Russian Doll Renormalization Group Flows

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Based in part on collaborations with:
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Related work in the last few years:
Bedaque, Hammer and van Kolck; Glazer and Wilson; Braaten et. al.

(RUTGERS, October 2003)

Outline

I. General properties

II. Cyclic Regime of Kosterlitz-Thouless Flows

III. Russian Doll superconductors

Russian Doll RG: General Properties

Renormalization Group:

Given a hamiltonian

$$H(g, L)$$

where g are couplings and L a length scale, this hamiltonian has the same spectrum as

$$H(g(L'), L')$$

as long as g depends appropriately on L.

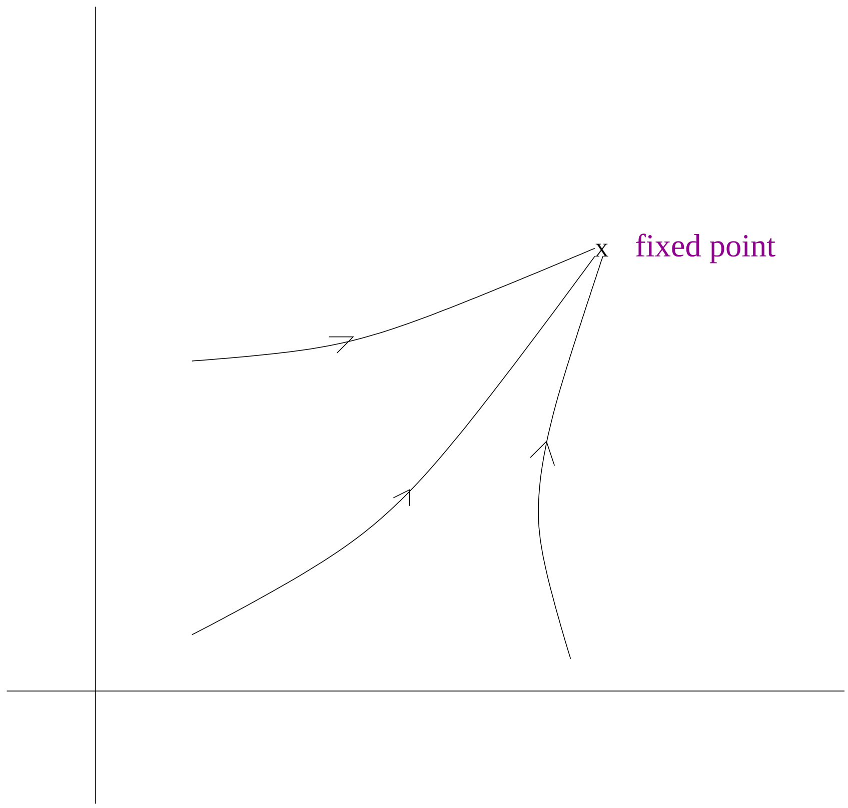
How g depends on L is encoded in the beta functions:

