponds to an $\mathbf{SU}(2) \subset G$.
- The standard 't Hooft-Polyakov monopole in the BPS limit can then be embedded to G.
- A monopole with mass $\tau\alpha^* \cdot \Phi$ where $\alpha^* = 2\alpha/|\alpha|^2$ is called the co-root.

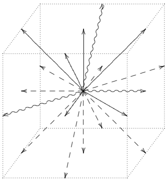
Co-roots of G form roots of ${}^LG$.

$G \leftrightarrow {}^LG$ $\qquad$ $\alpha \leftrightarrow \alpha^* = 2\alpha/|\alpha|^2$

$\tau \leftrightarrow -1/(n_G\tau)$ $\qquad$ W-boson $\leftrightarrow$ monopoles

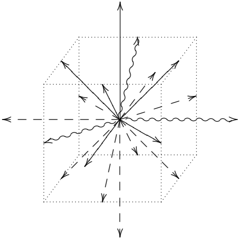
SO(7)

$\pm e_i \pm e_j, \pm e_i$



USp(6)

$\pm m_i \pm m_j, \pm 2 m_i$



$n_G = 1$: simply-laced ; $n_G = 2, 3$: non-simply-laced

$\mathcal{N} = 2$: Argyres-Seiberg-Gaiotto dualities

$\mathcal{N} = 2$ $\mathbf{SU}(N)$ with $2N$ hypermultiplets in the fundamental: $\beta = 0$.

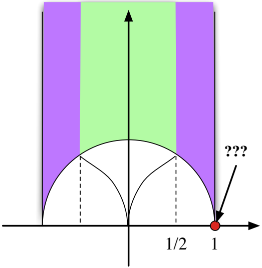
$$\tau = \frac{8\pi i}{g^2} + \frac{\theta}{\pi}$$

doesn't run, and can be tuned.

- Invariant under $\tau \to \tau + 2$
- Invariant under $\tau \to \tau + 1$ for $\mathbf{SU}(2)$: doublet = anti-doublet.
- Analysis using the Seiberg-Witten curve: invariant under $\tau \to -1/\tau$

Problem at $\tau = 1$

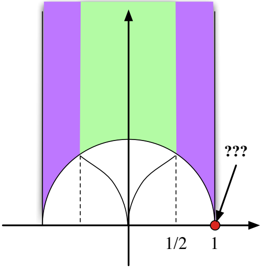
$$\tau \to -1/\tau\, ;\, \tau \to \tau + 1$$



For $\mathbf{SU(2)}$ with 4 flavors, $\tau \to \tau + 1$.
Things are OK: it's self-dual. [Seiberg-Witten, 1994]

Problem at $\tau = 1$

$$\tau \to -1/\tau \,;\, \tau \to \tau + 2$$

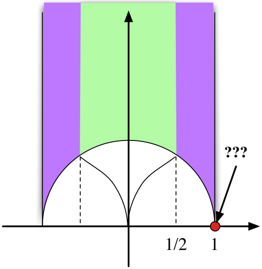


Argyres-Seiberg, 2007

SU(3) with 6 flavors at τ $\leftrightarrow$ **SU(2) at $\tau' = 1/(1-\tau)$** with **one doublet hypermultiplet**, and also coupled to **Minahan-Nemeschansky's SCFT with E_6 flavor symmetry**.

Problem at $\tau = 1$

$$\tau \to -1/\tau \,;\, \tau \to \tau + 2$$



Gaiotto 2009, Chacaltana-Distler 2010

$SU(N)$ with $2N$ flavors at τ $\leftrightarrow$ **$SU(2)$ at $\tau' = 1/(1-\tau)$** with **one doublet hypermultiplet**, and also coupled to **a strange SCFT with $SU(2) \times SU(2N)$ flavor symmetry**.

Explanation via 6d theory

- All these dualities can be understood by compactifying 6d $\mathcal{N} = (2, 0)$ theory on a Riemann surface, possibly with punctures. [Gaiotto,2009]
- The 6d theory is either A_{N-1}, D_N or E_N, i.e. **simply-laced**.
- To get 4d **non**-simply-laced theory, one needs a twist. [Vafa,1997]
- Today's objective: explore this system, confirm old dualities, and find new ones.
- also, review of the simply-laced cases at the same time.

