Duality walls in 5d gauge theories

Hee-Cheol Kim

Perimeter Institute

Based on
arXiv:1506.03871 with Davide Gaiotto (Perimeter Institute)

Introduction

A large class of BPS domain walls has been studied in 4d maximal SUSY gauge theories.

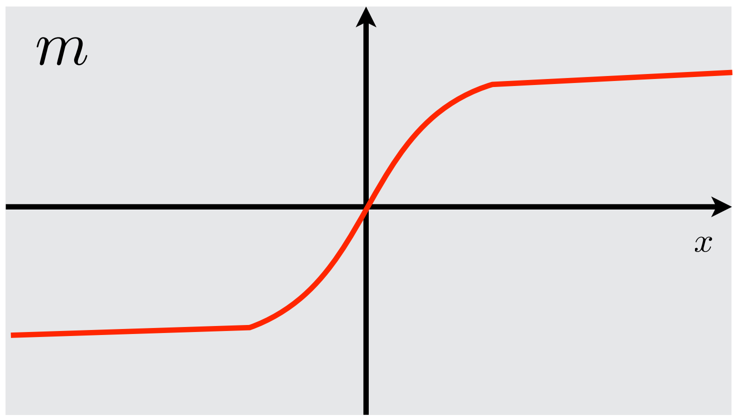
- AdS/CFT, Boundary conditions, S-duality, Branes, …

[Bak, Gutperle, Hirano 03], [Clark, Freedman, Karch, Schnabl 04], [Clark, Karch 05], [D'Hoker, Estes, Gutperle 07], [Bak, Gutperle, Hirano 07], [Gaiotto, Witten 08], …

We are interested in the BPS domain walls in 5d N=1 gauge theories.

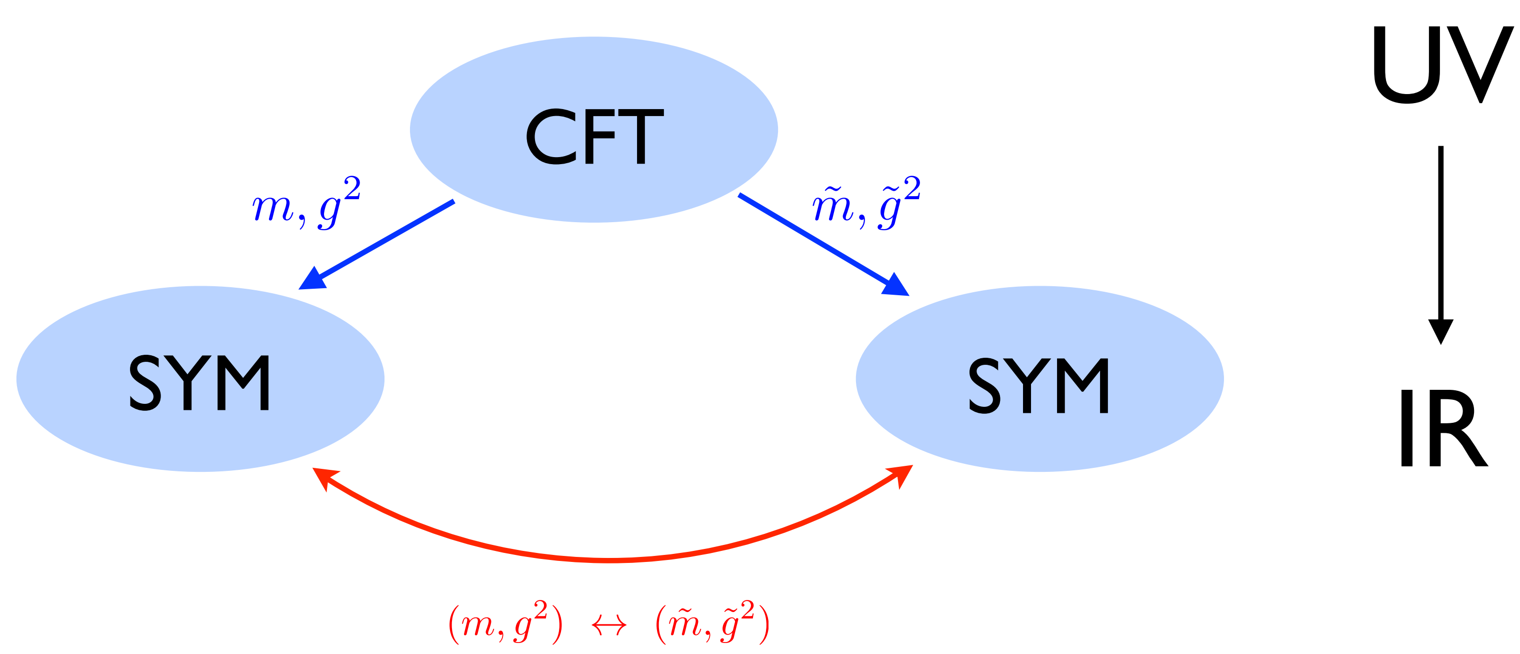
We focus on Janus-like domain walls (or interfaces)

- Coupling or mass parameter varies as a function of coordinate.

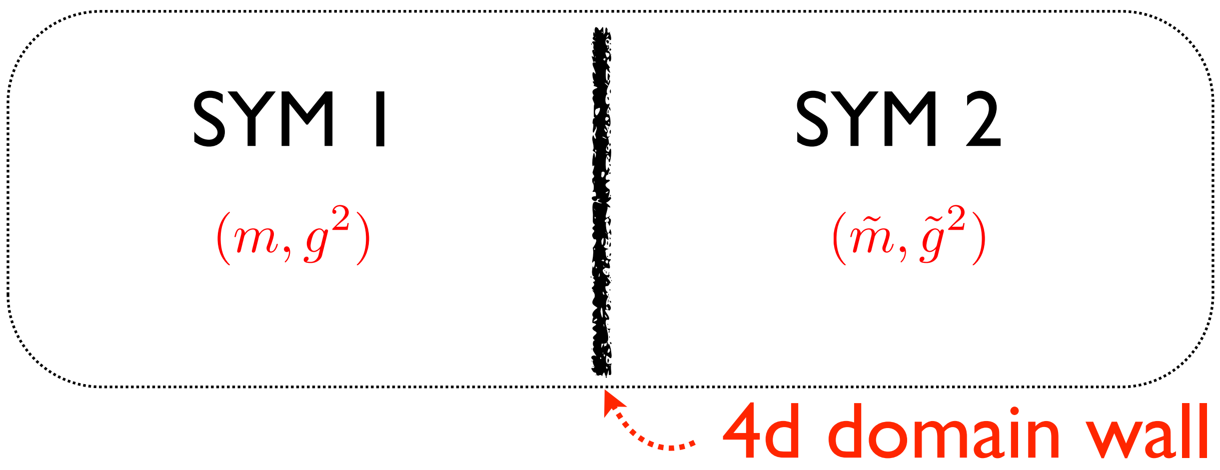


Introduction

We consider certain 5d SUSY theories which have CFT fixed points in UV and have relevant deformations to SYMs in IR.