$$\boxed{\frac{dg}{dl} = \beta(g)}$$

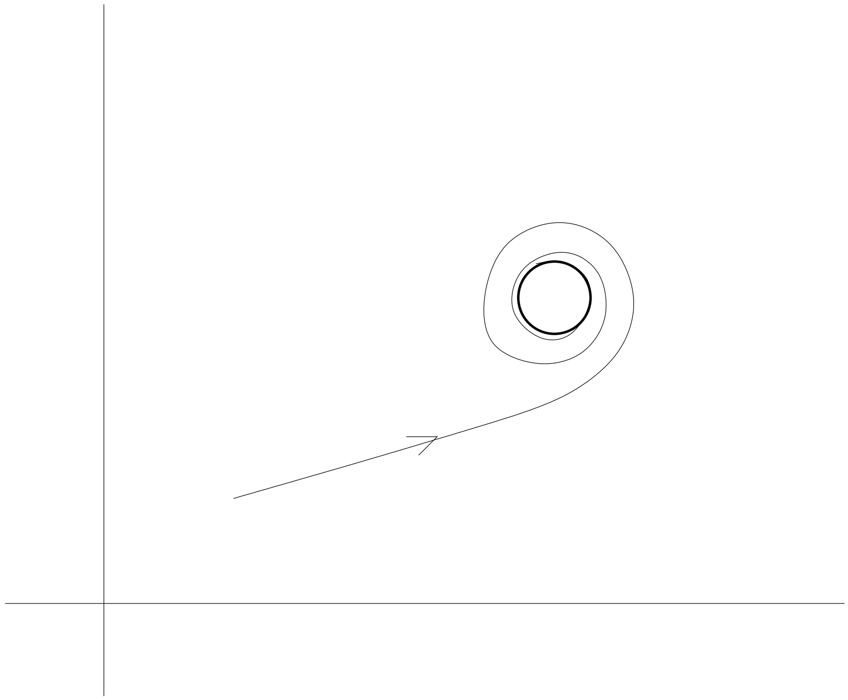
$$L = e^l \qquad \text{is the RG scale}$$

- The renormalization group flows are analogous to dynamical systems, with l the "RG time".
- Usually low energy properties correspond to an

infra-red fixed point of the RG flow



- Are other kinds of flows possible, e.g. limit cycles, chaos??



A Russian Doll renormalization group trajectory we define to be one that is cyclic, i.e. the couplings return to their initial values after a finite RG time l:

$$g(e^{\lambda}L) = g(L)$$

$$L = e^{l} \qquad \text{is the RG scale}$$

Here, λ, the period of the RG flows, is a fixed model-dependent constant.

- Implications for the spectrum: periodicity in the spectrum of eigen-energies as a function of scale. I.e. self-similarity of the spectrum upon a discrete scale transformation:

$$\{E(g, L)\} = \{E(g, e^{\lambda}L)\}$$

- Implications for the S-matrix:

$$S(e^{\lambda}E_{cm}) = S(E_{cm})$$

Kosterlitz-Thouless flows at one-loop

The Kosterlitz-Thouless flows arise in a multitude of systems, and are thought to be well understood.

- Arise as the continuum limit of XXZ Heisenberg chain:

$$H = \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right)$$

- Arise as perturbations of Luttinger liquids. Here the currents are fermion bilinears: $J^a = \psi^\dagger \sigma^a \psi$.

As a continuum field theory, it corresponds to anisotropic current-current interactions for $su(2)$. The action is

$$S = S_{free} + \int \frac{d^2x}{2\pi} \left(4g_\perp (J^+ \overline{J}^- + J^- \overline{J}^+) - 4g_\parallel J_3 \overline{J}_3 \right)$$
$$= S_{free} + 4 - \text{fermion interactions}$$

To one loop the beta functions are well-known:

