Recent Results in the Conformal Bootstrap

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Outline

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Conformal Bootstrap

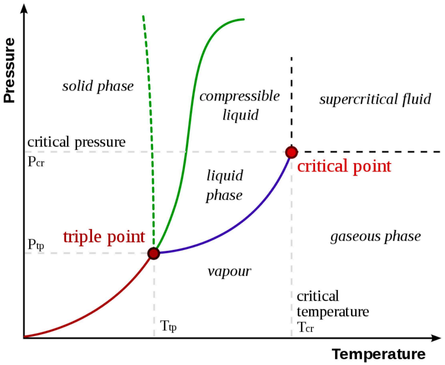
- The conformal bootstrap aims to use basic consistency conditions to map out and solve the space of CFTs
  - Conformal symmetry
  - Associativity of the OPE (crossing symmetry)
  - Unitarity
- Beautiful success story in 2D
  [Ferrara, Gatto, Grillo '73; Polyakov '74; Belavin, Polyakov, Zamolodchikov '84]
- Great progress in $D > 2$ starting in 2008
  [Rattazzi, Rychkov, Tonni, Vichi '08; ...]

Motivations

Many motivations to learn about CFTs in $D > 2$:

- $3D$: Condensed Matter and Statistical Systems at Phase Transitions
- $4D$: Scenarios for Physics Beyond the Standard Model
- Structure of QFT and space of CFTs
- AdS/CFT Correspondence (precise way to study quantum gravity)

Example: 3D Ising Model



- CFTs describe condensed matter systems at phase transitions
  - Liquid-vapor critical point in fluids (^{3}He, O_2, Ar, Xe, ...)
  - Continuum limit of 3D spin lattice: $H = \sum_{\langle ij \rangle} \sigma_i \cdot \sigma_j$ at T_c

  $\rightarrow$ Both described by 3D Ising CFT: scalar with ϕ^4 interaction

Single Correlator Bootstrap

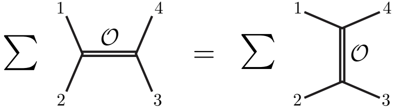
Simplest bootstrap involves evaluating scalar 4-point functions with OPE:

$$
\begin{aligned}
&\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \\
&\qquad = \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 C_I(x_{12},\partial_2) C_J(x_{34},\partial_4)\langle \mathcal{O}^I(x_2)\mathcal{O}^J(x_4)\rangle \\
&\qquad \equiv \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2\, g_{\Delta,\ell}(u,v)
\end{aligned}
$$

- $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ conformally-invariant cross ratios
- $g_{\Delta,\ell}(u,v)$ conformal block, labeled by $\Delta = \dim(\mathcal{O})$ and $\ell = \text{spin}(\mathcal{O})$ (known in any D using explicit formulas or recursion relations)

Crossing Symmetry

- $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ is symmetric under permutations of x_i
- Switching $x_1 \leftrightarrow x_3$ after OPE gives the crossing relation:



$$v^{\Delta_\phi} \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(u,v) \;=\; u^{\Delta_\phi} \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(v,u)$$

- This is a *constraint* on the spectrum of primary Δ's, ℓ's, and $\lambda_{\mathcal{O}}$'s

Crossing Symmetry

Convenient to write as a sum rule (separating out $\phi \times \phi \sim \mathbb{1} + \ldots$)

$$0 = \underbrace{F_{0,0}(u,v)}_{\text{unit op.}} + \underbrace{\sum \lambda_{\mathcal{O}}^2 F_{\Delta,\ell}(u,v)}_{\text{everything else}}$$

where

$$F_{\Delta,\ell}(u,v) \equiv v^{\Delta_\phi} g_{\Delta,\ell}(u,v) - u^{\Delta_\phi} g_{\Delta,\ell}(v,u).$$

How Does Crossing Symmetry Lead to CFT Bounds?