- Duality domain wall :



Introduction

We will propose duality walls, which involve

- boundary conditions
- new 4d degrees of freedom
- 4d superpotentials
- test through explicit partition functions.

We expect to learn

- non-perturvative dynamics at UV fixed point from IR physics.
- close relation between 5d duality and 4d duality.
- new dualities.

Outline

1. Introduction.

2. Duality walls in SU(N) gauge theories.

3. Test with partition functions.

4. Duality walls in SU(N) with flavours.

5. Sp(N) $\leftrightarrow$ SU(N+1) duality and domain walls.

6. Conclusion

Basics of 5d SUSY gauge theories

$\mathcal{N}=1$ gauge theories in 5d

- Vector multiplet $(A_\mu, \phi; \lambda)$
- Hypermultiplet $(q^A; \psi)$
- Preserve 8 SUSY

There is a topological $U(1)_I$ associated to instanton number symmetry :

$$J_I = *\mathrm{Tr} F \wedge F$$

Thus, full symmetry is

- $SO(5)$ Lorentz symmetry times $SU(2)_R$ R-symmetry.
- G : gauge symmetry.
- $G_F \times U(1)_I$: flavour symmetry.

Basics of 5d SUSY gauge theories

5d gauge theories are non-renormalizable. However, for certain SUSY theories, we expect non-trivial UV fixed points exist.

- QFT analysis
- Branes and string duality, (p,q) five-brane web
- M-theory on CY3

[Seiberg 96], [Morrison, Seiberg 96], [Douglas, Katz,Vafa 96], [Intriligator, Morrison, Seiberg 97], [Aharony, Hanany 97], [Aharony, Hanany, Kol 97], [DeWolfe, Hanany, Iqbal, Katz 99], …

Effective gauge coupling is 1-loop exact : $\frac{1}{g_{eff}^2} = \frac{1}{g_0^2} + c|\phi|$

Note that, when $c > 0$, we can remove a scale by $g_0 \to \infty$ and interacting CFT fixed point can be attained at $\phi \to 0$.

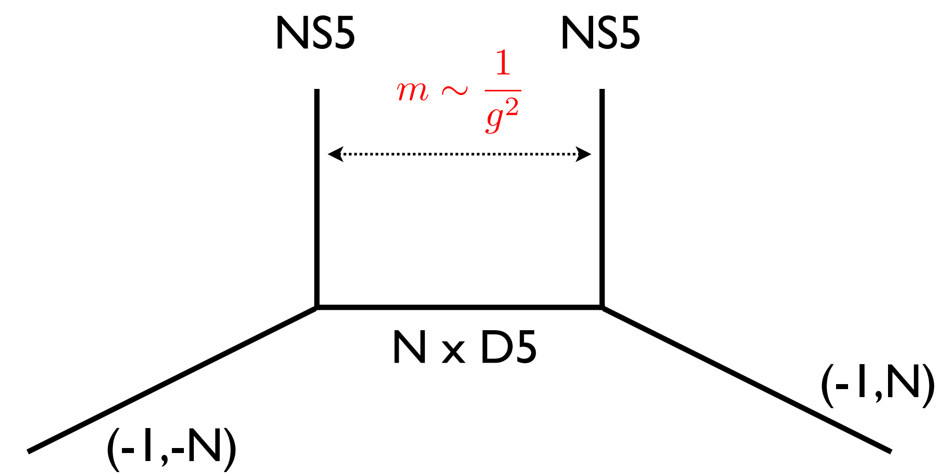
Some UV fixed points enjoy global symmetry enhancement.

(Ex : $SU(2)$, $N_f = 5, 6, 7$ have enhanced E_6, E_7, E_8 symmetries)

Duality walls in SU(N) theories

$\mathcal{N}=1\ SU(N)_N$ gauge theory

Five-brane web construction
(at the origin of Coulomb branch)



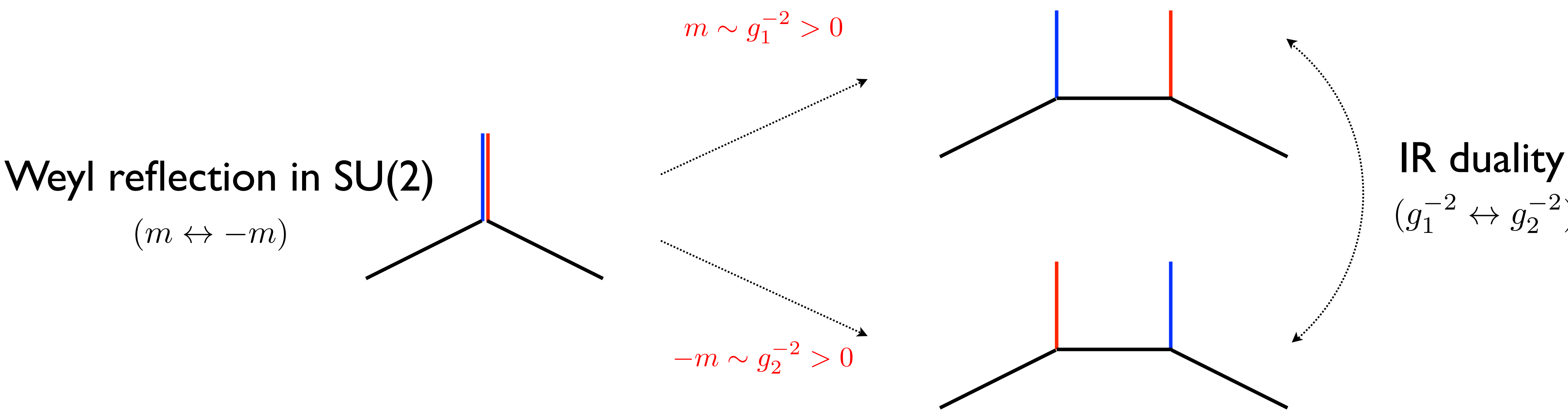
- $SU(N)$ gauge theory on N D5-branes with classical Chern-Simons coupling $\kappa = N$.

