S-duality, the 4d Superconformal Index and 2d Topological QFT

Leonardo Rastelli

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arXiv:0910.2225,arXiv:1003.4244, and in progress

Yang Institute for Theoretical Physics, Stony Brook

Rutgers, November 9 2010

A new paradigm for 4d $\mathcal{N}=2$ susy gauge theories (Gaiotto, . . .)

Compactification of the $(2,0)$ 6d theory on a 2d surface Σ, with punctures. $\Longrightarrow$
$\mathcal{N}=2$ superconformal theories in four dimensions.

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- Space of complex structures Σ = parameter space of the 4d theory.
- Moore-Seiberg groupoid of Σ = (generalized) 4d S-duality

Vast generalization of "$\mathcal{N}=4$ S-duality as modular group of T^2".

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Vast generalization of "$\mathcal{N}=4$ S-duality as modular group of T^2".

6=4+2: beautiful and unexpected 4d/2d connections. For ex.,

- Correlators of Liouville/Toda on Σ compute the 4d partition functions (on S^4)

In this talk we will uncover another surprising connection:

- A protected 4d quantity, the superconformal index, is computed by topological QFT on Σ.

A "microscopic" 2d definition of the TQFT still lacking. We will define it in terms of its abstract operator algebra.

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A "microscopic" 2d definition of the TQFT still lacking. We will define it in terms of its abstract operator algebra.
Index = twisted partition function on $S^3 \times S^1$. Independent of the gauge theory moduli and invariant under S-duality.
It encodes the protected spectrum of the 4d theory. Useful tool.

- Computing the index in different duality frames gives very non-trivial checks of Gaiotto's dualities.
- Conversely, assuming S-duality we will explicitly compute the index of 4d theories lacking a Lagrangian description.
- Surprising connection with elliptic hypergeometric function, an active area of mathematical research.

Outline

Review of superconformal index

$\mathcal{N}=2$: A_1 generalized quivers
- S-duality for $SU(2)$ theories
- The index of the A_1 theories and TQFT interpretation.
- Index as elliptic hypergeometric integral

$\mathcal{N}=2$: A_2 generalized quivers
- Index of E_6 theory
- A_2 TQFT

$\mathcal{N}=4$ index

$\mathcal{N}=1$ index

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The Superconformal Index [Romelsberger; Kinney, Maldacena,Minwalla, Raju 2005]

The SC Index counts (with signs) the (semi)short multiplets, up to equivalence relations that sets to zero $\oplus_i$ Short$_i$ =Long.

$$\mathcal{I}\,(t, v\,, y\,, \ldots) = \mathrm{Tr}(-1)^F t^{2(\Delta+j_2)} y^{2\,j_1} v^{-(r+R)} \ldots .$$

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Consider a 4d SCFT. On $S^3 \times \mathbb{R}$ (radial quantization), $Q^\dagger = S$.

- The superconformal algebra implies (taking $Q = \bar{Q}_{2+}$)

$$2\,\{S,\,Q\} = \Delta - 2\,j_2 - 2\,R + r \equiv H \geq 0\,.$$

  where E is the conformal dimension, (j_1, j_2) the $SU(2)_1 \otimes SU(2)_2$ Lorentz spins, and $(R\,,r)$ the quantum numbers under the $SU(2)_R \otimes U(1)_r$ R-symmetry.

- The SC index is the Witten index

$$\mathcal{I} = Tr(-1)^F\, e^{-\beta H + M}$$

  Here M is a generic combination of charges (weighted by chemical potentials) which commutes with S and Q.

- States with $H > 0$ come in pairs, boson + fermion, and cancel out,so $\mathcal{I}$ is β-independent.

The Index as a Matrix Integral

If the theory has Lagrangian description there is a simple recipe to compute the index.

- One defines a single-letter partition function as the index evaluated on the set of the basic objects (letters) in the theory with $H=0$ and in a definite representation of the gauge and flavor groups:

$$f^{\mathcal{R}_j}(t,y,v),$$

where $\mathcal{R}_j$ labels the representation.

- Then the index is computed by enumerating the gauge-invariant words,

$$\boxed{\mathcal{I}(t,y,v,\mathbf{V})=\int[d\mathbf{U}]\,\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_j f^{\mathcal{R}_j}(t^n,y^n,v^n)\cdot\chi_{\mathcal{R}_j}(\mathbf{U}^n,\mathbf{V}^n)\right)},$$

Here $\mathbf{U}$ is the matrix of the gauge group, $\mathbf{V}$ the matrix of the flavor group and $\mathcal{R}_j$ label representations of the fields under the flavor and gauge groups.

