The arrow of time, black holes, and quantum mixing of large N Yang-Mills theories

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based on

Guido Festuccia, HL, to appear

The arrow of time and space-like singularities

While equations of general relativity are *time symmetric* , one often finds solutions with an *intrinsic time direction* , due to the presence of *spacelike singularities* , like

FRW cosmologies, gravitational collapse .

Gravitational collapse: the direction of time appears to be *thermodynamic* in nature.

The Big Bang singularity has also long been speculated to be related to the thermodynamic arrow of time observed in daily life. Gold, Penrose, ...

Gravitational collapse in AdS spacetime

AdS/CFT:

quantum gravity in AdS	$\Leftrightarrow$	$\mathrm{Yang-Mills\ on}\ S^3$
classical gravity	$\Leftrightarrow$	large N
a classical mass	$\Leftrightarrow$	excited state of energy $O(N^2)$
gravitational collapse	$\Leftrightarrow$	thermalization
black hole	$\Leftrightarrow$	thermal equilibrium

Large N limit

An SYM theory on S^3 is a *bounded* many-body quantum mechanical system. At *finite* N,

- it has a *discrete* spectrum
- it is time *reversible*

Heuristically, the large N limit for a SYM theory in a (highly) excited state of energy of $O(N^2)$ is like a *thermodynamic limit.*

One expects a direction of time to emerge in this limit (*notoriously difficult to prove*).

Observables

The simplest observables are (spatial dependence suppressed)

$$G_+(t) = \frac{1}{Z}\text{Tr}\left(e^{-\beta H}O(t)O(0)\right)$$

and its Fourier transform $G_+(\omega)$.

1. O has dimension of order $O(1)$, corresponding to fundamental string states in AdS.

2. β is small enough

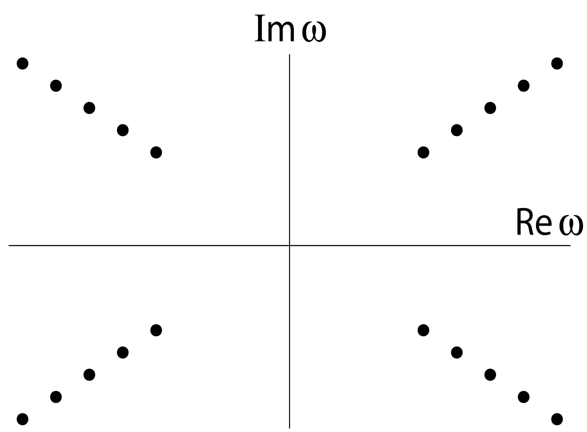
3. At finite N, oscillatory and recurring.

4. Direction of time emerges if $G_+(t) \to 0$ as $t \to \infty$ (*mixing*).

Analytic properties of $G_+(\omega, l)$ at strong coupling

G. Festuccia, HL

- $G_+(\omega, l)$ has a *continuous* spectrum with $\omega \in (-\infty, +\infty)$.
- Simple poles in the complex ω-plane (for l not too large)



- Direction of time: The coordinate space $G_+(t, l)$ *decays exponentially* with time.

- The *black hole singularity* is encoded in behavior of $G_+(\omega, l)$ at the *imaginary infinity* of the ω-plane.

  1. $G_+(\omega, l)$ decays exponentially as $\omega \to \pm i\infty$.

  2. Derivatives of $G_+(\omega, l)$ over l evaluated at $l = 0$ are divergent as $\omega \to \pm i\infty$.

- At finite N,

$$G_+(\omega) = 2\pi \sum_{m,n} e^{-\beta E_m} \rho_{mn} \delta(\omega - E_n + E_m)$$

  is a sum of delta functions supported on the real axis.

Spacelike singularities and thermalization?

It would desirable to have a qualitative understanding for:

- underlying physics for the emergence of a direction of time in the large N limit

- Does thermalization happen at weak coupling? e.g. does $G_+(\omega, l)$ has a continuous spectrum at weak coupling?

- connection between spacelike singularities and thermalization?