Crossing relation for real scalar ϕ:

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- Make an assumption: all scalars have dimension $\Delta > \Delta_{\text{min}}$
- Search for a linear functional α such that

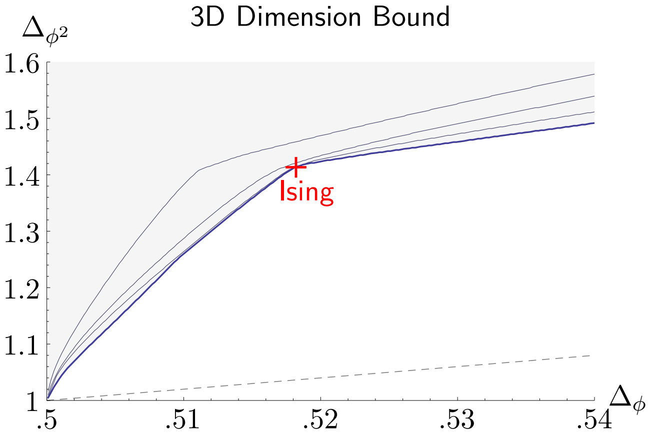
$$\begin{aligned} \alpha(F_{0,0}) &= 1, \quad \text{and} \\ \alpha(F_{\Delta,\ell}) &\geq 0, \quad \text{for all other } \mathcal{O} \in \phi \times \phi. \end{aligned}$$

- If you find one, the assumption is ruled out!

CFT Bounds

- Can be solved with linear or semidefinite programming techniques [Rattazzi, Rychkov, Tonni, Vichi '08; DP, Simmons-Duffin, Vichi '11]
- Many nice results between 2008-2014 following this approach in (2-6)D, as well as generalizations to SUSY and other global symmetries
- Here I will focus on the 3D story...

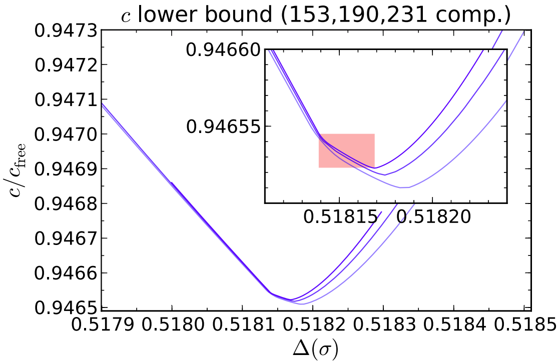
3D Dimension Bounds



[El-Showk, Paulos, DP, Rychkov, Simmons-Duffin, Vichi, '12]

- 3D Ising dimensions from numerical simulations:
  $\Delta_\sigma \simeq 0.51813(5)$, $\Delta_\epsilon \simeq 1.41275(25)$ [Hasenbusch '10]

c-minimization and Spectrum Extraction



[El-Showk, Paulos, DP, Rychkov, Simmons-Duffin, Vichi, '14]

- Under the conjecture that the central charge $\langle TT \rangle \propto c$ is minimized, a precise spectrum in $\sigma \times \sigma \sim \mathbb{1} + \epsilon + \epsilon' + \ldots$ can be extracted:

$$\Delta_\sigma \simeq 0.518154(15),\ \Delta_\epsilon \simeq 1.41267(13),\ \Delta_{\epsilon'} = 3.8303(18),\ \ldots$$