- $\chi_{\mathcal{R}_j}(\mathbf{U})$ is the character of the group element in representation $\mathcal{R}_j$.
- The measure of integration $[d\,\mathbf{U}]$ is the invariant Haar measure.

$$\int[d\mathbf{U}]\prod_{j=1}^{n}\chi_{\mathcal{R}_j}(\mathbf{U})=\#\text{of singlets in }\mathcal{R}_1\otimes\cdots\otimes\mathcal{R}_n\,.$$

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- Index of E_6 theory
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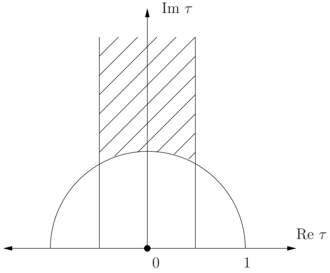
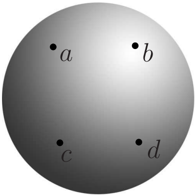
$\mathcal{N}=4$ index

$\mathcal{N}=1$ index

S-duality for $\mathcal{N}=2$ $SU(2)$ SYM with $N_f=4$

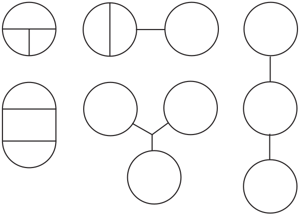




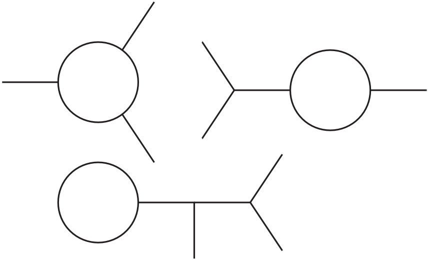
- S-duality $\tau \to -\frac{1}{\tau}$ is accompanied by an $SO(8)$ triality transformation
- $\mathbf{2} \sim \bar{\mathbf{2}}$ and thus we have eight $\mathcal{N}=1$ χsf in fundamental of $SU(2)$.
- Generalized quivers: internal edges = gauge groups; external edges = flavour groups; vertices = Tri-Fundamental χsf.
- Triality permutes the four $SU(2)$ flavor factors.
  On the diagrams this is implemented as channel crossing.

Generalized $SU(2)$ quivers

Some examples:



(a)



(b)

The generalized quivers in (**a**) arise from different pairs-of-paint decomposition of the same Riemann surface. The corresponding 4d theories are related by S-dualities. They must have the same superconformal index. The same applies to (**b**).

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The index for the A_1 theories

The index is read off from the quiver

$$\mathcal{I} = \int \left[\prod_{I=1}^{N_G} dU_I\right] \qquad e^{\sum_{I\in Edges}\sum_{n=1}^{\infty}\frac{1}{n} f_{adj}(t^n, y^n, v^n)\, \chi_{adj}(U_I^n)}$$

$$e^{\sum_{\{I,J,K\}\in Vertices}\sum_{n=1}^{\infty}\frac{1}{n} f_{3-fund}(t^n, y^n, v^n)\, \chi_{3-fund}(U_I^n, U_J^n, U_K^n)}$$

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Define a "metric" and "structure constants"

$$C_{\mathbf{U}_I\mathbf{U}_J\mathbf{U}_K} = e^{\sum_{n=1}^{\infty}\frac{1}{n}f_{3-fund}(t^n,y^n,v^n,..)\,\chi_{3-fund}(U_I^n,U_J^n,U_K^n)},$$
$$\eta^{\mathbf{U}_I\mathbf{U}_J} = e^{\sum_{n=1}^{\infty}\frac{1}{n}f_{adj}(t^n,y^n,v^n,..)\,\chi_{adj}(U_I^n)}\,\hat{\delta}(U_I,U_J).$$

so that the index can be written as

$$\boxed{\mathcal{I} = \prod_{\{I,J,K\}\in\mathcal{V}} C_{\mathbf{U}_I\mathbf{U}_J\mathbf{U}_K} \prod_{\{M,N\}\in\mathcal{G}} \eta^{\mathbf{U}_M\mathbf{U}_N}},$$

where indices are contracted by integration over the Haar measure.
"N_F–point correlator, with the quiver as a Feynman diagram".