$$L = \frac{1}{g^2} F \wedge *F + \frac{\kappa}{24\pi^2} A \wedge F \wedge F + \cdots$$

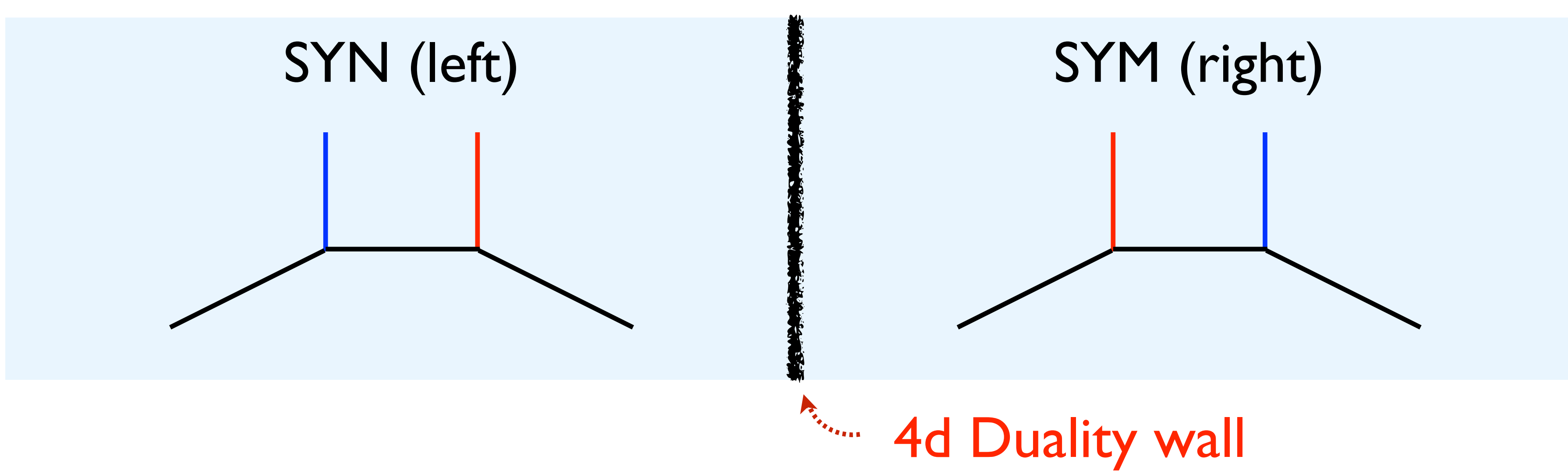
- Instanton symmetry $U(1)_I$ is enhanced to $SU(2)$ at UV fixed point, which comes from two parallel NS5-branes.

5d Duality in IR gauge theories

Mass deformation of UV CFT leads to different IR gauge theories.



We propose a 1/2-BPS domain wall connecting IR dual gauge theories.



Boundary condition and boundary d.o.f

Neumann boundary condition at the interface :

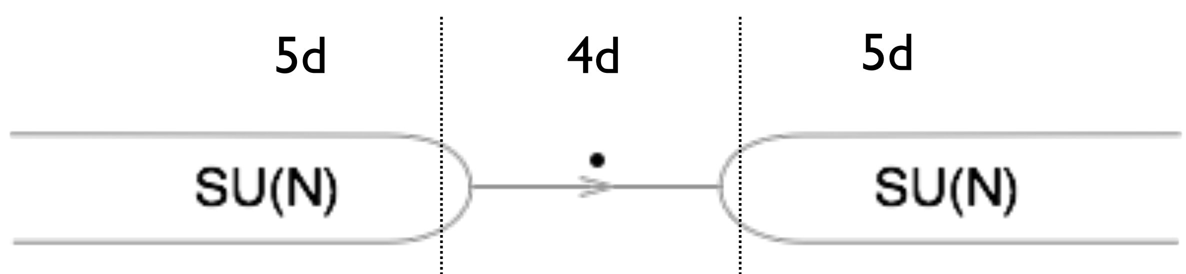
- $F_{5i}|_\partial = 0$
- Half-BPS
- Gauge symmetry survives at the boundary

We then couple new 4d degrees of freedom

- 4d $\mathcal{N}=1$ matter content :

	$SU(N)_l$	$SU(N)_r$	$U(1)_R$	$U(1)_B$
q	N	$\bar{N}$	0	$1/N$
b	1	1	2	-1

- Superpotential : $W = b\ \mathrm{det} q$



Consistency requires that boundary gauge anomaly must be cancelled.

Boundary condition and boundary d.o.f

- 4d $\mathcal{N}=1$ matter content :

	$SU(N)_l$	$SU(N)_r$	$U(1)_R$	$U(1)_B$
q	N	$\bar{N}$	0	$1/N$
b	1	1	2	-1

- Strong constraints by anomaly cancellation
  1. Cubic anomaly (of unit N) from 4d matters is cancelled by bulk classical Chern-Simons term at $\kappa = N$.
  2. Boundary $U(1)_R \subset SU(2)_R$ is fixed by mixed 't Hooft anomaly.
  3. Anomaly-free $U(1)_\lambda \subset U(1)_B \times U(1)_{I_l} \times U(1)_{I_r}$ glues instanton symmetries in both sides.