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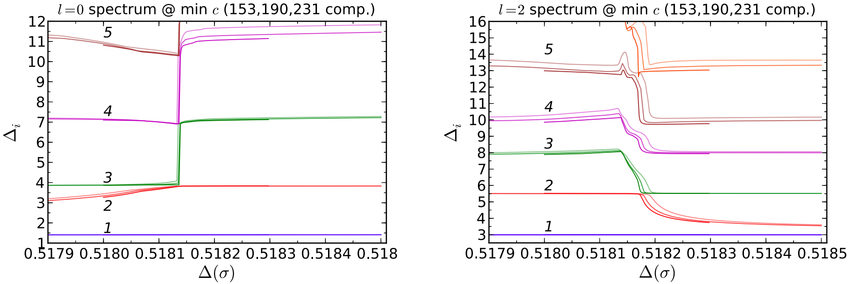
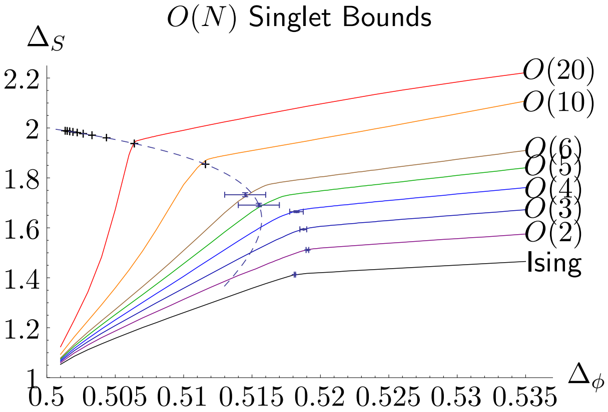


[El-Showk, Paulos, DP, Rychkov, Simmons-Duffin, Vichi, '14]

- Operators merge and disappear from the spectrum!
- Reminiscent of null states in 2D

3D $O(N)$ Bounds



[Kos, DP, Simmons-Duffin '13]

- Extension to $\langle \phi_i \phi_j \phi_k \phi_l \rangle$, where ϕ_i is $O(N)$ vector
- OPE $\phi_i \times \phi_j \sim \mathbb{1} + S + T_{ij} + \dots$ contains singlets and two-index tensors

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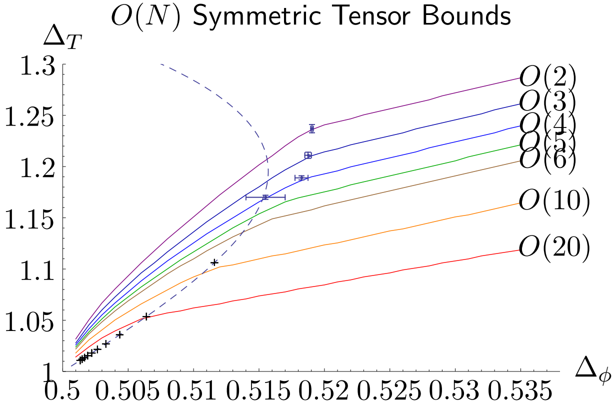

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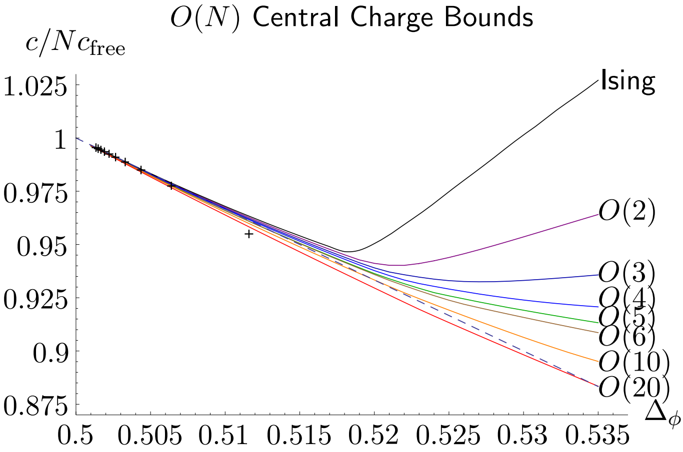

[Kos, DP, Simmons-Duffin '13]

- In general c bounds do not show a minimum

Outline

Missing Operators?