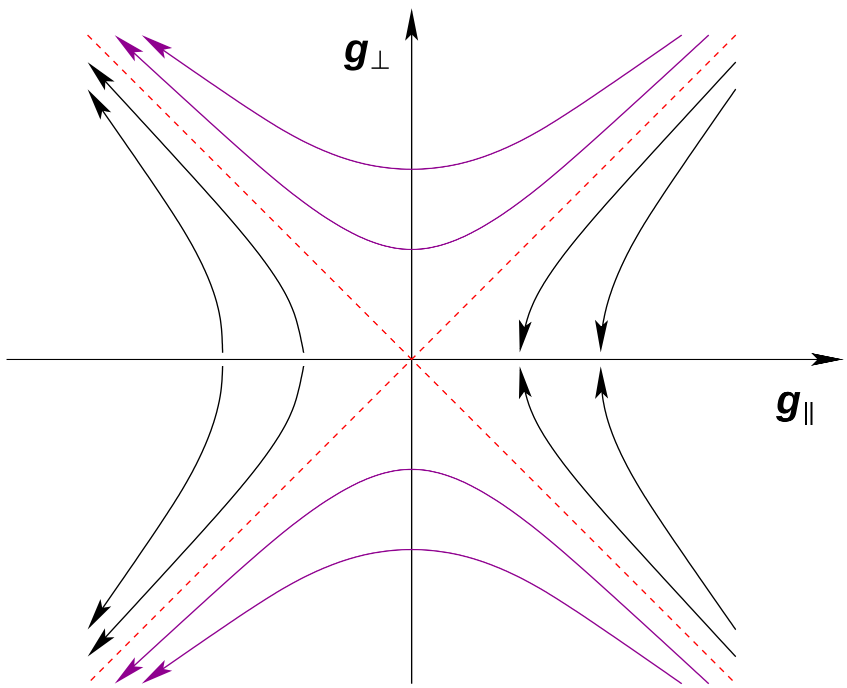
$$\frac{dg_{\|}}{dl} = -4g_{\perp}^2, \qquad \frac{dg_{\perp}}{dl} = -4g_{\perp}g_{\|}$$

There exists the RG invariant:

$$Q = g_{\|}^2 - g_{\perp}^2 \quad \equiv \quad -\frac{h^2}{16}$$

h is the main parameter of the model, and is a measure of the anisotropy. (h is related to Δ of the spin chain).

$$\frac{2\pi}{h} = \frac{\pi^2}{\cosh^{-1}(\Delta)}$$


$g_\perp$
$g_\parallel$

Eliminating $g_\perp$ and defining h as above one gets

$$\frac{dg_{||}}{dl} = -4(g_{||}^2 + \frac{h^2}{16})$$

The coupling $g_{||}$ as a function of scale $L = \exp(l)$ is:

$$g_{||} = -\frac{h}{4}\tan(h(l - l_0))$$

Thus one observes the periodicity:

$$g_{||}(e^\lambda L) = g_{||}(L), \qquad \lambda_{1-loop} = \frac{\pi}{h}$$

DOES THIS BEHAVIOR PERSIST NON-PERTURBATIVELY??

All-orders βeta function for general current-current perturbations:

Definition of the models:

$$S = S_{G_k} + \int \frac{d^2x}{2\pi} \sum_A g_A \mathcal{O}^A$$

G_k is a level k current algebra for the (super) group G, with currents J^a.

$$\mathcal{O}^A \equiv d^A_{ab} J^a \overline{J}^b$$

Also define the purely chiral operator:

$$T^A \equiv d^A_{ab} J^a J^b$$

The βeta function depends on some structure constants $C, D, \widetilde{C}$ which are easily computed in the cft.

$$\mathcal{O}^A(z,\overline{z})\mathcal{O}^B(0) \sim \frac{1}{z\overline{z}} C^{AB}_C \; \mathcal{O}^C(0)$$
$$T^A(z)\mathcal{O}^B(0) \sim \frac{1}{z^2}\left(2kD^{AB}_C + \widetilde{C}^{AB}_C\right)\mathcal{O}^C(0)$$

To two loops:

$$\beta_{g_A} = -\frac{1}{2} g_B g_C C^{BC}_A - \frac{k}{2} g_B g_C g_D D^{BC}_E \widetilde{C}^{ED}_A + ...$$

All-orders formula: (with Gerganov and Moriconi)

$$\beta_g = -\tfrac{1}{2} C(g', g')(1 + k^2 D^2/4) + \tfrac{k^3}{8} C(g' D, g' D) D - \tfrac{k}{2} \widetilde{C}(g' D, g)$$

- $g =$ a row vector
- $D = \text{matrix}, \quad D^A_B = \sum_C D^{AC}_B g_C$
- $C(a,b) =$ a row vector, $C(a,b)_A = \sum_{B,C} a_B b_C C^{BC}_A$
- $g' = g/(1 - k^2 D^2/4)$