Banks, Fischler

Plan

1. Discrete spectrum (no thermalization) at any finite order in perturbation theory

2. Perturbation theory breaks down in the long time limit (planar expansion is divergent)

3. a statistical approach: *direction of time does emerge at any nonzero* λ

4. Conclusions

Relevant theories

We consider generic matrix quantum mechanical systems

$$S = N\text{tr} \int dt \left[\sum_\alpha \left(\frac{1}{2}(D_t M_\alpha)^2 - \frac{1}{2}\omega_\alpha^2 M_\alpha^2 \right) \right] + S_{int}[M_\alpha] \qquad (1)$$

1. More than one matrices, $U(N)$ gauge symmetry

2. the theory has a mass gap and a unique vacuum.

3. S_{int} can be written as a sum of integrals of *single-trace* operators and is controlled by a coupling constant λ.

$\mathcal{N} = 4$ SYM on S^3 in an example of (1).

Observables

The simplest observables are (possible spatial dependence suppressed)

$$G_+(t) = \frac{1}{Z}\text{Tr}\left(e^{-\beta H} O(t) O(0)\right)$$

and its Fourier transform $G_+(\omega)$.

1. O gauge invariant operator of dimension of order $O(1)$.

2. When $\beta \sim O(1)$ is small enough, the free energy $F = -\frac{1}{\beta}\log Z$ is of order $O(N^2)$ (always the case with more than two matrices).

Planar Perturbation theory

1. Free theory correlation functions are always oscillatory, and have discrete spectral functions. *No thermalization* .

2. In perturbation theory (planar limit)

$$G_+(t,\lambda) = \sum_{n=0}^{\infty} \lambda^n G_+^{(n)}(t)$$

with typical terms in $G_+^{(n)}(t)$

$$g_{kl}(\beta) t^l e^{ik\omega_0 t},$$

$$l = 0, 1, \cdots n, \quad k = n - 2\Delta, \cdots n + 2\Delta, \quad \omega_0 = \frac{1}{R}$$

No thermalization at weak coupling?

We will argue that, however, the perturbative expansion has a *zero radius of convergence in λ* as $t \to \infty$ and cannot be used to understand the *long time* behavior.

Breakdown of planar perturbation theory at finite temperature

Consider a toy example (to simplify combinatorics):

$$S = \frac{N}{2}\mathrm{tr}\int dt\ \left[(D_t M_1)^2 + (D_t M_2)^2 - \omega_0^2(M_1^2 + M_2^2) + \lambda M_1 M_2 M_1 M_2\right]$$

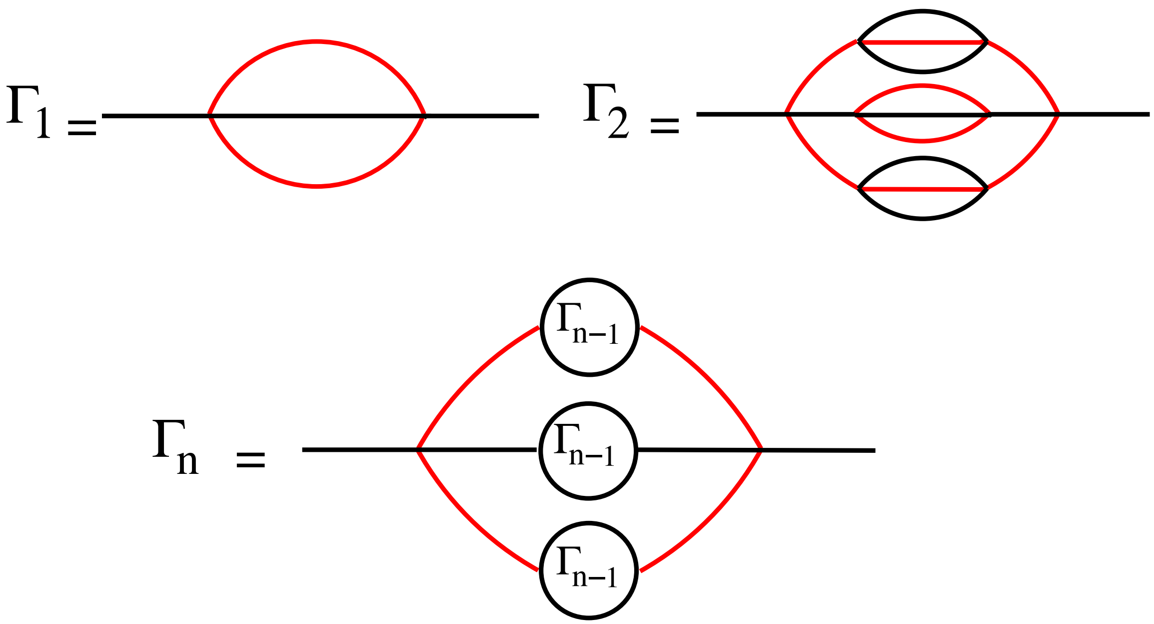
To illustrate our argument it is enough to look at the propagator for M_1 at finite T (we will also ignore the singlet condition)