Studies so far failed to access parts of the operator spectrum:

- In 3D Ising, only saw $\mathbb{Z}_2$-even operators in $\sigma \times \sigma \sim \mathbb{1} + \epsilon + \dots$
- In $O(N)$ models, only saw $O(N)$ singlets S and two-index tensors T_{ij}

In fact, there are good reasons to expect that the unaccessed operators are important...

Non-perturbative Equations of Motion

- In ϕ^4 theory, one expects an equation of motion like $\partial^2\phi \sim \phi^3 + \ldots$
- This means that the ϕ^3 operator becomes a descendant, and is removed from the primary spectrum
- The consequence is that there is a large *gap* in the $\mathbb{Z}_2$-odd spectrum, along with many other relations between operators (e.g., $\phi\partial^2\phi \sim \phi^4$)

It is very important to understand the role of these gaps (and operator relations) in the context of the conformal bootstrap!

Mixed Correlators

- To probe gaps, one must consider mixed correlators like $\langle \sigma\sigma\epsilon\epsilon \rangle$
- However, the expansion

$$\langle \overbracket{\sigma\sigma}\overbracket{\epsilon\epsilon} \rangle \quad \sim \quad \sum_{\mathcal{O}} \lambda_{\sigma\sigma\mathcal{O}} \lambda_{\epsilon\epsilon\mathcal{O}} g_{\Delta,\ell}(u,v)$$

  does not have positive coefficients, so we cannot use the same logic
- In fact, it turns out that the positivity constraints must be phrased in terms of positive semidefinite matrices (SDP is manditory)

Mixed Correlators

- The positivity properties can be made manifest by considering the system $\{\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle\}$, which leads to 5 sum rules:

$$\sum_{\mathcal{O}^+} \begin{pmatrix} \lambda_{\sigma\sigma\mathcal{O}} & \lambda_{\epsilon\epsilon\mathcal{O}} \end{pmatrix} \vec{V}_{+,\Delta,\ell}(u,v) \begin{pmatrix} \lambda_{\sigma\sigma\mathcal{O}} \\ \lambda_{\epsilon\epsilon\mathcal{O}} \end{pmatrix} + \sum_{\mathcal{O}^-} \lambda^2_{\sigma\epsilon\mathcal{O}} \vec{V}_{-,\Delta,\ell}(u,v) = 0,$$

where $\vec{V}_{\pm,\Delta,\ell}(u,v)$ are 5-vectors and $\vec{V}_{+,\Delta,\ell}(u,v)$ is a 2×2 matrix

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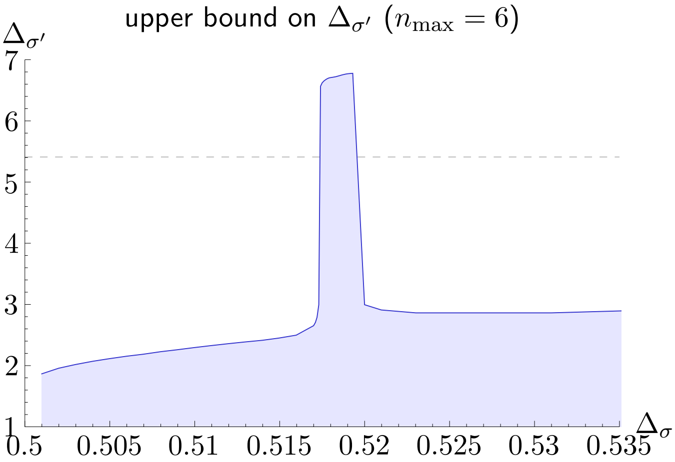
- Bounds follow from applying a 5-vector of functionals $\vec{\alpha}$ such that

$$\begin{aligned} \begin{pmatrix} 1 & 1 \end{pmatrix} \vec{\alpha}\cdot\vec{V}_{+,0,0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= 1, \\ \vec{\alpha}\cdot\vec{V}_{+,\Delta,\ell} &\succeq 0, \quad \text{for all } \mathbb{Z}_2\text{-even operators } \mathcal{O}^+, \\ \vec{\alpha}\cdot\vec{V}_{-,\Delta,\ell} &\geq 0, \quad \text{for all } \mathbb{Z}_2\text{-odd operators } \mathcal{O}^-. \end{aligned}$$