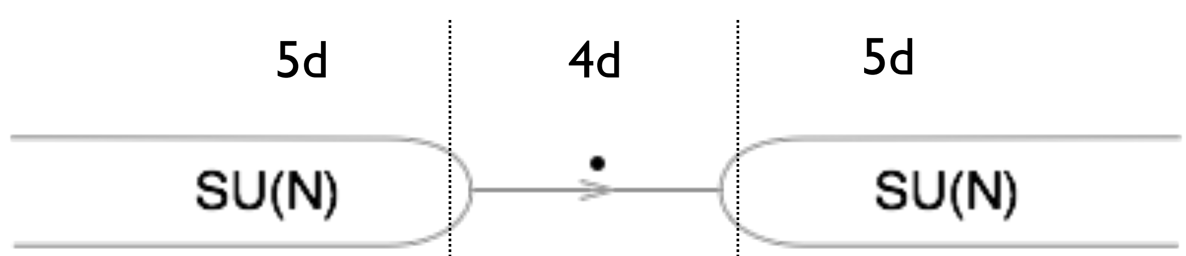
	$U(1)_\lambda$	$U(1)_B$	$U(1)_{I_l}$	$U(1)_{I_r}$
q	$1/N$	$1/N$	0	0
I_l	1	0	1	0
I_r	-1	0	0	1

Duality wall

Anomaly-free $U(1)_\lambda$ glues together two instanton symmetries on two sides of the wall with opposite signs.

	$U(1)_\lambda$	$U(1)_B$	$U(1)_{I_l}$	$U(1)_{I_r}$
q	$1/N$	$1/N$	0	0
I_l	1	0	1	0
I_r	-1	0	0	1

Therefore, duality wall exchanges gauge couplings

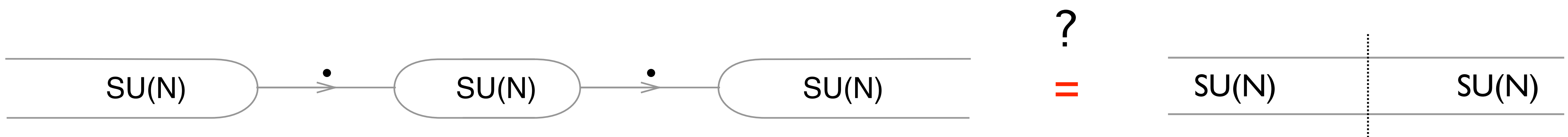


gauge coupling : $\frac{1}{g^2} \sim m$ $\qquad$ $\frac{1}{\tilde{g}^2} \sim -m$

* Duality wall implements $\mathbb{Z}_2$ action in SU(2) global symmetry of UV CFT.

Composition of duality walls

Consistency check :



- 4d theory is now $SU(N)$ SQCD with $N_f = N$ and
$$W = b\,\det q + \tilde{b}\,\det\tilde{q}$$

- "Seiberg dual" theory consists of a meson $M = \tilde{q}q$ and baryons $B = \det q,\ \tilde{B} = \det\tilde{q}$ with a constraint $\det M - B\tilde{B} = \Lambda^{2N}$ and superpotential : $W = b\,B + \tilde{b}\,\tilde{B}$.

➡ Trivial interface

SUSY indices with Duality walls

SUSY index with duality wall

We now see a more non-trivial check with supersymmetric indices in the presence of the duality wall.

- Superconformal index (SCI)

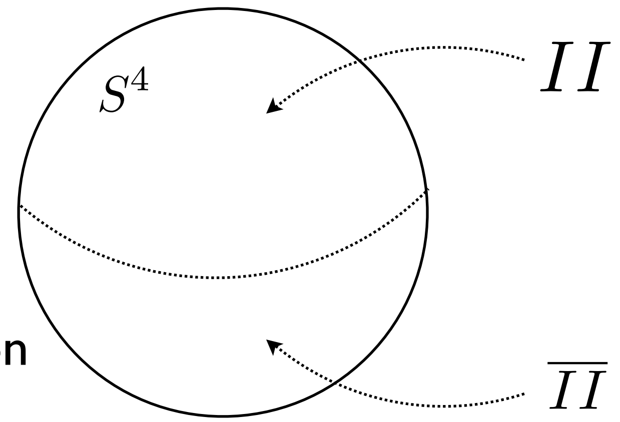
$$I(w_a, \mathfrak{q}; p, q) = \mathrm{Tr}(-1)^F p^{j_1+R} q^{j_2+R} \prod_a w_a^{F_a} \, \mathfrak{q}^k$$

  - j_1, j_2, R are Cartan generators of $SO(2,5) \times SU(2)_R$.
  - F_a are Cartans of flavour symm. and k is instanton number.
  - SCI is equivalent to twisted partition function on $S^1 \times S^4$.

[S.-S Kim, H.-C Kim, K. Lee 12], [Terashima 12]

- SCI factorizes into two "hemisphere" indices by localization.

$$I(w_a, \mathfrak{q}; p, q) = \langle II | II \rangle = \oint d\mu_z \overline{II(z, w_a, \mathfrak{q}; p, q)} II(z, w_a, \mathfrak{q}; p, q)$$

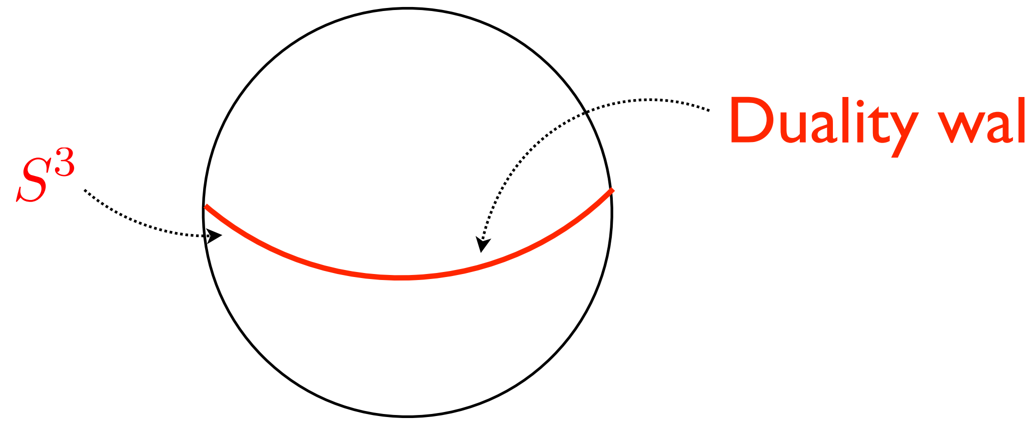


$II = Z_{\mathrm{pert}} \cdot Z_{\mathrm{inst}}$: Hemisphere index
= Partition function on $S^1 \times \mathbb{R}^4$ with Omega deformation

z : gauge holonomy

SUSY index with duality wall

We can insert a duality wall at the equator (with 1/2-SUSY)



The superconformal index with the interface simply becomes

$$I = \langle II(\mathfrak{q}^{-1})|I^{4d}(\mathfrak{q})|II(\mathfrak{q})\rangle = \oint d\mu_z d\mu_{z'} \overline{II(z,\mathfrak{q}^{-1};p,q)} I^{4d}(z,z',\mathfrak{q};p,q) II(z',\mathfrak{q};p,q)$$

where $I^{4d}(z,z',\mathfrak{q};p,q)$ is the contribution from 4d d.o.f at the interface (which also depends on the boundary condition).