$$\begin{aligned} D_F(t) &= \frac{1}{Z(\beta)}\mathrm{Tr}\left(e^{-\beta H} T(M_1(t) M_1(0))\right) \\ &= \sum_{n=0}^{\infty} \lambda^n D_F^{(n)}(t) \\ &= D_F^{(0)}(t) \sum_{n=1}^{\infty} c_n \lambda^n t^n + \cdots \end{aligned}$$

Breakdown of planar perturbative expansion II

Strategy: to identify a family of diagrams and show that the sum of them lead to a divergent sum.

We consider the following set of diagrams:



Breakdown of planar perturbative expansion III

These graphs appear at orders $d_1 = 2, d_2 = 8, d_3 = 26, \cdots$ of perturbation theory

$$d_i = 3d_{i-1} + 2 = 3^i - 1, \qquad i = 1, 2 \cdots$$

Summing them gives rise to

$$\sum_i \Gamma_i(t) \approx D_F^{(0)}(t) \sum_{i=1}^{\infty} (-1)^i \left(\frac{\lambda t}{h_c}\right)^{d_i} + \cdots$$

Barring any unforseen cancellations, this implies that the perturbative expansion has a zero radius of convergence in the large t limit.

Note $h_c \to \infty$, as $T \to 0$.

Breakdown of planar perturbative expansion IV

Thus we expect:

- The radius of convergence of λ at large t

$$\lambda_c(t) \sim \frac{1}{t}$$

- Essential singularity at $\lambda = 0$ in momentum space

$$D_F(\omega) \sim e^{i\omega \frac{qc}{\lambda}}$$

Breakdown of planar perturbative expansion V

The argument is very general. Should be applicable to generic matrix quantum mechanical systems with more than one matrices, including $\mathcal{N} = 4$ SYM on S^3.

However, the argument is not foolproof since we have not demonstrated that cancellation will not happen even though we do not expect it to happen.

Given that the perturbation theory breaks down we need to develop new non-perturbative methods to understand the *long time* behavior of real time correlation functions in the *large N limit.*

Let us first give a simple argument *why the breakdown is expected* .