TQFT interpretation

(a) (b)

TQFT interpretation of the structure constants $C_{\alpha\beta\gamma}$ and of the metric $\eta_{\alpha\beta}$

TQFT interpretation

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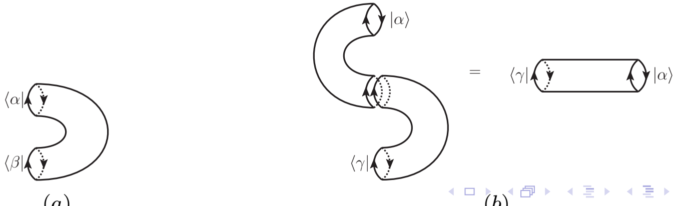
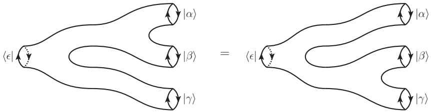
- The structure constants and the metric have to satisfy a set of axioms, which guarantee independence of correlators from the way one decomposes the Riemann surface into "*pairs of pants*".
- Most of the axioms are simply verified, they reduce to the statement that the indices are lowered/raised with the metric.



The one non-trivial condition is associativity of the algebra

$$C_{\alpha\beta}{}^{\delta}\, C_{\delta\gamma}{}^{\epsilon} = C_{\beta\gamma}{}^{\delta}\, C_{\delta\alpha}{}^{\epsilon}$$



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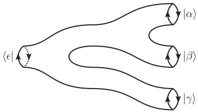
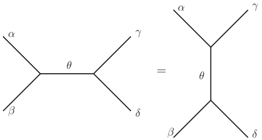


Associativity of the algebra is equivalent to invariance of the index under channel crossing of the graph and thus is implied by S-duality



Crucially depends on the field content.

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- Index of E_6 theory
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$\mathcal{N}=4$ index

$\mathcal{N}=1$ index

Index of a chiral superfield = elliptic Gamma function

- Mathematicians have a name for the index of the chiral superfield: elliptic Gamma function

$$\Gamma(z;p,q) \equiv \prod_{j,k\geq 0} \frac{1-z^{-1}\,p^{j+1}q^{k+1}}{1-z\,p^j q^k}\,.$$

- The index of a χsf is (*Dolan and Osborn - 2008*)

$$\exp\left[\sum_{k=1}^{\infty}\frac{1}{k}\,f^{chi}\left(t^k,\,v^k,\,y^k\right)\right] = \Gamma\left(\frac{t^2}{\sqrt{v}};p,q\right)\,, \quad p=t^3y,\; q=t^3y^{-1}.$$

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- The Elliptic Beta integral is a generalization of the celebrated Euler Beta integral (*Spiridonov - 2001*)

$$\kappa\oint\frac{dz}{2\pi i z}\frac{\prod_{i=1}^{6}\Gamma(t_i\,z^{\pm 1};p,q)}{\Gamma(z^{\pm 2};p,q)}=\prod_{i<j}\Gamma(t_it_k;p,q)\rightarrow\int_0^1 dt\,t^{\alpha-1}(1-t)^{\beta-1}=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,.$$

Elliptic Cookbook

Recall the character of the (anti)fundamental representation of $SU(n)$

$$\chi_f = \sum_{i=1}^{n} a_i, \qquad \chi_{\bar{f}} = \sum_{i=1}^{n} \frac{1}{a_i}, \qquad \prod_{i=1}^{n} a_i = 1 .$$

- The index of a chiral multiplet in fundamental of $SU(n)$

$$\prod_{i=1}^{n} \Gamma\left(\frac{t^2}{\sqrt{v}}\, a_i^{\pm 1}; p, q\right)$$

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$$\prod_{i=1}^n \Gamma\left(\frac{t^2}{\sqrt{v}}\,a_i^{\pm 1};\,p,q\right)$$

- When an $SU(n)$ symmetry is gauged we add a vector multiplet and integrate over the gauge group

$$\frac{\left[2\,\Gamma(t^2\,v;p,q)\,\kappa\right]^{n-1}}{n!}\oint_{\mathbb{T}_{n-1}}\prod_{i=1}^{n-1}d\mu(a_i)\prod_{i\neq j}\frac{\Gamma(t^2\,v\,a_i/a_j;p,q)}{\Gamma(a_i/a_j;p,q)}\dots\Bigg|_{\prod_{i=1}^n a_i=1}$$

* For brevity we will often omit the parameters p and q from the expression of the Gamma function.