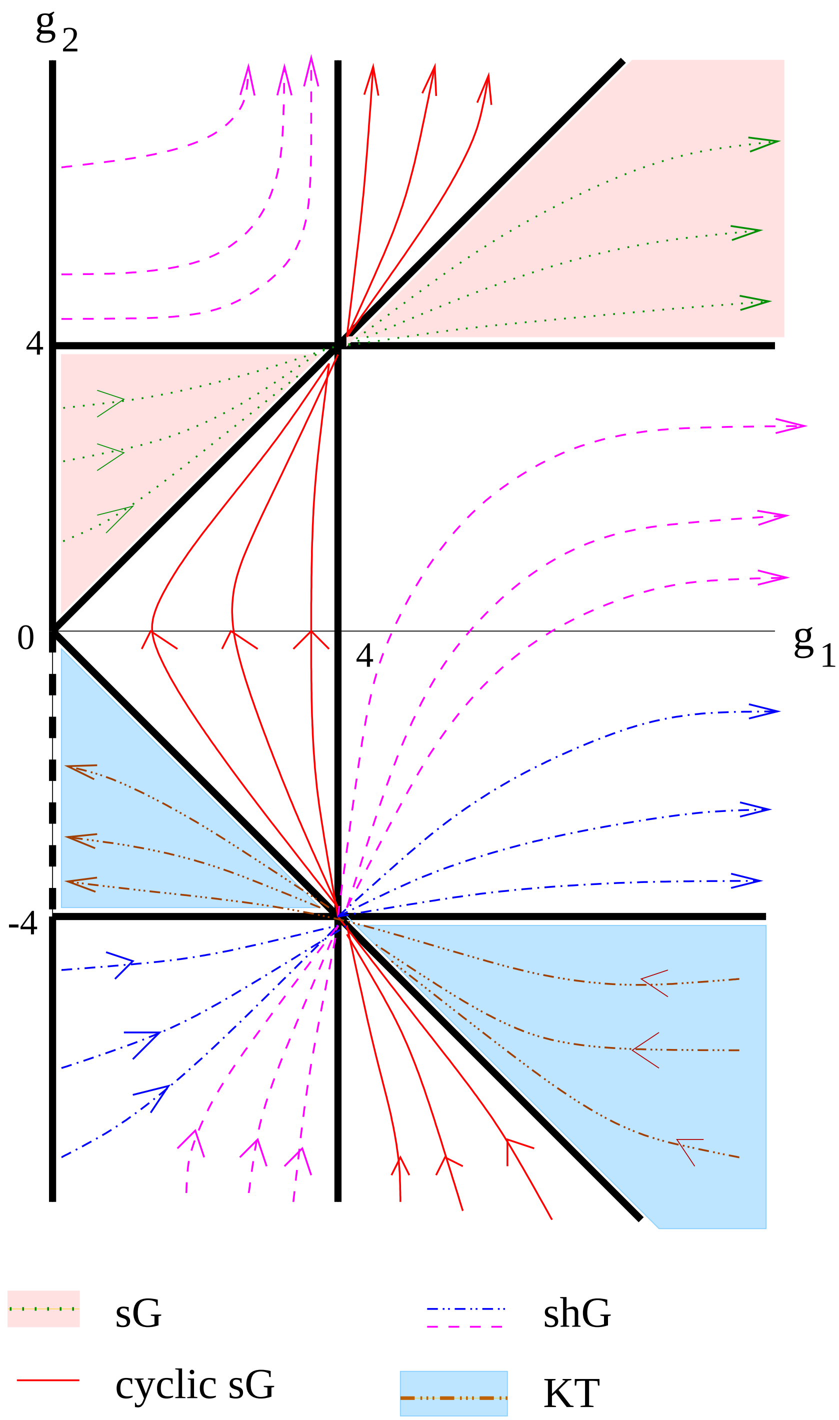
All orders beta function for Anisotropic $su(2)$:

$$\frac{dg_{\parallel}}{dl} = \frac{-4g_{\perp}^2(1+g_{\parallel})^2}{(1-g_{\perp}^2)^2}$$

$$\frac{dg_{\perp}}{dl} = \frac{-4g_{\perp}(g_{\parallel}+g_{\perp}^2)}{(1-g_{\perp}^2)(1-g_{\parallel})}$$