Contents

Contents

6d $\mathcal{N} = (2, 0)$ theory

- Decoupled from gravity. Has 16 supercharges. (as many as 4d $\mathcal{N} = 4$.) Chiral. Conformal.
- Comes with types $G = A_{N-1}$, D_N and $E_{6,7,8}$, but **not** gauge theory.
- Can be given a vev $\vec{a}$: r dimensional vector.

  $\longrightarrow$ a theory of r abelian two-forms ω_i with $d\omega_i = \star d\omega_i$.

- Strings couple to ω_i. Tension is given by $\alpha \cdot a$, α is a root of G.

6d A_{N-1} theory

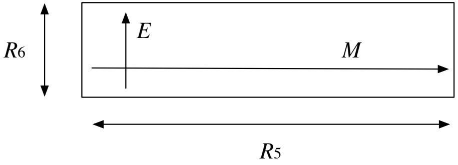
- Take N almost-coincident M5-branes, take the IR limit. Remove the center-of-mass mode.
- Vevs a given by the position of M5-branes $a = (a_1, a_2, \ldots, a_N)$.
- M2-branes can have an end on i-th and j-th M5-branes
  → String with tension $a_i - a_j$. Root: $\alpha_{ij} = e_i - e_j$.
- Compactify on an S^1 of radius R_6
  → 5d gauge $\mathbf{SU}(N)$ theory on N D4-branes:

$$S = \int d^5x \frac{1}{g^2} F_{\mu\nu} F_{\mu\nu}$$

$1/g^2 \sim 1/R_6$.

- String wrapped on S^1 → W-bosons
- String not wrapped on S^1 → Monopole strings.

Compactification on the torus



- Take 6d theory of type G, give a vev a, put it on a torus.

$$\int d^5x \frac{1}{g_{5d}^2} F_{\mu\nu}F_{\mu\nu} = \int d^4x \frac{R_5}{g_{5d}^2} F_{\mu\nu}F_{\mu\nu} = \int d^4x \frac{R_5}{R_6} F_{\mu\nu}F_{\mu\nu}$$

$\longrightarrow$ 4d coupling is $\tau = iR_5/R_6$.

- Strings wrapped around E
  $\longrightarrow$ particles of mass $R_6\alpha \cdot a$: W-bosons.
- Strings wrapped around M:
  $\longrightarrow$ particles of mass $R_5\alpha \cdot a$: Monopoles.
- Invariance under $\tau \to -1/\tau$ manifest: $R_5 \leftrightarrow R_6$.

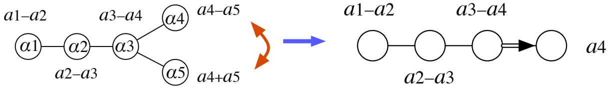
How can I get 4d SO(odd)?

- 6d D_N on S^1 $\longrightarrow$ 5d $\mathbf{SO}(2N)$ theory.
- Impose

$$\Phi(x_5 = R_5) = P\Phi(x_5 = 0)P^{-1}$$

where $P = \mathbf{diag}(1, 1, \ldots, 1, -1)$: parity
$\longrightarrow$ 4d $\mathbf{SO}(2N-1)$ theory.

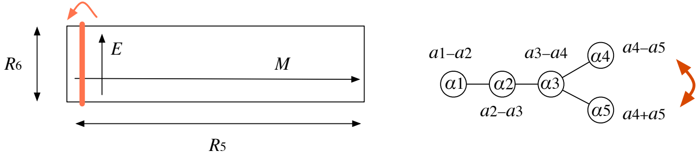
- In terms of the Dynkin diagram



i.e. set $a_5 = 0$, identify under the diagram automorphism.

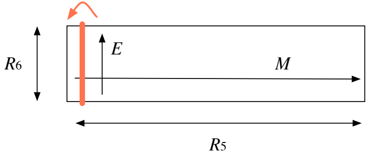
- Parity exchanges left- and right-handed spinors.

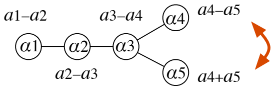
What’s the 6d interpretation?



- Across the red line, the string of type α_4 $\longrightarrow$ the string of type α_5.
- To have constant vev, we need to set $a_5 = 0$.

What's the 6d interpretation?