Duality wall action on hemisphere index

Contribution from 4d d.o.f at interface

	$SU(N)_l$	$SU(N)_r$	$U(1)_R$	$U(1)_B$
q	N	$\bar{N}$	0	$1/N$
b	1	1	2	-1

$$\Rightarrow \quad I^{4d} = \frac{\prod_{i,j=1}^{N} \Gamma(\lambda^{1/N} z_i / z'_j)}{\Gamma(\lambda)}$$

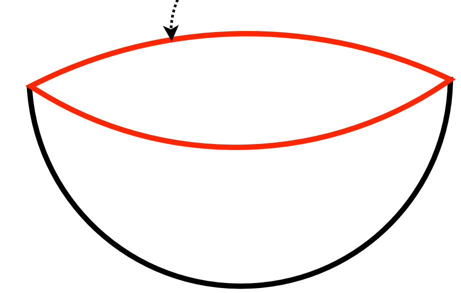
$z_i : SU(N)_l$
$z'_i : SU(N)_r$
$\lambda : U(1)_\lambda$

($\Gamma(x)$: Elliptic gamma function)

We can couple this 4d index to hemisphere index :

$$\hat{D}II^N(z,\lambda) \equiv \oint \prod_{i=1}^{N-1} \frac{dz'_i}{2\pi i z'_i} \frac{I^{4d}(z,z',\lambda)}{\prod_{i,j}^{N} \Gamma(z'_i/z'_j)} II^N(z'_i,\lambda)$$

4d SU(N) vectormultiplet



- Here, we identify $U(1)_\lambda$ fugacity with instanton number fugacity (or gauge coupling) as $\mathfrak{q} = \lambda$.

Duality wall action on hemisphere index

Contribution from 4d d.o.f at interface

	$SU(N)_l$	$SU(N)_r$	$U(1)_R$	$U(1)_B$
q	N	$\bar{N}$	0	$1/N$
b	1	1	2	-1

$$\Rightarrow \quad I^{4d} = \frac{\prod_{i,j=1}^{N} \Gamma(\lambda^{1/N} z_i / z'_j)}{\Gamma(\lambda)}$$

$z_i : SU(N)_l$
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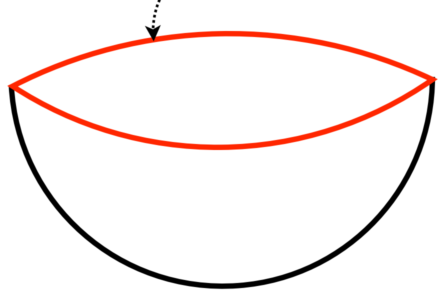
($\Gamma(x)$: Elliptic gamma function)

We can couple this 4d index to hemisphere index :

$$\hat{D} II^N(z, \lambda) \equiv \oint \prod_{i=1}^{N-1} \frac{dz'_i}{2\pi i z'_i} \frac{I^{4d}(z, z', \lambda)}{\prod_{i,j}^{N} \Gamma(z'_i/z'_j)} II^N(z'_i, \lambda)$$

4d SU(N) vectormultiplet



Duality wall is conjectured to exchange the gauge coupling, therefore, we claim that

$$\hat{D} II^N(z_i, \lambda) = II^N(z_i, \lambda^{-1})$$

Duality wall action on hemisphere index

- Duality wall : $\hat{D} II^N(z_i,\lambda) = II^N(z_i,\lambda^{-1})$ [Gaiotto, H.-C Kim 15]

- The hemisphere index is actually given by a series expansion in instanton number. Thus, this is a very surprising claim since the index $II^N(z,\lambda)$ is expanded by $\lambda^{k\geq 0}$, while the dual index $\hat{D}II^N(z,\lambda)$ is expanded by $(\lambda^{-1})^{k\geq 0}$.

- Can be checked in $x \equiv (pq)^{1/2}$ expansion.
  - Numerical checks for $N = 2,3,4$ at least up to x^4 order.

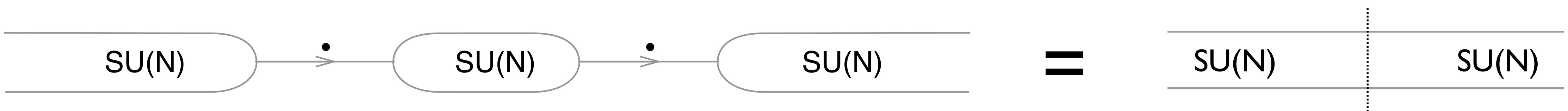
- More surprisingly, assuming $II = Z_{\text{pert}} \cdot Z_{\text{inst}} = Z_{\text{pert}} \cdot (1+\mathcal{O}(x))$, the integral equation

$$\oint \prod_{i=1}^{N-1} \frac{dz_i'}{2\pi i z_i'} \frac{I^{4d}(z,z',\lambda)}{\prod_{i,j}^N \Gamma(z_i'/z_j')} II^N(z_i',\lambda) = II^N(z,\lambda^{-1})$$

uniquely determines the instanton partition function Z_{inst} in x expansion!!