There continues to be an RG invariant:

$$Q = \frac{g_{\perp}^2 - g_{\parallel}^2}{(1+g_{\parallel})^2(g_{\perp}^2-1)} \equiv -\frac{h^2}{16}$$



g 2
4
0
-4
4
g 1
sG
cyclic sG
shG
KT

Eliminating $g_\perp$:

$$\frac{dg_{||}}{dl} = \frac{4\left((g_{||}+1)^2 Q - 1\right)\left(g_{||}^2 - (1+g_{||})^2 Q\right)}{(1-g_{||})^2}$$

The above equation is easily integrated and the solution explicitely has the cyclic property:

$$g_{||}(e^{\lambda}L) = g_{||}(L), \qquad \lambda = \frac{2\pi}{h} = 2\lambda_{1-loop}$$

Massive verses Massless:

In the isotropic limit $h = 0$, there are *two different* theories:

- $g_\perp = g_\parallel$. IR fixed point. Massless theory. O(3) sigma model at $\vartheta = \pi$.

- $g_\perp = -g_\parallel$. UV fixed point. Massive theory. sine-Gordon at the su(2) invariant point.

- When $h \neq 0$, expect two possibilities: massive verses massless

Spectrum and S-matrices

The cyclic regime of the KT flows can be formally mapped onto the sine-Gordon theory:

$$S = \int \frac{d^2x}{4\pi} \; \frac{1}{2}(\partial\phi)^2 + \Lambda \cos b\phi$$

$$b^2 = \frac{2}{1 + ih/2}$$

This theory has a quantum affine symmetry $\mathcal{U}_q(\widehat{sl(2)})$ with $q = -exp(-\pi h/2)$ *real*.

Requiring the S-matrix to commute with $\mathcal{U}_q(\widehat{sl(2)})$ fixes it up to an overall scalar factor, subject to constraints of crossing and unitarity, for which there are minimal solutions.

Massive case:

Spectrum: a massive doublet of charge ± 1 spinons. Agrees with the low energy limit of the spin-chain.

Relativistic dispersion relation:

$$E = m \cosh \beta, \qquad p = m \sinh \beta$$

Exact spinon-spinon S-matrix:

$$S(\beta, h) = \zeta^{-1} \prod_{n=0}^{\infty} \frac{(1 - q^{4+4n}\zeta^{-2})(1 - q^{2+4n}\zeta^{2})}{(1 - q^{4+4n}\zeta^{2})(1 - q^{2+4n}\zeta^{-2})}$$

$$\zeta = e^{-i\beta h/2}, q = -e^{-\pi h/2}$$

- The S-matrix is an analytic extension of the sine-Gordon one with a different overall scalar factor. Satisfies all the constraints.
- Matches the XXZ spin chain S-matrix at low energies, with $\Delta < -1.$ Differs at high energies

because of relativistic dispersion, related to explicite lattice cut-off of the spin chain.

Massless case:

Dispersion:

$$E = \frac{m}{2} e^{\beta}, \qquad p = \frac{m}{2} e^{\beta} \qquad \mathrm{right-movers}$$

$$E = \frac{m}{2} e^{-\beta}, \qquad p = -\frac{m}{2} e^{-\beta} \qquad \mathrm{left-movers}$$

The scattering of left with right movers:

$$S_{LR} = \text{as before}$$

Periodic properties of the S-matrix:

Though the arguments leading to the S-matrix were entirely independent of the RG, the S-matrix has periodic properties correctly predicted by the RG, i.e. of precisely the correct period $\lambda = 2\pi/h$:

$$S(\beta + 2\lambda) = S(\beta)$$

- In the massless case, the above is valid at all energies.
- In the massive case: only at high energies, indicating a UV limit cycle.