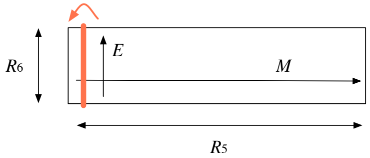
Particles are:

- $\alpha_{1,2,3}$ wrapped on E
- α_4 wrapped on E **is** α_5 wrapped on M: slide along M!
- $\alpha_{1,2,3}$ wrapped on M
- "α_4 wrapped on M" doesn't make sense. $(\alpha_4 + \alpha_5)$ on M

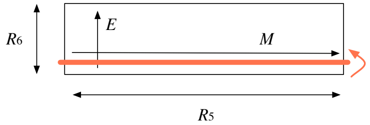
Masses are then

$$\begin{array}{rll} \text{W-bosons: } & R_6 \times (a_1 - a_2), \ldots, (a_3 - a_4), a_4 & \mathbf{SO}(9). \\ \text{Monopoles: } & R_5 \times (a_1 - a_2), \ldots, (a_3 - a_4), 2a_4 & \mathbf{USp}(8). \end{array}$$

S-duality of SO(odd)



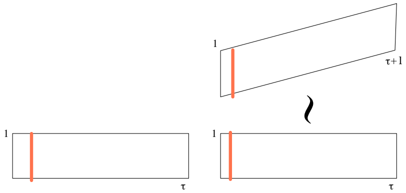
- 6d D_{2N} theory on S^1 gave 5d $\mathbf{SO}(2N)$ theory.
- Further compactification on S^1 with the twist gave 4d $\mathbf{SO}(2N-1)$.



- The role of cycles interchanged.
  $\longrightarrow$ 6d D_{2N} theory on S^1 with this twist = 5d $\mathbf{USp}(2N-2)$.

Periodicity of τ

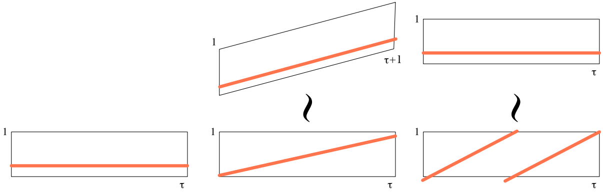
For $\mathbf{SO}(2N-1)$:



$$\tau_{\text{torus}} = \tau_{\text{gauge}}[\mathbf{SO}(\text{odd})].$$

Periodicity of τ

For $\mathbf{USp}(2N-2)$:

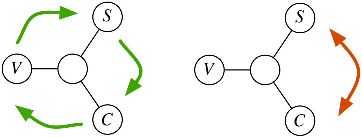


$$\tau'_{\text{torus}} = 2\tau'_{\text{gauge}}[\mathbf{USp}].$$

Recall $\tau_{\text{torus}} = \tau_{\text{gauge}}[\mathbf{SO}(\text{odd})]$. Note $\tau'_{\text{torus}} = -1/\tau_{\text{torus}}$.

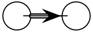
$$\tau_{\text{gauge}}[\mathbf{USp}] = -\frac{1}{2\tau_{\text{gauge}}[\mathbf{SO}(\text{odd})]}$$

D_4 i.e. $\mathbf{SO}(8)$ has another outer automorphism:



permuting vector $\mathbf{8}_V$, spinor $\mathbf{8}_S$ and conjugate spinor $\mathbf{8}_C$.

Twisting by this outer autmorphism gives G_2:



I'll come back to this...

Contents

$\mathcal{N} = 2$ from 6d theory

- 6d theory has 16 supercharges. T^2 doesn't break any.
- We can preserve half of the supercharges on a generic Riemann surface C. $\longrightarrow$ $\mathcal{N} = 2$ in 4d
- To preserve SUSY, a is a one-form on C.
- The vev of the 6d theory, a, can vary holomorphically on C.
- We can also put half-BPS codimension-two defects.
- The vev of the 6d theory, a, can vary meromorphically on C.

$\mathcal{N} = 2$ from 6d theory

- The vev a can vary meromorphically on C.
- Tensions of the string: $\alpha \cdot a$.
- When going around the loop, a doesn't have to come back to itself; the set $\alpha \cdot a$ needs to.
- e.g. For A_{N-1} theory, $a = (a_1, \ldots, a_N)$ and $\alpha_{ij} \cdot a = a_i - a_j$. There can be permutation of $i = 1, \ldots, N$.
- Let $\Phi(z) = \mathbf{diag}(a_1, \ldots, a_N)$ and let

  $$\mathbf{det}(x - \Phi(z)) = x^N + \phi_2(z)x^{N-2} + \phi_3(z)x^{N-3} + \cdots + \phi_N(z).$$

  $\phi_k(z)$ is easier to deal with.
- a is a one-form. $\phi_k(z)$ is a degree-k multi-differential.
- a has dimension 1 in 4d. $\phi_k(z)$ has dimension k in 4d.