Duality wall action on hemisphere index

- Analytic proof of $\hat{D}^2 = I$



- There is an integral formula (elliptic Fourier transform)

[Spiridonov, Warnaar 04]

$$\oint d\mu_{z'} \frac{\prod_{i,j}^N \Gamma(\lambda^{1/N} z_i/z'_j)}{\Gamma(\lambda)\prod_{i,j}^N \Gamma(z'_i/z'_j)} \oint d\mu_{z''} \frac{\prod_{i,j}^N \Gamma(\lambda^{-1/N} z'_i/z''_j)}{\Gamma(\lambda^{-1})\prod_{i,j}^N \Gamma(z''_i/z''_j)} f(z'') \sim f(z)$$

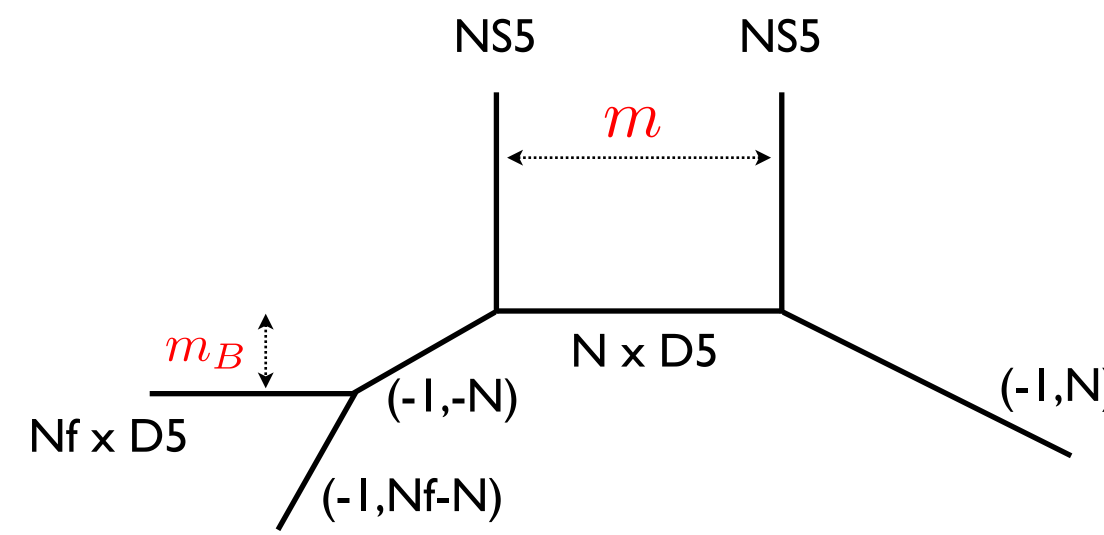
(note: $I^{4d} = \dfrac{\prod_{i,j=1}^N \Gamma(\lambda^{1/N} z_i/z'_j)}{\Gamma(\lambda)}$)

- This proves $\hat{D}\hat{D}II(z,\lambda) = II(z,\lambda)$.

Duality walls with flavours

SU(N) gauge theory with flavours

Five-brane web construction



- $SU(N)$ gauge theory with CS coupling at $\kappa = N - N_f/2$.
- IR gauge coupling is identified as $g^{-2} \sim m + \frac{N_f}{2} m_B$, where m_B is the mass parameter for the overall $U(1)_f \subset U(N_f)$ flavor symmetry.
- UV fixed point has an enhanced $SU(2)$ global symmetry and m is the corresponding mass deformation.

We propose a duality interface which exchanges $m \leftrightarrow -m$.

Boundary conditions and domain wall

Boundary conditions :

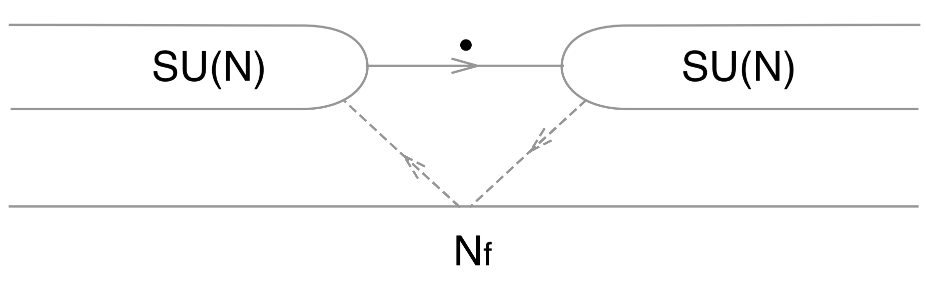
- Vector multiplet : Neumann b.c. $F_{5i}|_\partial = 0$
- Hypermultiplet $\Phi = (X, Y)$: $X|_\partial = 0$, $\partial_5 Y|_\partial = 0$

We couple this to the same 4d $\mathcal{N} = 1$ system

- matter content :

	$SU(N)_l$	$SU(N)_r$	$U(1)_R$	$U(1)_B$
q	N	$\bar{N}$	0	$1/N$
b	1	1	2	-1

- Superpotential : $W = b\,\mathrm{det}q + Y q X'$



Duality wall with flavours

- 4d $\mathcal{N}=1$ matters :

	$SU(N)_l$	$SU(N)_r$	$U(1)_R$	$U(1)_B$
q	N	$\bar{N}$	0	$1/N$
b	1	1	2	-1

- Superpotential : $W = b\,\mathrm{det}q + YqX'$

Cubic anomaly $N - N_f/2$ at the interface is cancelled by bulk CS-term.

We find anomaly free U(1) global symmetries as (in terms of fugacities)

	q	I_l	I_r	X	X'
fugacity	$\lambda^{1/N}$	$\lambda w^{-N_f/2}$	$\lambda^{-1}(w')^{-N_f/2}$	w	w'

(with U(1) fugacities $w = \lambda^{1/N} w'$ and $$e^{-\frac{4\pi^2}{g^2}} = \lambda w^{-N_f/2}\,, \quad e^{-\frac{4\pi^2}{(g')^2}} = \lambda^{-1}(w')^{-N_f/2}\,, \quad e^{m_B} = w\,,\ e^{m'_B} = w'$$)

Duality wall action on hemisphere index

Hemisphere index of the boundary condition $F_{ij}|_\partial = 0\ ,\ X|_\partial = 0\ ,\ Y|_\partial \neq 0$

$$II^{N,N_f}(z_i, w_a, \mathfrak{q}; p, q) = \frac{(pq;p,q)_\infty^{N-1} \prod_{i\neq j}^N (pqz_i/z_j;p,q)_\infty}{\prod_{i=1}^N \prod_{a=1}^{N_f} (\sqrt{pq} z_i/w_a;p,q)_\infty} Z_{\text{inst}}^{N,N_f}(z_i, w_a, \mathfrak{q}; p, q)$$

from hypermultiplet

$(x;p,q)_\infty$: q-Pochhammer symbol

Duality wall action on the hemisphere index

$$\hat{D} II^{N,N_f}(z, w, \lambda) \equiv \oint \prod_{i=1}^{N-1} \frac{dz'_i}{2\pi i z'_i} \frac{I^{4d}(z, z', \lambda)}{\prod_{i,j}^N \Gamma(z'_i/z'_j)} II^{N,N_f}(z'_i, w, \lambda)$$

We claim that $\hat{D} II^{N,N_f}(z_i, w, \lambda) = II^{N,N_f}(z_i, w', \lambda^{-1})$ (with $w = \lambda^{1/N} w'$)

- $\hat{D}\ :\ \lambda \to \lambda^{-1}$
- Numerical checks for several small N, N_f

Again, this integral relation of duality wall uniquely determines the full instanton partition function with fund. hypers in $x = (pq)^{1/2}$ expansion.