The index of the $SU(2)$ generalized quivers in terms of elliptic Gamma functions

The index of $N_f = 4$ $SU(2)$ gauge theory can be written as

$$\kappa \, \Gamma\left(t^2 v\right) \oint \frac{dz}{2\pi i\, z} \, \frac{\Gamma(t^2\, v\, z^{\pm 2})}{\Gamma(z^{\pm 2})} \, \Gamma(\frac{t^2}{\sqrt{v}} a^{\pm 1} b^{\pm 1} z^{\pm 1}) \, \Gamma(\frac{t^2}{\sqrt{v}} c^{\pm 1} d^{\pm 1} z^{\pm 1}).$$

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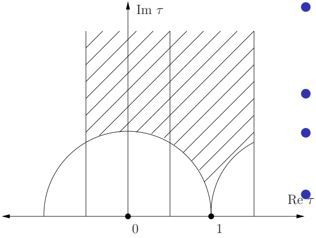
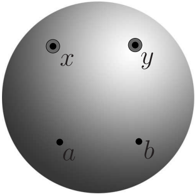
This integral was recently shown to be invariant under exchanging a and c (more generally, under the Weyl group of F_4) (van de Bult 2009)

This checks associativity of the A_1 TQFT, or equivalently, S-duality for the index of the A_1 theories

Outline

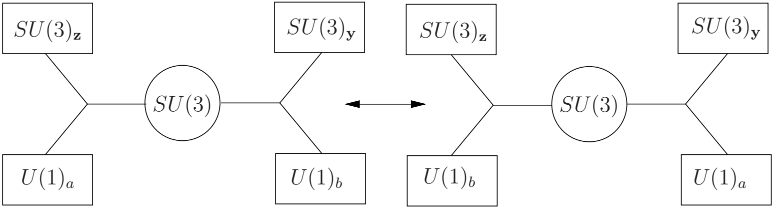
A_2 generalized quivers





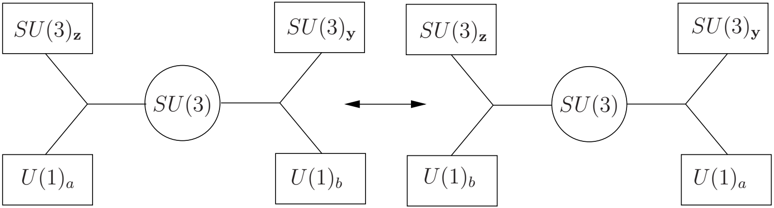
- Generalized quivers: internal edges = $SU(3)$ gauge groups; external edges = flavour groups, either $U(1)$ or $SU(3)$; vertices = hypermultiplets.
- Basic example $N_f = 6$ $SU(3)$ SYM
- S-duality group generated by $\tau \to -\frac{1}{\tau}$ and $\tau \to \tau + 2$.
- Three possible degenerations of the four-punctured sphere: different types of punctures collide (2 possibilities), or two like punctures collide.
- "Usual" S-duality is the equivalence of the two degenerations when different types of punctures collide (interchange of the two flavor $U(1)$ or $SU(3)$ factors)
- Argyres-Seiberg duality brings us to the frame when two like punctures collide. The theory consists of an $SU(2)$ vector multiplet coupled to a fundamental hyper and to a strongly coupled rank-one SCFT with E_6 flavor symmetry.

Weakly-coupled frame



$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}}=\frac{2}{3}\kappa^2\Gamma(t^2v)^2\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\,x_i}\frac{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{a\,z_i}{x_j}\right)^{\pm1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(b\,y_i\,x_j\right)^{\pm1}\right)\prod_{i\neq j}\Gamma\left(t^2v\,\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}.$$

Weakly-coupled frame



$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} = \frac{2}{3}\kappa^2\Gamma(t^2 v)^2 \oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\, x_i}\frac{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{a\, z_i}{x_j}\right)^{\pm 1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(b\, y_i\, x_j\right)^{\pm 1}\right)\ \prod_{i\neq j}\Gamma\left(t^2 v\,\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}.$$

S-duality implies symmetry under $a \quad \leftrightarrow \quad b$

Checked perturbatively in t and analytically proved for $t=v$.

Using (Rains 2003)

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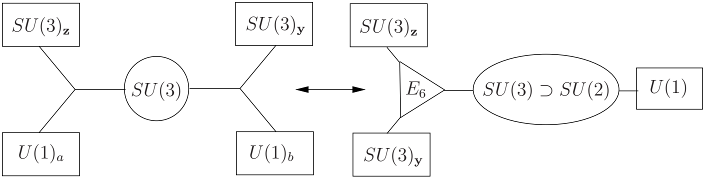
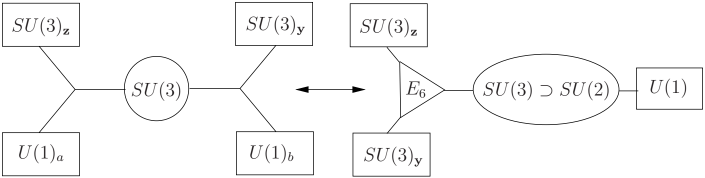
Strongly-coupled frame