$\mathcal{N} = 4$ theory again

A_{N-1} theory:

- $\Phi(z) = (a_1, \ldots, a_N)$ and

$$0 = x^N + \phi_2(z)x^{N-2} + \phi_3(z)x^{N-3} + \cdots + \phi_N(z).$$

- $\phi_k(z)$ is holomorphic on the torus and without singularity. Constant.
- The dimensions $2, 3, \ldots, N$ are those of the invariants of $\mathbf{SU}(N)$.

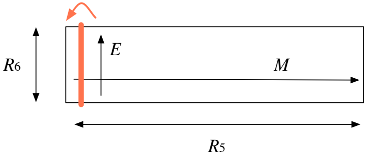
D_N theory:

- $\Phi(z) = (a_1, -a_1, \ldots, a_N, -a_N)$ and

$$0 = x^N + \phi_2(z)x^{N-2} + \phi_4(z)x^{N-4} + \cdots + \phi_{2N-2}(z)x^2 + \tilde{\phi}_N(z)^2.$$

- $\tilde{\phi}_N$ corresponds to the Pfaffian $a_1 a_2 \cdots a_N$.
- The dimensions $2, 3, \ldots, 2N-2$ and N are those of the invariants of $\mathbf{SO}(2N)$.

$\mathcal{N} = 4$ theory again

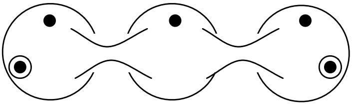


D_N theory with twist:

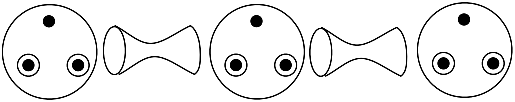
- Operators $\phi_2(z), \phi_4(z), \ldots, \phi_{2N-2}(z)$, and $\tilde{\phi}_N(z)$.
- $\tilde{\phi}_N$ corresponds to the Pfaffian $a_1 a_2 \cdots a_N$.
- $\tilde{\phi}_N \to -\tilde{\phi}_N$ across the twist line. No constant mode.
- The dimensions $2, 3, \ldots, 2N-2$ are the dimension of invariants of $\mathbf{SO}(2N-1)$ and $\mathbf{USp}(2N-2)$.
- To distinguish them requires more analysis, which we just did.

$\mathcal{N} = 2$ theory

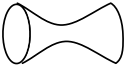
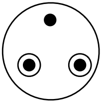
Consider A_{N-1} theory on



Components are



Composed of

 : $\mathbf{SU}(N)$ vector;  : $N \times N$ hypermultiplets

Each $\mathbf{SU}(N)$ has $2N$ hypers in the fundamental representation!

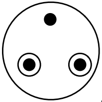
Punctures

- at $z = 0$: carry a $\mathbf{U}(1)$ flavor symmetry.

$$\phi_2(z) \sim \frac{(dz)^2}{z},\ \phi_3(z) \sim \frac{(dz)^3}{z},\ \ldots,\ \phi_N(z) \sim \frac{(dz)^N}{z}$$

$\odot$ at $z = 0$: carry an $\mathbf{SU}(N)$ flavor symmetry.

$$\phi_2(z) \sim \frac{(dz)^2}{z},\ \phi_3(z) \sim \frac{(dz)^3}{z^2},\ \ldots,\ \phi_N(z) \sim \frac{(dz)^N}{z^{N-1}}$$

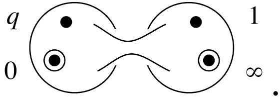
Consider  . Put two $\odot$ at $z = 0$ and $z = 1$; $\bullet$ at $w = 1/z = 0$.

$$\phi_k \sim \frac{P(z)}{z^{k-1}(1-z)^{k-1}}(dz)^k \sim \frac{P(w^{-1})}{w^2}(dw)^k$$

i.e. $P(z) = 0$. You can't turn on any Coulomb branch vev if you just have $N \times N$ hypermultiplets.

Punctures

Consider



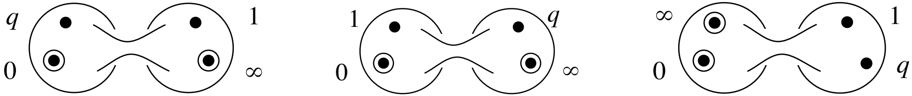
$$\phi_k \sim \frac{P_k(z)}{z^{k-1}(z-1)(z-q)}(dz)^k \sim \frac{P(w^{-1})}{w^{k-1}}(dw)^k$$

i.e. $P(z) = c_k$.
We can turn on Coulomb branch operators c_k, $k = 2, 3, \ldots, N$.
Agrees with what we expect for $\mathbf{SU}(N)$ with $2N$ fundamental hypermultiplets.

