Localization of gauge theory: exact results for circular supersymmetric Wilson loop operators

Vasily Pestun

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Erickson-Semenoff-Zarembo/Drukker-Gross conjecture

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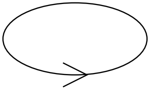

- N. Drukker and D. J. Gross, An exact prediction of N $=$ 4 SUSYM theory for string theory, hep-th/0010274.
- J. K. Erickson, G. W. Semenoff and K. Zarembo, Wilson loops in N $=$ 4 supersymmetric Yang-Mills theory, Nucl. Phys. B 582 (2000) 155, hep-th/0003055.

The formula

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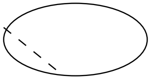

In Feynman gauge

$$\langle A_\mu(x) A_\nu(x') \rangle = \frac{1}{4\pi^2} \frac{g_{\mu\nu}}{(x - x')^2}$$

$$\langle \Phi(x) \Phi(x') \rangle = \frac{1}{4\pi^2} \frac{1}{(x - x')^2}$$

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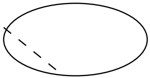

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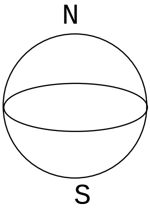
Hence

$$\langle A_\mu(\phi)\dot{x}^\mu A_\nu(\phi')\dot{x}^\nu + i\Phi(\phi)i\Phi(\phi')\rangle = \frac{1}{4\pi^2 r^2}\frac{cos(\phi-\phi')-1}{4sin^2\frac{\phi-\phi'}{2}} = -\frac{1}{8\pi^2 r^2}$$

The $\mathcal{N} = 4$ SYM on S^4

Consider a four sphere S^4 with the Wilson loop placed on the equator



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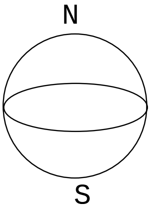

In $d = 10$ $\mathcal{N} = 1$ SYM notations $A_M = \{A_\mu, \Phi_A\}$. The action is

$$S = \frac{1}{2g_{YM}^2} \int \sqrt{g} d^4x \left(\frac{1}{2} F_{MN}^2 - \Psi \Gamma^M D_M \Psi + \frac{2}{r^2} \Phi_A \Phi^A \right)$$

The gauge-fixed path integral is well defined on S^4. No IR divergencies.

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Compute it!

$\mathcal{N} = 4$ superconformal symmetry

The action is invariant under the fermionic supersymmetry

$$\delta A_M = \varepsilon \Gamma_M \psi$$
$$\delta \psi = \frac{1}{2} F_{MN} \Gamma^{MN} \varepsilon + \frac{1}{2} \Gamma_{\mu A} \Phi^A \nabla^\mu \varepsilon$$

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where ε is a conformal Killing spinor on S^4

$$\nabla_\mu \varepsilon = \Gamma_\mu \tilde{\varepsilon}$$
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The idea of localization

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$$\int_M \frac{\omega^n}{(2\pi)^n n!} e^{iH(\phi)} = i^n \sum_{p \in F} \frac{e^{iH(\phi)}}{\prod \alpha_i^p(\phi)}$$

where (M, ω) is a symplectic manifold, $H : M \to g^*$ is the moment map for the Hamiltonian action of a torus G on M ($i_\phi \omega = dH(\phi)$ for any $\phi \in g$).

Atiyah-Bott-Berline-Vergne localization formula

More generally, if $Q\alpha = 0$ on a G-manifold M then

$$\int_M \alpha = \int_F \frac{i_F^* \alpha}{e(N_F)}$$

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$\alpha \in \Omega(M) \otimes S(g^*)$ is a differential form on M valued in a functions on g

$F \overset{i}{\hookrightarrow} M$ is the fixed point set of G acting on M

$e(N_F)$ is the Euler class of the normal bundle of F in M.

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Hence $Q^2 = 0$ on G-invariant objects

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$$\frac{d}{dt}\int e^{S+tQV} = \int \{Q, V\} e^{S+tQV} = \int \{Q, V e^{S+tQV}\} = 0$$

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$$Q^2 = \mathcal{L}_v + R$$

$\mathcal{L}_v$ is a Lie derivative along the vector field $v^M = \varepsilon\Gamma^M\varepsilon$ generating rotations of S^4 and gauge transformation $[V^M A_M, \cdot]$.

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R is an $SU(2)^R_L \subset SO(4)$-R-symmetry transformation; it acts nontrivially on four scalars $\Phi_5, \dots \Phi_8$ and fermions of $\mathcal{N} = 2$ vector multiplet

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0	χ^R	0	1/2	0	1/2	$+$
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- $(A_1 \dots A_4, \Phi_9, \Phi_0,\ \psi^L, \psi^R)$ is $\mathcal{N} = 2$ vector multiplet
- $(\Phi_5 \dots \Phi_8,\ \chi^L, \chi^R)$ is $\mathcal{N} = 2$ hyper multiplet

Killing spinor

$$\varepsilon = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}(\hat{\varepsilon}_s + x^\mu \Gamma_\mu \hat{\varepsilon}_c)$$

where $\hat{\varepsilon}_s = (1, 0, \ldots, 0)$ and $\hat{\varepsilon}_c = \frac{1}{2r}\Gamma_{12}\hat{\varepsilon}_s$
The North and the South poles are the fixed points of Q^2 acting S^4.
The transformation Q^2 is

- an anti-self-dual Lorentz $SU(2)_L$ rotation of 12-plane and 34-plane
- gauge transformation by $[i\Phi_0 + \Phi_9 \cos\theta]$
- an R-symmetry rotation in the $SU(2)_L^R$ group