The E_6 SCFT has no Lagrangian description

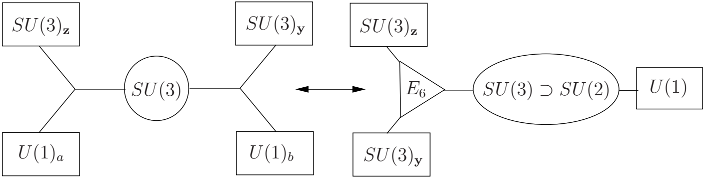
Strongly-coupled frame



The E_6 SCFT has no Lagrangian description

Let $C^{(E_6)}(\mathbf{x},\mathbf{y},\mathbf{z})$ denote the index of rank one E_6 SCFT.

Strongly-coupled frame



The E_6 SCFT has no Lagrangian description

Let $C^{(E_6)}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ denote the index of rank one E_6 SCFT.

- In the strongly-coupled frame, the index reads

$$\hat{\mathcal{I}}(s, r; \mathbf{y}, \mathbf{z}) = \kappa \, \Gamma(t^2 v) \oint_{\mathbb{T}} \frac{de}{2\pi i \, e} \frac{\Gamma(t^2 v e^{\pm 2})}{\Gamma(e^{\pm 2})} \Gamma\left(\frac{t^2}{\sqrt{v}} e^{\pm 1} s^{\pm 1}\right) C^{(E_6)}\left((e, r), \mathbf{y}, \mathbf{z}\right) .$$

- Argyres-Seiberg duality implies

$$\boxed{\hat{\mathcal{I}}(s, r; \mathbf{y}, \mathbf{z}) = \mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} \qquad s = (a/b)^{3/2}, \qquad r = (a\, b)^{-1/2}}.$$

where $\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}}$ is the index in the weakly-coupled frame.

Inverting the $SU(2)$ integral

$$\hat{\mathcal{I}}\,(s,r;\mathbf{y},\mathbf{z}) \quad = \quad \kappa\, \oint_{\mathbb{T}} \frac{de}{2\pi i\, e}\, \frac{\Gamma(\frac{t^2}{\sqrt{v}} e^{\pm 1}\, s^{\pm 1})}{\Gamma(\frac{t^4}{v})\Gamma(e^{\pm 2})}\, \Gamma(t^2 v e^{\pm 2})\, C^{(E_6)}\,((e,\, r), \mathbf{y}, \mathbf{z})\ .$$

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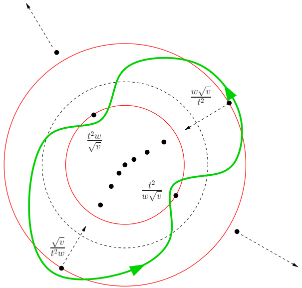
Inversion formula: Under certain assumptions the following holds: (Spiridonov-Warnaar 2004)

$$\hat{f}(w) = \kappa \oint_{C_w} \frac{ds}{2\pi i\, s}\, \delta\left(s, w; \left(\frac{t^2}{\sqrt{v}}\right)^{-1}\right) f(s) \Rightarrow f(s) = \kappa \oint_{\mathbb{T}} \frac{de}{2\pi i\, e}\, \delta\left(e, s; \frac{t^2}{\sqrt{v}}\right) \hat{f}(e)\,.$$

The integration contour C_w is a deformation of the unit circle



The index of the E_6 SCFT

Using the inversion formula we obtain the index of the E_6 SCFT

$$C^{(E_6)}((w,\,r),\mathbf{y},\mathbf{z}) = \frac{2\,\kappa^3\Gamma(t^2 v)^2}{3\,\Gamma(t^2 v\,w^{\pm 2})}\oint_{C_w}\frac{ds}{2\pi i\,s}\,\frac{\Gamma(\frac{\sqrt{v}}{t^2}w^{\pm 1}\,s^{\pm 1})}{\Gamma(\frac{v}{t^4},\,s^{\pm 2})}\times$$

$$\times\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\,x_i}\frac{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{\frac{1}{3}}\,z_i}{x_j\,r}\right)^{\pm 1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{-\frac{1}{3}}\,y_i\,x_j}{r}\right)^{\pm 1}\right)\prod_{i\neq j}\Gamma\left(t^2 v\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}.$$