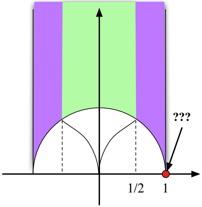
$$\log q \sim \tau$$

Limits

q can be close to either 0, 1 or ∞.



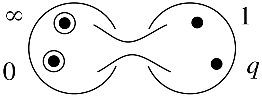
Corresponds to the three cusps of $\tau = i\infty$, 0 and 1.



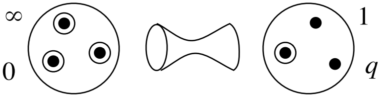
Makes it clear why one of the three limits is different when $N > 2$.
For $N = 2$, $\bullet$ and $\odot$ are the same, and thus it's always dual to itself.

Dual of SU(3) with six fundamentals

Decompose



into



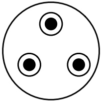
The allowed divergences at $z = 1$ and $z = q$ are very mild

→ Only $\phi_2(z)$ can be sizable at the neck

→ $\mathbf{SU(2)}$ vector.

The right hand side gives a doublet hypermultiplet of this $\mathbf{SU(2)}$.
What's the left hand side?

The theory T_3



Recall $\phi_k(z) \sim (dz)^k / z^{k-1}$.

$$\phi_2(z) = \frac{P_2(z)}{z(1-z)}(dz)^2 \sim \frac{P_2(w^{-1})}{w^2}(dw)^2$$

i.e. $P_2 = 0$.

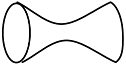
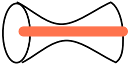
$$\phi_3(z) = \frac{P_3(z)}{z^2(1-z)^2}(dz)^3 \sim \frac{P_3(w^{-1})}{w^2}(dw)^3$$

i.e. $P_3 = c_3$.

- Has only one Coulomb branch operator, of dimension 3.
- Not a gauge theory, which always has a dimension-two $\mathbf{tr}\,\phi^2$!
- Minahan-Nemeschansky's E_6 theory does the job.
- $\mathbf{SU}(3)^3 \subset E_6$ can be manifestly seen.

D_N theory

The basic building blocks are

 : $\mathbf{SO}(2N)$ vector  : $\mathbf{USp}(2N-2)$ vector

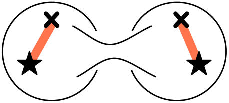
and

 : $\mathbf{2N \times (N-1)}$ hypermultiplets

$\times$: no flavor symmetry
$\odot$: an $\mathbf{SO}(2N)$ flavor symmetry
$\star$: an $\mathbf{USp}(2N-2)$ flavor symmetry

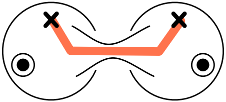
D_N theory

This combination



is the $\mathbf{SO}(2N)$ with $\mathbf{2}N - \mathbf{2}$ fundamental hypermultiplet.

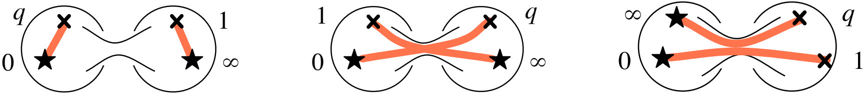
This combination



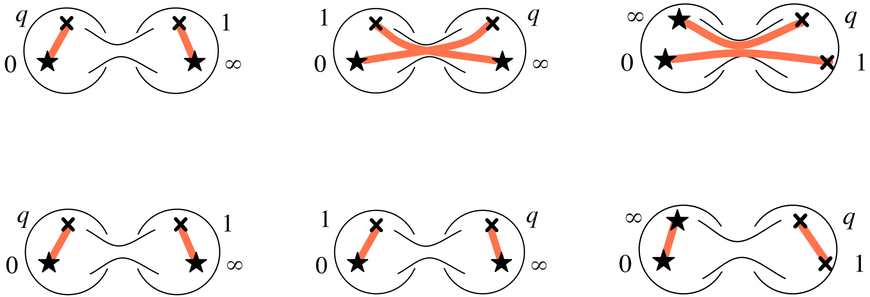
is the $\mathbf{USp}(\mathbf{2}N - \mathbf{2})$ with $\mathbf{2}N$ fundamental hypermultiplet.