Mixed Correlator Bounds

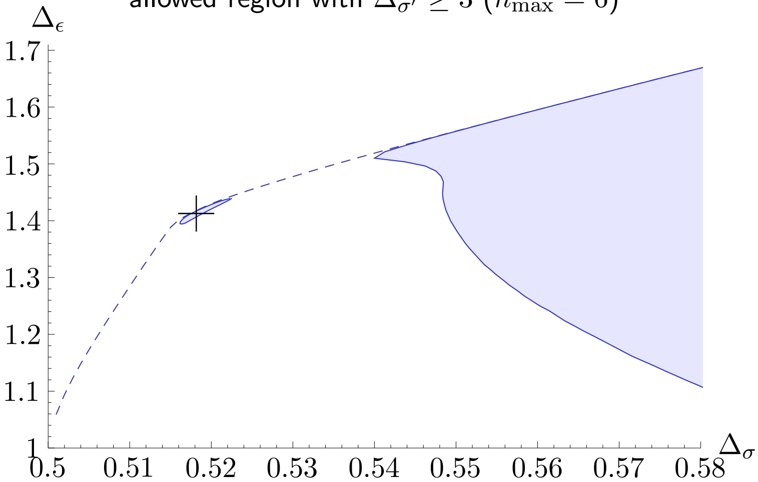


[Kos, DP, Simmons-Duffin '14]

- Imposing a gap in $\mathbb{Z}_2$-odd spectrum restricts Δ_σ to a small interval!

Mixed Correlator Bounds

allowed region with $\Delta_{\sigma'} \geq 3$ ($n_{\max} = 6$)



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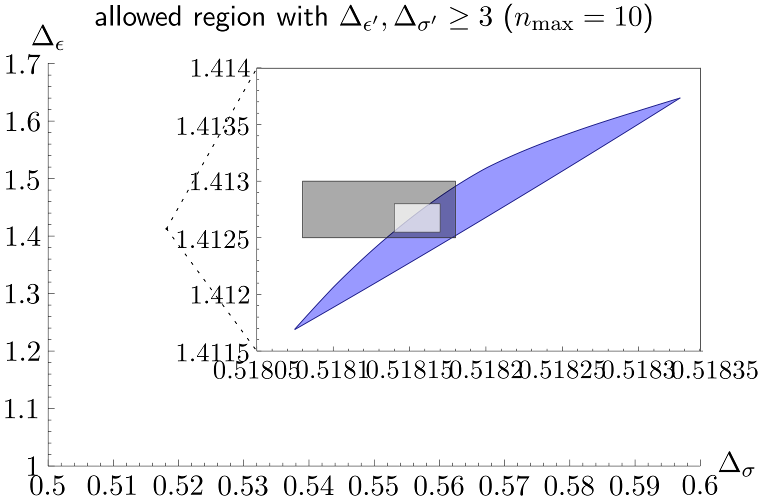

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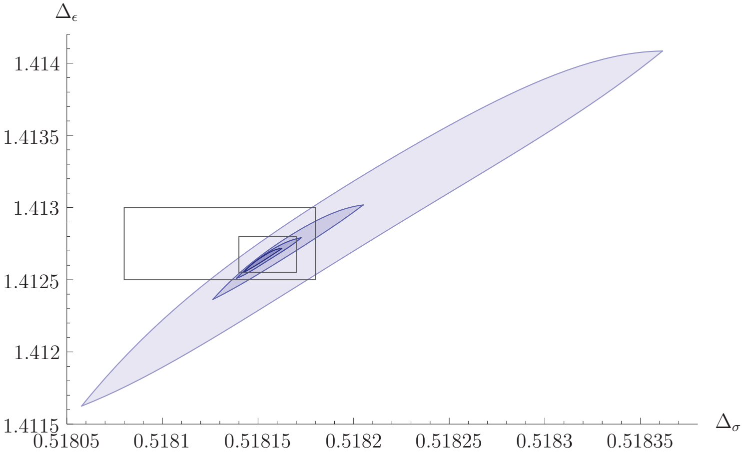