Features of energy spectrum: free theory

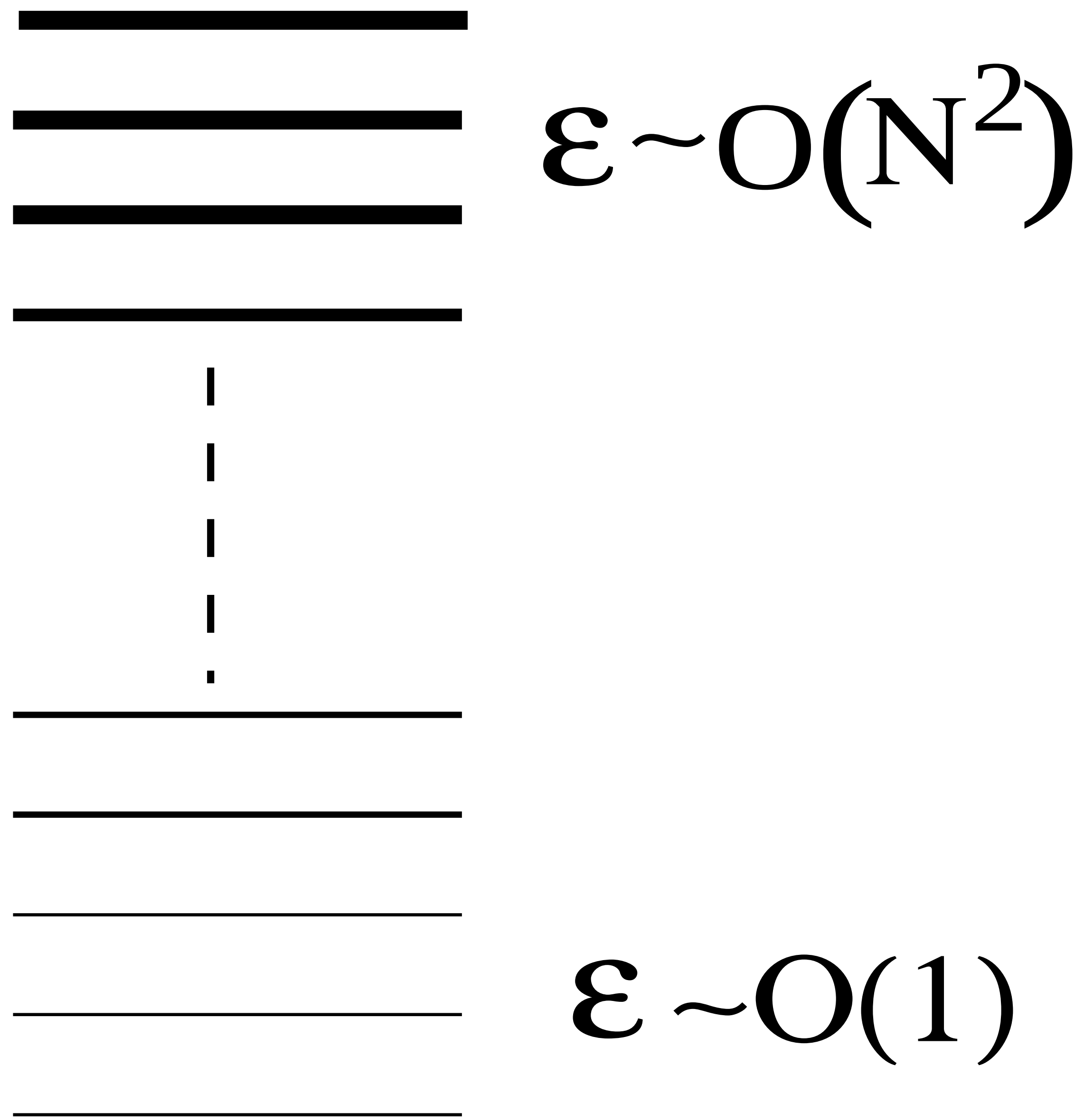
In free theory limit, the spectrum of $\mathcal{N} = 4$ SYM on S^3 has the following features:

1. The energy levels are equally spaced, with spacing $1/2R$.

2. Typical levels are degenerate with degeneracy:

$$D(E) \sim O(1), \quad E \sim O(1);$$

$$D(E) \sim e^{O(N^2)}, \quad E \sim O(N^2)$$


ε ~O(N2)
ε ~O(1)

Features of energy spectrum: weakly coupled

Now turning on a tiny, but nonzero λ:

1. A naive application of degenerate perturbation theory suggests that of order $e^{O(N^2)}$ states mix under the perturbation.

2. Expect:

- degeneracy broken
- level spacing of order $e^{-O(N^2)}$.