The beta function is zero for both!

S-duality of SO$(2N)$ with $2N-2$ fundamentals



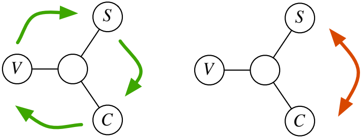
S-duality of $SO(2N)$ with $2N-2$ fundamentals



The limit $q \to \infty$ is dual to $q \to 0$.
The limit $q \to 1$ gives something different.

$\mathbf{SO}(8)$

Recall the Dynkin diagram of $\mathbf{SO(8)}$:



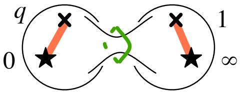
Operators: ϕ_2, ϕ_4, $\tilde{\phi}_4$ and ϕ_6.

$$\mathbf{8}_V \to \mathbf{8}_S \to \mathbf{8}_C \to \mathbf{8}_V$$
$$\mathbf{8}_S \leftrightarrow \mathbf{8}_C$$

$\mathbb{Z}_3$ rotates ϕ_4 and $\tilde{\phi}_4$ by 120°. Invariant part: ϕ_2 and ϕ_6: G_2.

$\mathbf{SO}(8)$

Let's consider



composed of

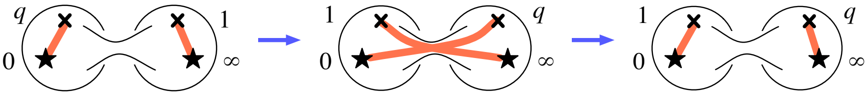
3 hypers in 8-dim rep. + $\mathbf{SO}(8)$ vector + 3 hypers in 8-dim rep.

Which 8-dim representations? → The $\mathbb{Z}_3$ twist line makes it $\mathbf{8}_V + \mathbf{8}_S$.

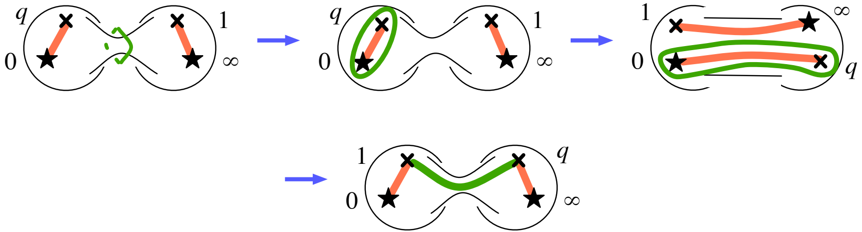
Still $\beta = 0$.

SO(8)

For **SO**(8) with six hypers in $\mathbf{8}_V$,

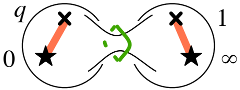


For **SO**(8) with three hypers in $\mathbf{8}_V$ and three in $\mathbf{8}_S$,

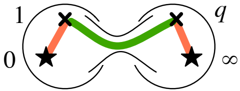


$\mathbf{SO}(8)$

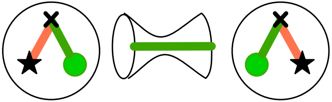
Originally we had



Coulomb branch operators: u_2, u_4, $\tilde{u}_4$ and u_6.



composed of



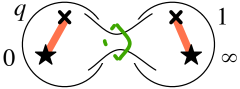
The gauge group is G_2, with invariants u_2 and u_6.
Each three-punctured sphere should have one operator u_4.

SO(8)

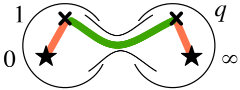
- has one dimension-4 operator. Not a gauge theory!
- has $\mathbf{USp(6)} \times G_2$ flavor symmetry.
- Minahan-Nemeschansky's E_7 theory does the job.
- $\mathbf{USp(6)} \times G_2$ is one of the special maximal subgroups of E_7.

$\mathbf{SO}(8)$

$\mathbf{SO}(8)$ with three hypers in $\mathbf{8}_S$ and three hypers in $\mathbf{8}_V$



is dual to G_2 with two copies of Minahan-Nemeschansky's E_7 theory



Predicted by [Argyres-Wittig,0712.2028]

Contents

Summary

- $\mathcal{N} = \mathbf{2}$ theory with vanishing β function has tunable coupling $\boldsymbol{\tau}$.
- 6d theory helps us understand the behavior at strong coupling.
- To get non-simply-laced theory, need to use the **outer-automorphism twist**.