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The action and the operator are Q-invariant.

$$QS = 0, \quad QW(C) = 0$$

Off-shell closure of Q_ε for $\mathcal{N} = 4$ SYM on S^4

Add 7 auxiliary scalar field K_i as in [Berkovits '93] for $d = 10$ $\mathcal{N} = 1$ SYM on R^{10}

The action $\int \sqrt{g} d^4 x \mathcal{L}$ where

$$\mathcal{L} = \frac{1}{2} F_{MN} F^{MN} - \Psi \Gamma^M D_M \Psi + \frac{2}{r^2} \Phi_A \Phi^A + K_i K_i$$

is invariant under

$$\begin{aligned} \delta A_M &= \varepsilon \Gamma_M \Psi \\ \delta \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \varepsilon + \frac{1}{2} \Gamma_{\mu A} \Phi^A \nabla^\mu \varepsilon + i K_i \nu_i \\ \delta K_i &= i \nu_i \Gamma^M D_M \Psi \end{aligned}$$

where $\{\nu_i\}$ is a set of 7 Mayorana-Weyl fermions satisfying algebraic equations

$$\varepsilon \Gamma^M \nu_i = 0, \quad \nu_i \Gamma^M \nu_j = \delta_{ij} \varepsilon \Gamma^M \varepsilon$$

Q-exact deformation of the action

$$S \to S + tQV$$

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After integrating out K, at the limit $t \rightarrow \infty$, only the zero mode of Φ_0 has finite quadratic action!
The path integral is localized to the locus $\Phi_0 = const$, all other fields vanish.

The quadratic term

Be careful with integrating out K. The deformed action

$$S + tQV = \frac{1}{2g_{YM}^2} \int (\cdots + \frac{2}{r^2}\Phi_0^2 + K^2 + t((K + \frac{1}{r}\Phi_0)^2 + \dots))$$

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In this way we obtain the Matrix Model action

$$S_{MM}[\Phi_0 = a] = \frac{1}{2g_{YM}^2} \times \frac{8}{3}\pi^2 r^4 \times \frac{3}{r^2} a^2 = \frac{8\pi^2 r^2}{g_{YM}^2}\Phi_0^2$$

Coincides with Erickson-Semenoff-Zarembo/Drukker-Gross matrix model.

The one-loop determinant

At $t \to \infty$ limit we need also to compute the determinant for the fluctuations of the fields with the action $S + tQV$ near the dominant configuration $\Phi_0 = const$. Similarly to Duistermaat-Heckman formula

$$\int_M \frac{\omega^n}{(2\pi)^n n!} e^{iH(\phi)} = i^n \sum_{p \in F} \frac{e^{iH(\phi)}}{\prod \alpha_i^p(\phi)}$$

the determinant $Z_{1-loop}(a)$ can be computed as a certain product of weights of Q^2 acting to the tangents space to all fields at the locus $\Phi_0 = a$.

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This is a linear problem; it can be treated by the Atiyah-Singer theorem.

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where $s_i(A_M) = (\nu_i, \frac{1}{2} F_{MN} \Gamma^{MN} \varepsilon + \frac{1}{2} \Gamma_{\mu A} \Phi^A \nabla^\mu \varepsilon)$ are the "equations". We further define $H_i = K_i + s_i(A_M)$

The Q-complex

In new notations the transformations look like

$$\delta A_M = \psi_M$$
$$\delta \psi_M = \mathcal{R} \cdot A_M$$
$$\delta \chi_i = H_i$$
$$\delta H_i = \mathcal{R} \cdot \chi_i$$

where $\mathcal{R}$ stands for the Q^2 action on fields.
Then

$$Z_{1-loop} = \frac{\det \mathcal{R}|_{H_i}}{\det \mathcal{R}|_{A_M}}$$

To compute the ratio of determinants of $\mathcal{R}$ acting on the set of fields A_M and χ_i we use the Atiyah-Singer index theorem to compute the $\mathcal{R}$-equivariant character

$$ind = \mathrm{tr}_{A_M}\, e^{\mathcal{R}} - \mathrm{tr}_{H_i}\, e^{\mathcal{R}}$$

Each term is infinite dimensional and is not well-defined on its own. But the difference is well-defined.

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In the present problem the equations $s_i(A_M)$ are not elliptic, but they are transversally elliptic with the respect to $U(1)$ rotations of S^4.
The index is then well defined as a distribution on the group $U(1)$.

The generating function for the index is

$$-\frac{1+q^2}{(1-q)^2}$$

for the $\mathcal{N}=2$ vector multiplet, and

$$\frac{2q}{(1-q)^2}$$

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In the case of $\mathcal{N}=2$ theory with a matter hypermultiplet in representation R we have

$$Z_{1-loop}^{\mathcal{N}=2,W}(ia_E) = \frac{\prod_{\alpha \in \mathrm{weights(Ad)}} H(i\alpha \cdot a_E/\varepsilon)}{\prod_{w \in \mathrm{weights}(W)} H(iw \cdot a_E/\varepsilon)}.$$

Here $H(z)$ is the related to the Barnes G-function ('superfactorial') as

$$H(z) = G(1+z)G(1-z) = e^{-(1+\gamma)z^2} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)^n e^{\frac{z^2}{n}}.$$

Point instanton corrections and conclusion

Gauge theory:

$$Z_{S^4} = \int da Z_{1-loop} |Z^{inst}_{Nekr}(r^{-1}, r^{-1}, a)|^2$$

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$$Z_{BH} = |Z_{top}|^2$$

The relation?