Finite Size Effects

$E(R)$ = ground state energy on a cylinder of circumference R.

Define the effective Virasoro central charge$c_{\rm eff}(R)$:

$$E(R) = -\frac{\pi}{6}\frac{c_{\rm eff}(R)}{R}$$

$c_{\rm eff}$ tracks the RG flow. Usually $c(r)$ is an uneventful function smoothly interpolating between UV and IR fixed point values of c.

RG equations for $c_{\rm eff}$:

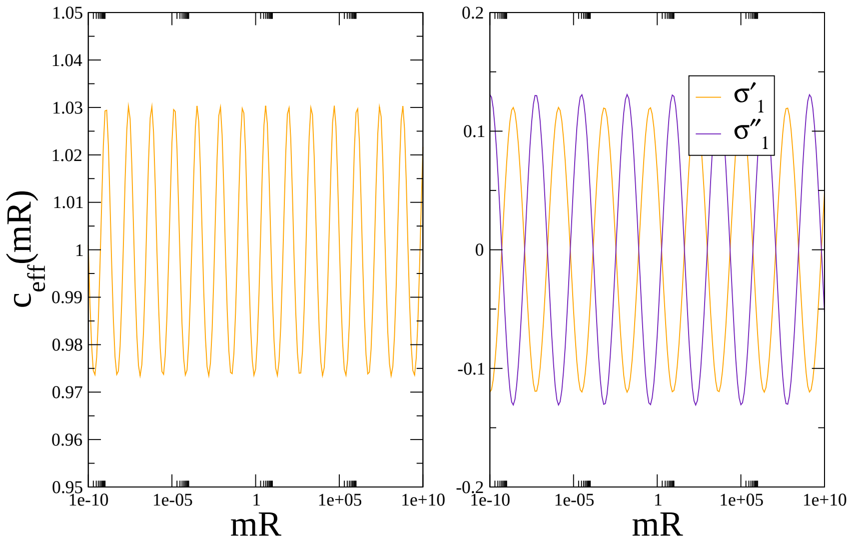
Since $c_{\rm eff}$ is related to a one-point function, it obeys RG scaling equations.

$$\langle T^\mu_\mu \rangle = \frac{2\pi}{R}\frac{d}{dR}(RE(R)) = -\frac{\pi^2}{3R^2}\frac{dc_{\rm eff}}{d\log R}$$

Simple RG arguments imply:

$$c'_{\rm eff}(e^{\lambda} R) = c'_{\rm eff}(R), \qquad c'_{\rm eff} \equiv \frac{d c_{\rm eff}(R)}{d \log R}$$

TBA analysis of $c_{\rm eff}$:



Approximate analytic result for the massless case:

$$c_{\rm eff}(mR) \approx 1 + \frac{24}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q^{2n})^2}{(1+q^{-2n})} \Im m \left(e^{2ihns} \zeta_1^2(-inh) \right)$$

$$s = \log 2\pi/mR$$

$$\zeta_1(z) \equiv (1-2^{-z})\zeta(z), \qquad \zeta = {\rm Riemann zeta}$$

This is log-periodic behaviour.

- In the massive case, this behaviour only appears in the deep UV.

A stringy solution to the S-matrix

The quantum affine symmetry only fixed the S-matrix up to an overall scalar factor. We investigated another solution to the S-matrix which is simply the analytic extension of usual sine-Gordon soliton S-matrix to the coupling $b^2 = \frac{2}{1+ih/2}$.
Satisfies algebraic unitarity but not real analyticity, which may be pathological.