Spectrum of protected operators from the index

$$C^{(E_6)} \equiv \sum_{k=0}^{\infty} a_k\, t^k\,.$$

$$\boxed{\begin{aligned}
a_0 =&1,\;\; a_1 t = a_2 t^2 = a_3 t^3 = 0,\;\; a_4 t^4 = \frac{t^4}{v}\chi^{E_6}_{\mathbf{78}},\;\; a_5 t^5 = 0, a_6 t^6 = -t^6\chi^{E_6}_{\mathbf{78}} - t^6 + t^6 v^3\\
a_7 t^7 =& \frac{t^7}{v}\left(y+\frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}} + \frac{t^7}{v}\left(y+\frac{1}{y}\right) - t^7 v^2\left(y+\frac{1}{y}\right)\\
a_8 t^8 =& \frac{t^8}{v^2}\left(\chi^{E_6}_{sym^2(\mathbf{78})} - \chi^{E_6}_{\mathbf{650}} - 1\right) + t^8 v + t^8 v\\
a_9 t^9 =& -t^9\left(y+\frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}} - 2t^9\left(y+\frac{1}{y}\right) + t^9 v^3\left(y+\frac{1}{y}\right)\\
a_{10} t^{10} =& -\frac{t^{10}}{v}(\chi^{E_6}_{\mathbf{78}}\,\chi^{E_6}_{\mathbf{78}} - \chi^{E_6}_{\mathbf{650}} - 1) + \frac{t^{10}}{v}\left(y^2+1+\frac{1}{y^2}\right)\chi^{E_6}_{\mathbf{78}}\\
&+\frac{t^{10}}{v}\left(y+\frac{1}{y}\right)^2 - t^{10} v^2\left(y+\frac{1}{y}\right)^2 .
\end{aligned}}$$

The index is E_6 covariant

Spectrum of protected operators from the index

$$\mathcal{I}\,(t,\,v\,,y\,,\dots)=\mathrm{Tr}(-1)^F\,t^{2(E+j_2)}y^{2\,j_1}\,v^{-(r+R)}\dots.$$

$$a_0=1,\quad a_1 t=a_2t^2=a_3t^3=0,\quad a_4t^4=\frac{t^4}{v}\chi^{E_6}_{\mathbf{78}},\quad a_5t^5=0,\quad a_6t^6=-t^6\chi^{E_6}_{\mathbf{78}}-t^6+t^6v^3,$$

$$a_7t^7=\frac{t^7}{v}\left(y+\frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}}+\frac{t^7}{v}\left(y+\frac{1}{y}\right)-t^7v^2\left(y+\frac{1}{y}\right),\ a_8t^8=\frac{t^8}{v^2}\left(\chi^{E_6}_{sym^2(\mathbf{78})}-\chi^{E_6}_{\mathbf{650}}-1\right)+t^8v+t^8v,$$

$$a_9t^9=-t^9\left(y+\frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}}-2t^9\left(y+\frac{1}{y}\right)+t^9v^3\left(y+\frac{1}{y}\right)$$

$$a_{10}t^{10}=-\frac{t^{10}}{v}(\chi^{E_6}_{\mathbf{78}}\,\chi^{E_6}_{\mathbf{78}}-\chi^{E_6}_{\mathbf{650}}-1)+\frac{t^{10}}{v}\left(y^2+1+\frac{1}{y^2}\right)\chi^{E_6}_{\mathbf{78}}+\frac{t^{10}}{v}\left(y+\frac{1}{y}\right)^2-t^{10}v^2\left(y+\frac{1}{y}\right)^2.$$

$$\mathbb{X}\to\frac{t^4/v-t^6}{(1-t^3y)(1-t^3/y)},\quad u\to\frac{t^6v^3-t^7v^2(y+\frac{1}{y})+t^8v}{(1-t^3y)(1-t^3/y)},\quad T\to\frac{-t^6+\frac{t^7}{v}(y+\frac{1}{y})+t^8v-t^9(y+\frac{1}{y})}{(1-t^3y)(1-t^3/y)}.$$

	E	r	R	j_1	j_2
$\mathbb{X}$	2	0	1	0	0
u	3	-3	0	0	0
T	2	0	0	0	0

Constraints : $(\mathbb{X}\otimes\mathbb{X})|_{\mathbf{650}\oplus\mathbf{1}}=0\,,\qquad \mathbb{X}\otimes u=0\,,\qquad \mathbb{X}\otimes T=0\,.$