3. These effects *cannot* be captured by Feynmann diagram type perturbation theory describe earlier at any finite order.

An alternative approach

Note that

$$G_+(t) = \frac{1}{Z}\mathrm{Tr}\left(e^{-\beta H}O(t)O(0)\right) = \frac{1}{Z}\sum_{i,j} e^{-\beta E_i + i(E_i - E_j)t}\rho_{ij}$$

$$\rho_{ij} = |\langle i|O(0)|j\rangle|^2 = |O_{ij}|^2$$

$|i\rangle$: full set of energy eigenstates of the interacting theory
$|a\rangle$: full set of energy eigenstates of the free theory

$$|i\rangle = c_{ia}|a\rangle$$

$$O_{ij} = \langle i|O(0)|j\rangle = \sum_{a,b} c^*_{ia}c_{jb}\langle a|O|b\rangle = \sum_{a,b} c^*_{ia}c_{jb}O_{ab}$$

A statistical approach I

Suppose we naively apply the degenerate perturbation theory to an energy level of $O(N^2)$:

1. Write $H = H_0 + V$.

2. Need to to diagonalize V in a subspace of size $e^{O(N^2)}$.

3. May approximate V by an (extremely) sparse random matrix.

4. The spread of diagonalized energy eigenvalues is of order $O(\lambda N)$. This implies that one should diagonalize a much larger matrix of energy spread of order $O(N)$.