Symmetry enhancement and 4d duality

- Example : $SU(2)$ gauge theory with $N_f = 2$ flavours which has symmetry enhancement $SO(4) \times U(1)_I \rightarrow SU(2) \times SU(3)$ at the UV fixed point.

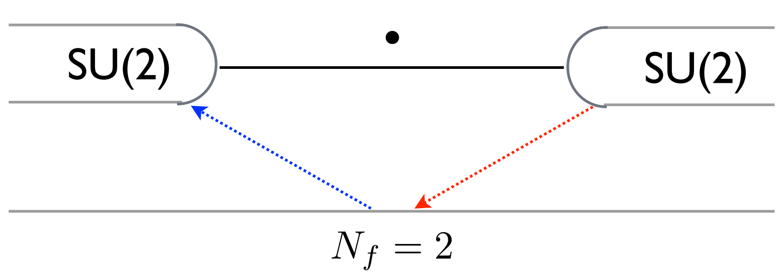
  [Seiberg 96]

- Enhanced $SU(3)$ involves S_3 permutation group which exchanges $U(1)_B \times U(1)_I \subset SU(3)$ charges.

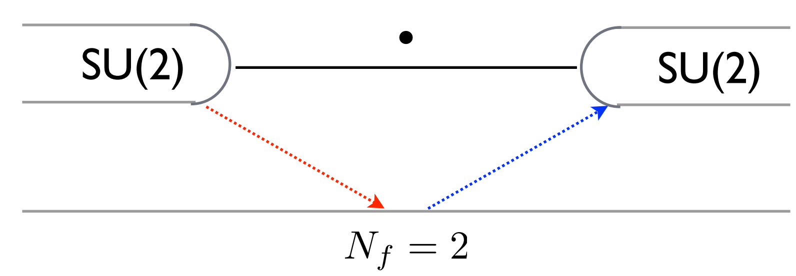
- Combinations of duality walls can realize full S_3 permutation group.

Let's define two different duality walls with two different b.c. for the hypermultiplets $\Phi_{a=1,2} = (Q_a, \tilde{Q}_a)$

$\hat{D}_1$:



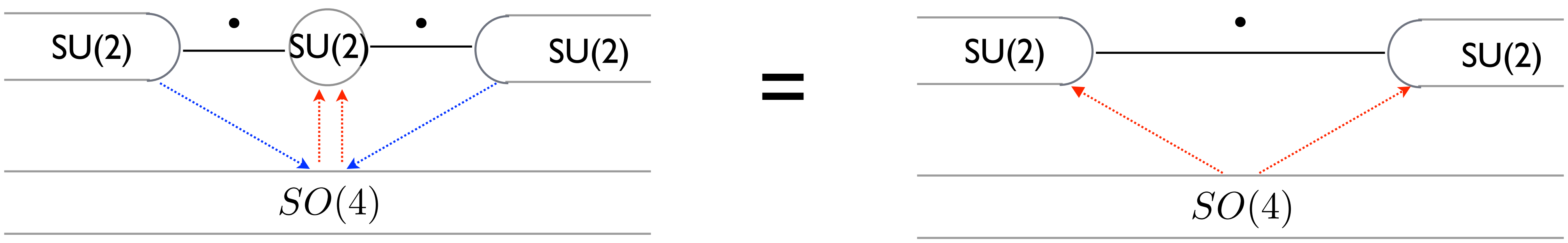
$\hat{D}_2$:



$(Q_a : \uparrow \ , \tilde{Q}_a : \downarrow \ )$

Symmetry enhancement and 4d duality

- Concatenation of two domain walls and 4d Seiberg duality shows



$\hat{D}_1 \hat{D}_2$ = $\hat{D}_2 \hat{D}_3$ (or $\hat{D}_3 \hat{D}_1$)

(where $\hat{D}_3 \; : \; (Q_a, \tilde{Q}_a) \leftrightarrow (\tilde{Q}_a, Q_a)$)

- Therefore, duality wall actions (with help of 4d Seiberg duality) implement Weyl permutations $\hat{D}_1$, $\hat{D}_2$, $\hat{D}_3 \subset S_3$ in the $SU(3)$ at the UV fixed point.

Sp(N) and SU(N+1) duality

Duality between Sp(N) and SU(N+1) theories

Duality between 1 . $Sp(N)$ gauge theory with N_f fundamental hypers.

2 . $SU(N+1)$ gauge theory with N_f fundamental hypers at CS-level $\kappa = N+3-N_f/2$. $(N_f < 2N+6)$

[Hayashi, S.-S Kim, K. Lee, Taki, Yagi 15], [Gaiotto, H.-C Kim 15]