Outline

Review of superconformal index

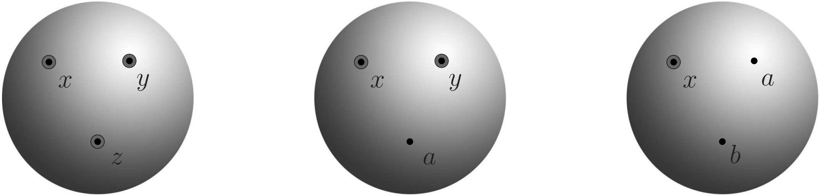
$\mathcal{N}=2$: A_1 generalized quivers
- S-duality for $SU(2)$ theories
- The index of the A_1 theories and TQFT interpretation.
- Index as elliptic hypergeometric integral

$\mathcal{N}=2$: A_2 generalized quivers
- Index of E_6 theory
- A_2 TQFT

$\mathcal{N}=4$ index

$\mathcal{N}=1$ index

$SU(3)$ TQFT



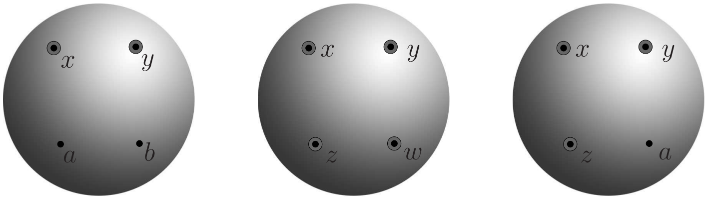
$C^{(333)}_{\mathbf{x},\mathbf{y},\mathbf{z}}$ $C^{(133)}_{a,\mathbf{x},\mathbf{y}}$ $C^{(113)}_{a,b,\mathbf{x}}$

rank **1** : $C^{(333)}_{\mathbf{x},\mathbf{y},\mathbf{z}}$: Index of E_6 SCFT.

rank **0** : $C^{(133)}_{a,\mathbf{x},\mathbf{y}}$: Index of a hypermultiplet.

"rank **-1**" : $C^{(113)}_{a,b,\mathbf{x}}$: An auxiliary construct to write the Argyres-Seiberg theory .

S-duality checks of the E_6 index



(a) (b) (c)

(a) $N_f = 6$ $SU(3)$ theory (in either of two S-dual frames), or Argyres-Seiberg theory.

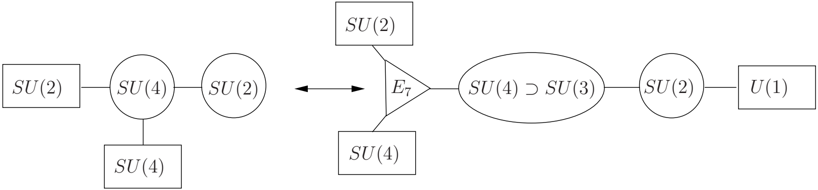
(b) Two E_6 theories "joined" by gauging an $SU(3)$ subgroup of the flavor symmetry.

(c) E_6 SCFT joined to hypers by an $SU(3)$ gauging.

We checked associativity perturbatively in t.

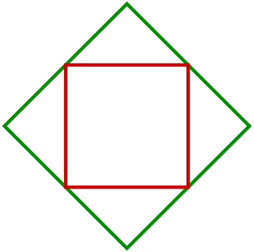
Higher rank

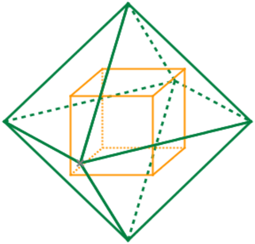
- Can in principle generalize the discussion to quivers with higher rank gauge groups.
- Get many intrinsically strongly coupled theories: E_7 SCFT, T_N theories ...
- To obtain the index of these higher rank theories have to learn to invert the superconformal tails.



Outline

32 supersymmetries: S-duality $SO(2n+1)/Sp(n)$





- The index on *root* system $\mathbf{X}$

$$\mathcal{I}_{\mathcal{N}=4} \sim \oint \prod_j \frac{dz_j}{2\pi i z_j} \prod_{\alpha\in\mathbf{X}} \frac{\Gamma(t^2\, e^{\alpha}; p, q)^3}{\Gamma(e^{\alpha}; p, q)},$$

  where we formally identify $z_i = e^{e_i}$.

- The root systems of $SO(2n+1)$ and $Sp(n)$ are

$$\begin{aligned} SO(2n+1) \quad &: \quad \mathbf{X} = \{\pm e_i,\ \pm e_i \pm e_j, i<j\} \\ Sp(n) \quad &: \quad \mathbf{X} = \{\pm 2\, e_i,\ \pm e_i \pm e_j, i<j\}, \end{aligned}$$

  and they define dual polyhedra.