Properties of the spectrum:

The S-matrix has poles corresponding to resonances of mass:

$$m_n = 2M_s \cosh \frac{n\pi}{h}, \qquad n = 1, 2, 3, .., \infty$$

Russian Doll property:

$$m_{n+2} \approx e^{\lambda} m_n, \qquad n \gg h/\pi$$

Closing the bootstrap leads particles of higher spin j with a stringy mass formula:

$$m_n^2(j) = 4M_s^2 \left(j^2 + (2j-1)\sinh^2 \frac{n\pi}{h} \right)$$

For h small:

$$m^2(j) \approx \frac{1}{\alpha'}(j - \alpha(0))$$

where the Regge slope and intercept are

$$\frac{1}{\alpha'} = 2M_s^2 e^{2\pi/h}, \qquad \alpha(0) = 1/2$$

Russian Doll Superconductors

(with J.-M. Román and G. Sierra)

BCS hamiltonian (in the pairing approximation)

$$H = \sum_{j=1}^{N} \varepsilon_j \; b_j^\dagger b_j \;\; - \;\; \sum_{j,j'=1}^{N} \; V_{jj'} \, b_j^\dagger b_{j'}$$

$$V_{jj'} = \begin{cases} G + i\Theta & \text{if} \quad \varepsilon_j > \varepsilon_{j'} \\ G & \text{if} \quad \varepsilon_j = \varepsilon_{j'} \,, \\ G - i\Theta & \text{if} \quad \varepsilon_j < \varepsilon_{j'} \end{cases} \qquad \begin{matrix} G = g\delta \\ \Theta = h\delta \end{matrix} .$$

- $b_j = c_{j,-} c_{j,+}$, $b_j^\dagger = c_{j,+}^\dagger c_{j,-}^\dagger$
- The ε_j are equally spaced energy levels $-\omega < \varepsilon_j < \omega$ with level spacing 2δ.
- This kind of hamiltonian (with $\Theta = 0$) is used to describe very small superconducting grains.
- Simplest extension of BCS that breaks time reversal.

Gap equation:

$$\widetilde{\Delta}_j = \sum_{j' \neq j} V_{jj'} \frac{\widetilde{\Delta}_{j'}}{E_{j'}}, \qquad \widetilde{\Delta}_j \equiv \Delta_j e^{i\phi_j}$$

Continuum limit:

$$\widetilde{\Delta}(\varepsilon) = g \int_{-\omega}^{\omega} \frac{d\varepsilon'}{2} \frac{\widetilde{\Delta}(\varepsilon')}{E(\varepsilon')} + ih \left[\int_{-\omega}^{\varepsilon} - \int_{\varepsilon}^{\omega} \right] \frac{d\varepsilon'}{2} \frac{\widetilde{\Delta}(\varepsilon')}{E(\varepsilon')},$$

where $\widetilde{\Delta}(\varepsilon) = \Delta(\epsilon) e^{i\phi(\varepsilon)}$, $E = \sqrt{\varepsilon^2 + \Delta^2}$.

Solving for the gap yields an infinite number of solutions Δ_n. They can be parameterized as follows:

$$\Delta_n = \frac{\omega}{\sinh t_n}, \quad t_n = t_0 + \frac{n\pi}{h}, \quad n = 0, 1, 2, \ldots,$$

where t_0 is the principal solution to the equation

$$\tan(ht_0) = \frac{h}{g}, \qquad 0 < t_0 < \frac{\pi}{2h}$$

The gaps satisfy $\Delta_0 > \Delta_1 > \cdots$.

RG equations:

Next we derive RG equations for our model. Let g_N, h_N denote the couplings for the hamiltonian H_N with N energy levels. The idea behind the RG method is to derive an effective hamiltonian H_{N-1} depending on renormalized couplings g_{N-1}, h_{N-1} by integrating out the highest energy levels ε_N or ε_1.