A statistical approach II

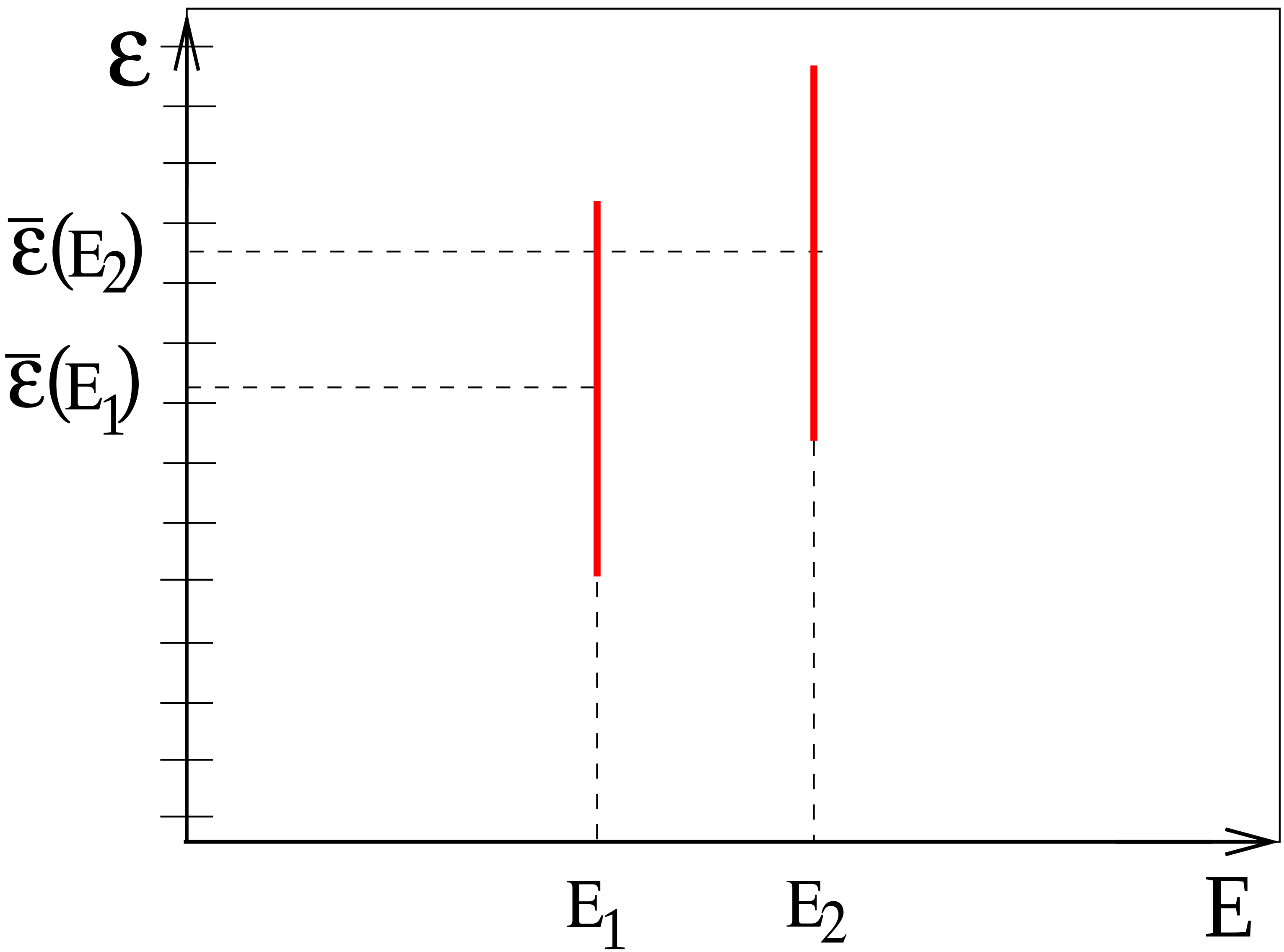
Introduce

$$\chi_i(\epsilon) = \sum_a |c_{ia}|^2 \delta(\epsilon - \epsilon_a)$$

$$\int d\epsilon \, \chi_i(\epsilon) = 1$$

$\chi_i(\epsilon)$ gives the distribution of free theory states of energy ϵ coupling to an exact eigenstate $|i\rangle$.

General large N scaling argument can be used to show that *the spread of $\chi_E(\epsilon)$ is of order $O(N)$,* when averaged over states of similar energy $E \sim O(N^2)$.


ε
$\bar{\varepsilon}(E_2)$
$\bar{\varepsilon}(E_1)$
E_1
E_2
E

A statistical approach III

In the absence of localization effect, c_{ia} can be treated as a random unit vector with support inside an energy shell of width $O(N)$.

The energy shells of two interacting theory states whose energy difference $\omega = E_i - E_j \sim O(1)$ overlap significantly.

O_{ab} can be treated as a sparse banded random matrix.

Then

$$O_{ij} = \langle i|O(0)|j\rangle = \sum_{a,b} c_{ia}^* c_{jb} \langle a|O|b\rangle = \sum_{a,b} c_{ia}^* c_{jb} O_{ab}$$

is supported for any $E_i - E_j \sim O(1)$.

Matrix elements

After averaging over states of similar energies, one finds

$$\rho_{E_1 E_2} = \frac{1}{\Omega(E)} A(\omega, E/N^2)$$

with

$$E = \frac{E_1 + E_2}{2}, \qquad \omega = E_1 - E_2$$

$A(\omega, E/N^2)$ is even, smooth function of ω.

As $\omega \to \pm\infty$

$$A_E(\omega) \propto e^{-\frac{1}{2}\beta(E)|\omega|}, \qquad \beta(E) = \frac{\partial \log \Omega(E)}{\partial E}$$

Wightman functions at weak coupling

One finds

$$G_+(\omega,\beta) = e^{\frac{\beta\omega}{2}} A(\omega,\mu_\beta)$$

$$\mu_\beta = \frac{E_\beta}{N^2}, \qquad \left.\frac{\partial S(E)}{\partial E}\right|_{E_\beta} = \beta$$

1. $G_+(\omega)$ has a continuous spectrum with $\omega \in (-\infty, +\infty)$.

2. The singularities of $G_+(\omega)$ should be finite distance away from the real axis in the complex ω-plane.

3. $G_+(t)$ should decay exponentially with time (*Direction of time*).

Unfortunately, our current techniques are not enough for obtaining more precise analytic structure of $A(\omega, E/N^2)$.

Summary

- Perturbation theory is inadequate for understanding the long time behavior of real-time correlation functions.

- In a statistical approach, we find $G_+(\omega)$ has a continuous spectrum and direction of time emerges.

Speculations

- High temperature phase of a weak coupled YM theory is dual to a stringy black hole in the large N limit.

- BH singularities may survive α' corrections

- The behavior of the matrix elements we found have long been argued to be a signature of quantum chaos.
  Peres, Feingold, Prozen, Wilkinson ....

- There could be a connection with BKL behavior near a space-like singularity.

Thank You