- $n=2$ $SO(5)$ and $Sp(2)$ are both squares.
- $n=3$ $SO(7)$ gives a cube and $Sp(3)$ is an octahedron.

Outline

8 Supersymmetries

Curious recipe : compute the index by counting the states in the UV, but with the IR charge assignments (Romelsberger)
Can be justified interpreting the index as the Witten index of the non-conformal theory on $S^3 \times \mathbb{R}$ interpolating betweeen UV and IR fixed points.

- Several Seiberg-dual pairs turn out to have the same index. (Romelsberger, Dolan Osborn, Spiridonov Vartanov)
- Remarkably, setting $v = t$ in the $\mathcal{N}=2$ index gives the $\mathcal{N}=1$ index of the SCFT obtained (in the IR) integrating out the chiral adjoints.
- We consider $\mathcal{N}=1$ SCFTs that have an AdS$_5$ dual. Closed formulas for the index of the SCFTs dual to $AdS_5 \times Y_{pq}$.
- We check toric duality of these theories
- We match the index of conifold gauge theory to gravity on $AdS \times T^{1,1}$

$Y^{p,q}$ quiver gauge theory

- $Y^{p,p}$ is $\mathbb{Z}_{2p}$ orbifold of $\mathcal{N}=4$
- $Y^{p,0}$ is $\mathbb{Z}_p$ orbifold of the conifold $Y^{1,0}$
- 4 types of field in $Y^{p,q}$ theory

	$U(1)_r$	Arrows
U	$1-\frac{1}{2}(x+y)$	
V	$1+\frac{1}{2}(x-y)$	
Z	x	
Y	y	

$$
\begin{aligned}
y_{p,q} &= \frac{1}{3q^2}\left\{-4p^2+2pq+3q^2+(2p-q)\sqrt{4p^2-3q^2}\right\}, \\
x_{p,q} &= \frac{1}{3q^2}\left\{-4p^2-2pq+3q^2+(2p+q)\sqrt{4p^2-3q^2}\right\}.
\end{aligned}
$$

Toric duality

- Example of toric duality

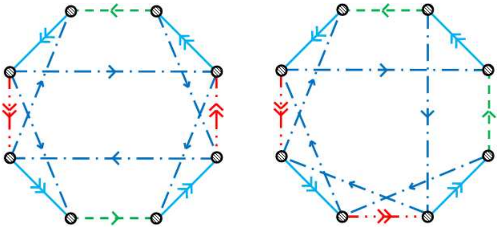


Figure: Different quiver diagrams for $Y^{4,2}$.

- Indices of toric-dual quivers are equal using Rains' identity

Toric duality

Figure: Action of Seiberg duality

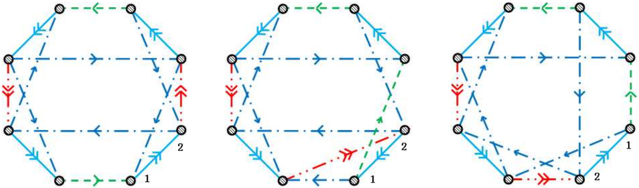


Figure: Example of the $Y^{4,2}$ quiver. Middle: Seiberg duality on node 1. Right: swap nodes 1 and 2.

Index of the conifold gauge theory and $AdS \times T^{1,1}$

- In the large N limit, we can compute quiver gauge theory index by saddle point approximation

$$\mathcal{I} = -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log[\det(1 - i(t^k, y^k))] \qquad \varphi(n) = \text{Euler Phi function}$$

- $i(x)$, index valued adjacency matrix of the quiver
- Conifold index:

$$\mathcal{I} = \frac{t^3 ab}{1 - t^3 ab} + \frac{t^3 \frac{a}{b}}{1 - t^3 \frac{a}{b}} + \frac{t^3 \frac{b}{a}}{1 - t^3 \frac{b}{a}} + \frac{t^3 \frac{1}{ab}}{1 - t^3 \frac{1}{ab}} - \frac{t^3 y}{1 - t^3 y} - \frac{t^3 \frac{1}{y}}{1 - t^3 \frac{1}{y}}$$

a, b are potentials of $SU(2)_a \times SU(2)_b$ global symmetry

- KK reduction on $T^{1,1}$, spectrum of scalar laplacian (*Nakayama*)
- On gravity side, contribution from graviton, gravitino and vector multiplets, exactly matches with gauge theory result

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- Possible to add line and surface operators

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Thank You