In the large N limit one can define a variable $s = \log N_0/N$, where N_0 is the initial size of the system.

$$\frac{dg}{ds} = (g^2 + h^2), \qquad s \equiv \log \frac{N_0}{N}.$$

The solution to the above equation is

$$g(s) = h \tan \left[hs + \tan^{-1}\left(\frac{g_0}{h}\right)\right], \quad g_0 = g(N_0).$$

$$g(s+\lambda) = g(s) \iff g(e^{-\lambda}N) = g(N), \quad \lambda \equiv \frac{\pi}{h}$$

Role of condensates in the cyclic RG:

In each cycle the coupling g jumps from ∞ to $-\infty$.
Note:

$$\Delta_0(g=+\infty) = \infty$$
$$\Delta_{n+1}(g=+\infty) = \Delta_n(g=-\infty)$$

Thus the condensate $|\psi_{\rm BCS}^{(n+1)}\rangle$ of one RG cycle plays the same role as $|\psi_{\rm BCS}^{(n)}\rangle$ of the next cycle

Scaling properties of the gaps:

$$\Delta_{n+1} \approx e^{-\lambda} \Delta_n, \qquad n \text{ large}$$

Numerical Work: One-Cooper Pair problem:

The bound states of one-Cooper pair problem are widely known to be the precursors to the BCS condensates. For the one-pair problem we can easily work at large system sizes N. In this problem one looks for eigenstates of the form

$$|\psi\rangle = \sum_j \psi_j \, b_j^\dagger |0\rangle$$

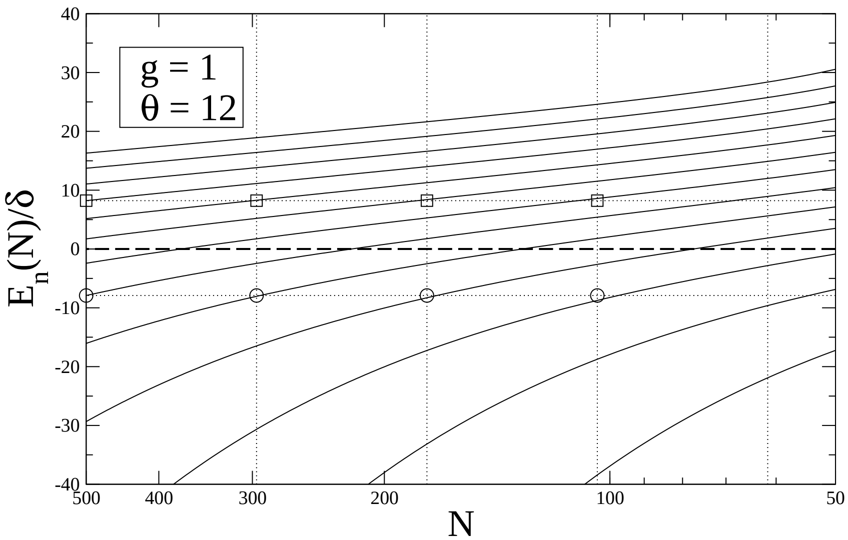
For this problem we find a very similar structure: an infinite number of bound states and a cyclic RG.

Bound state energies:

$$E_n = -\frac{\omega}{e^{t_n} - 1}, \qquad t_n = t_0 + \frac{2\pi n}{h}, \quad n \in \mathcal{Z},$$

where t_0 is the principal solution to the equation

$$\tan\left(\frac{1}{2} h t_0\right) = \frac{h}{g}, \qquad 0 < t_0 < \frac{\pi}{h}.$$



Exact eigenstates of one-Cooper pair Hamiltonian for N levels, from $N_0 = 500$ down to 50. We depict only the states nearest to zero. The vertical lines are at the values $N_n = e^{-n\lambda_1} N_0$. The dotted horizontal lines show the cyclicity of the spectrum.

Conclusions and open questions

- Cyclic RG trajectories are surprisingly commonplace
- Microscopic origins of the modified BCS theory?
- c-theorem?
- 3+1 d examples?
- Chaotic flows?