Pushing farther, the region keeps shrinking! [Simmons-Duffin '15]

$$\{\Delta_\sigma, \Delta_\epsilon\} = \{0.518151(6), 1.41264(6)\}$$

Mixed Correlator Lessons

- The 3D Ising CFT is *isolated* in the space of 3D CFTs with $\mathbb{Z}_2$ symmetry and 2 relevant operators
- It is a plausible conjecture that it is the only CFT with this property
- The conformal bootstrap can place the idea of *critical universality* on a rigorous footing

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- Extension to $O(N)$ symmetry in progress [Kos, DP, Simmons-Duffin, Vichi]
  - Challenge is many relevant operators:
    e.g. $\phi_i, \phi_i\phi_j, \ldots, \phi_i\phi_j\phi_k\phi_l\phi_m$ all relevant at large N
  - However, preliminary results show isolated regions from system:
    $\{\langle\phi_i\phi_j\phi_k\phi_l\rangle, \langle\phi_i\phi_j\phi^2\phi^2\rangle, \langle\phi^2\phi^2\phi^2\phi^2\rangle\}$

Outline

Spinning Bootstrap

- Another important direction is to extend the conformal bootstrap to external operators with spin
- E.g., one would like to include fermions, global symmetry currents, the stress tensor, higher spin operators, ...
- This brings two complications:
  - Multiple tensor structures in the 3- and 4-point functions
  - Need to calculate the conformal blocks
- Current work in progress: fermion bootstrap in 3D
  [Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby, in progress]

Fermion Bootstrap

- Consider 4-point functions $\langle\psi\psi\psi\psi\rangle$ of a Majorana fermion in 3D ($SO(2,1) \simeq Sp(2,\mathbb{R}) \to$ real two-component spinors)
- For now, we will also assume a parity symmetry: $(x,y) \to (-x,y)$
- To classify 3-point and 4-point structures, we can work in an embedding space, where $SO(3,2) \simeq Sp(4,\mathbb{R})$ is linearly realized

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Results:

- $\langle\psi\psi\mathcal{O}^{(\ell \text{ even})}\rangle$ has two structures of even parity and one of odd parity
- $\langle\psi\psi\mathcal{O}^{(\ell \text{ odd})}\rangle$ has one structure of odd parity
- $\langle\psi\psi\psi\psi\rangle$ has 5 independent tensor structures

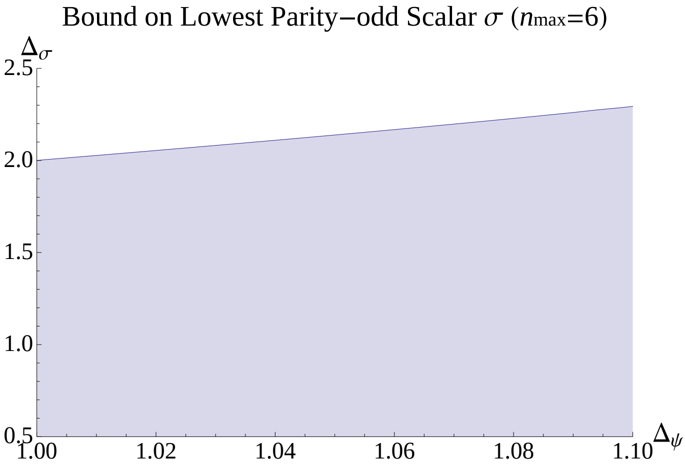
Fermion Bootstrap

- Crossing symmetry leads to a 5-vector of sum rules:

$$
\begin{aligned}
0 \;=\; & \sum_{\mathcal{O}_+,\,\ell_+} \begin{pmatrix} \lambda^1_{\mathcal{O}_+} & \lambda^2_{\mathcal{O}_+} \end{pmatrix} \vec{F}_{++,\Delta,\ell}(u,v) \begin{pmatrix} \lambda^1_{\mathcal{O}_+} \\ \lambda^2_{\mathcal{O}_+} \end{pmatrix} \\
& + \sum_{\mathcal{O}_-,\,\ell_+} (\lambda^3_{\mathcal{O}_-})^2 \vec{F}_{-+,\Delta,\ell}(u,v) + \sum_{\mathcal{O}_-,\,\ell_-} (\lambda^4_{\mathcal{O}_-})^2 \vec{F}_{--,\Delta,\ell}(u,v),
\end{aligned}
$$

- To calculate the conformal blocks, we express $\langle\psi\psi\mathcal{O}\rangle_a = D_a\langle\phi\phi\mathcal{O}\rangle$, which lets us relate $\int\langle\psi\psi\mathcal{O}\rangle_a\langle\tilde{\mathcal{O}}\psi\psi\rangle_b$ to $\int\langle\phi\phi\mathcal{O}\rangle\langle\tilde{\mathcal{O}}\phi\phi\rangle$
- Bounds follow from applying functionals α_I (again SDP is mandatory!)

Preliminary Fermion Bounds



- Bound converges to free fermion values: $\psi \times \psi \sim \mathbb{1} + \psi^2 + \ldots$

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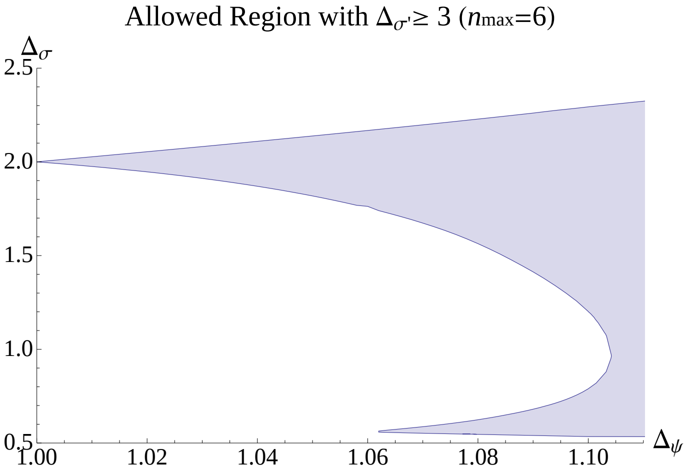

- Assuming a parity-odd gap carves out a large region

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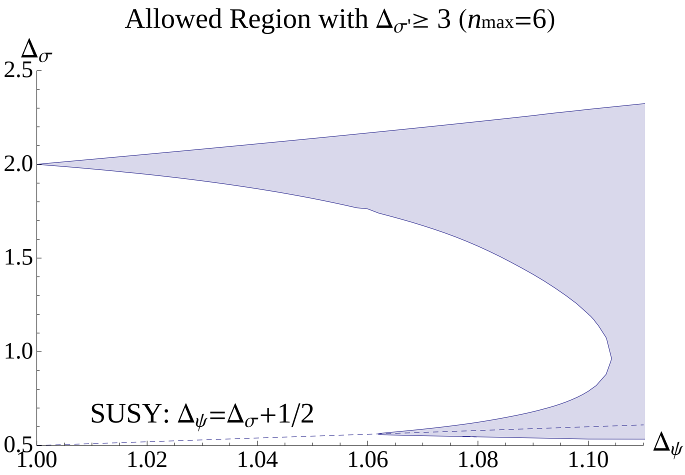

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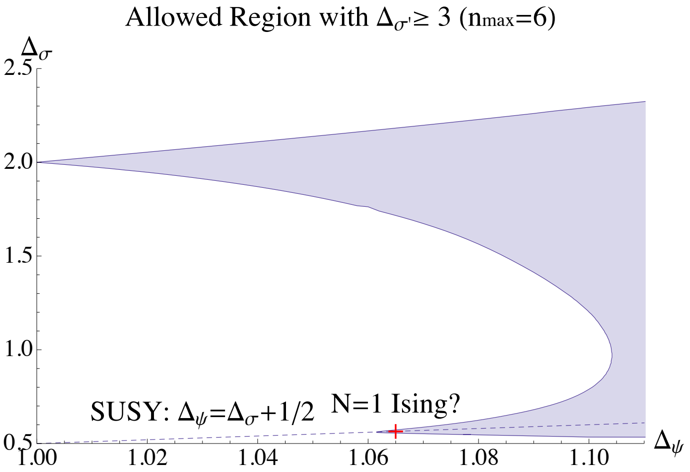
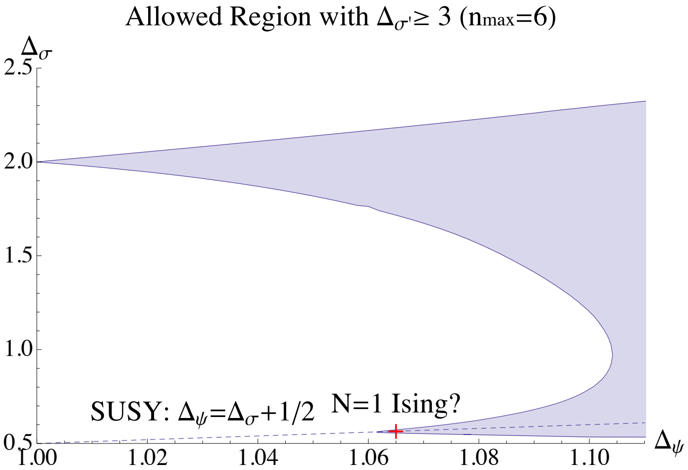


- Conjecture: Lower tip will converge to the 3D $\mathcal{N} = 1$ Ising model (perhaps with a slightly smaller σ' gap)
- SCFT with superpotential $W = \Sigma^3$ and $\Sigma = \sigma + \theta\psi + \theta^2\epsilon$

Preliminary Fermion Bounds



- Here it is at $\{\Delta_\sigma, \Delta_\psi\} \sim \{.562, 1.062\}$, not far from previous estimates ($\Delta_\sigma \sim .571$ [Grover, Sheng, Vishwanath '13], $\Delta_\sigma \gtrsim .565$ [Bashkirov '13])

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Future

- Extend spinning bootstrap to currents, stress tensor, higher spins, ...
- Larger systems of mixed correlators (all relevant operators!)
- Develop efficient algorithms for high-precision SDP
- Improve analytic arguments: large N, large ℓ, SUSY chiral algebras, ...

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- Larger systems of mixed correlators (all relevant operators!)
- Develop efficient algorithms for high-precision SDP
- Improve analytic arguments: large N, large ℓ, SUSY chiral algebras, . . .

$$\vdots$$

- Classify and map out space of CFTs in all dimensions!

Targets for the Bootstrap

- 2D CFTs with $c > 1$
- 3D Gross-Neveu Models (landscape of theories with fermions+scalars)
- 3D QED or QCD $+$ matter (monopole ops?)
- 3D Chern-Simons $+$ vector matter (connect to higher spin theory!)
- 4D QCD/SQCD in conformal window (archipelago for each N_f?)
- Classify space of 5D and 6D CFTs
- Existence of CFTs in $D > 6$?
- Conformal manifolds