Same dimension of Coulomb branch : $\dim \mathcal{M}_{\text{Coulomb}} = N$

Same global symmetry at UV fixed point : $SO(2N_f) \times U(1)_I$

- $SU(N)$ gauge theory has enhanced global symmetry as

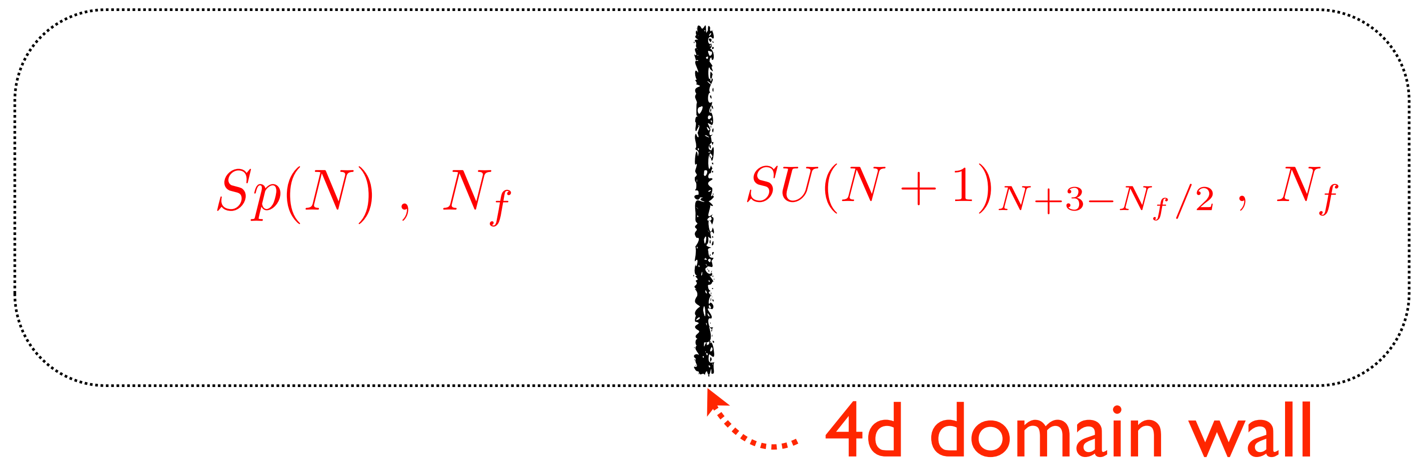
N_f	$SU(N)_{\pm(N+1-N_f/2)}$	N_f	$SU(N)_{\pm(N+2-N_f/2)}$
$\leq 2N$	$SU(N_f+1)\times U(1)$	$\leq 2N+1$	$SO(2N_f)\times U(1)$
$2N+1$	$SU(N_f+1)\times SU(2)$	$2N+2$	$SO(2N_f)\times SU(2)$
$2N+2$	$SU(N_f+2)$	$2N+3$	$SO(2N_f+2)$

- Can be seen from 1-instanton analysis [Yonekura 15], [Gaiotto, H.-C Kim 15]

- Or from (p,q) 5-Branes [Bergman, Zafrir 14,15], [Hayashi, S.-S Kim, K. Lee, Taki, Yagi 15]

Boundary condition and domain wall

We propose a duality wall :



We use a similar boundary conditions $F_{5i}|_\partial = 0\ ,\ \ X|_\partial = 0\ ,\ \ Y|_\partial \neq 0$

And couple it to 4d degrees of freedom at the interface

- 4d $\mathcal{N}=1$ matter content

	$Sp(N)$	$SU(N+1)$	$U(1)_R$	$U(1)_\lambda$
q	N	$N+1$	0	$1/2$
M	1	$N(N+1)/2$	2	-1

- Superpotential $W = \mathrm{Tr}\ qMq^T w + XqX'$

w : symplectic form of $Sp(N)$
X : chiral half of hypermultiplet in $SU(N+1)$
X' : chiral half of hypermultiplet in $Sp(N)$

- When N=1, it reduces to duality interface in previous SU(2) theory

We propose that this is the duality wall that interpolates Sp(N) and SU(N+1) gauge theories.

Duality wall action on hemisphere index

Duality action on the hemisphere index of $Sp(N)$ gauge theory.

$$\hat{D}II^{N_f}_{Sp(N)} = \oint d\mu_{z_i} \frac{I^{4d}(z,z',\lambda)}{\prod_{i>j}^{N}\Gamma(z_i^{\pm}z_j^{\pm})\prod_{i=1}^{N}\Gamma(z_i^{\pm 2})} II^{N_f}_{Sp(N)}(z_i,\mathfrak{q}_{Sp},w_a)$$

Contribution from 4d d.o.f : $I^{4d}(z,z',\lambda) = \frac{\prod_{i=1}^{N+1}\prod_{j=1}^{N}\Gamma(\sqrt{\lambda}z_i'z_j^{\pm})}{\prod_{i>j}^{N+1}\Gamma(\lambda z_i'z_j')}$

We claim that $\hat{D}II^{N_f}_{Sp(N)}(z_i,w_a,\mathfrak{q}_{Sp};p,q) = II^{N_f}_{SU(N+1)}(z_i',w_a',\mathfrak{q}_{SU};p,q)$

(with U(1) fugacities $w_a = \lambda^{1/2}w_a'\,,\ \mathfrak{q}_{Sp} = \lambda^{(N+1)/2}\prod_{a=1}^{N_f}(w_a)^{-1/2}\,,\ \mathfrak{q}_{SU} = \lambda^{-1}\prod_{a=1}^{N_f}(w_a')^{-1/2}$).

Checked this relation for $N=2$ at least up to x^5 order.

This integral equation can generate instanton partition functions of $SU(N)_{N+2-N_f/2}$ gauge theories, which we couldn't compute using standard ADHM analysis.

(A,C)-type Elliptic integral formula

Concatenation of two duality walls must be a trivial interface : $\hat{D}\hat{D} = I$

There are (A,C) and (C,A)-type inversion formulas [Spiridonov, Warnaar 04]

$$\oint d\mu_{z'} \Delta^{(A)}(z', x, \lambda) \oint d\mu_z \Delta^{(C)}(z, z', \lambda) f(z) = f(x) \ ,$$

$$\oint d\mu_z \Delta^{(C)}(z, x, \lambda) \oint d\mu_{z'} \Delta^{(A)}(z', z, \lambda) f(z') = f(x)$$

$\Delta^{(A)}$ and $\Delta^{(C)}$ are the index of 4d d.o.f :

$$\Delta^{(A)}(z', z, \lambda) \sim \frac{I^{4d}(z, 1/z', 1/\lambda)}{\prod_{i \neq j}^{N+1} \Gamma(z'_i/z'_j)} \ ,$$

$$\Delta^{(C)}(z, z', \lambda) \sim \frac{I^{4d}(z, z', \lambda)}{\prod_{i>j}^{N} \Gamma(z_i^{\pm} z_j^{\pm}) \prod_{i=1}^{N} \Gamma(z_i^{\pm 2})}$$

- This proves $\hat{D}\hat{D} = I$.
- Duality and domain wall action thus provides a physical interpretation of these elliptic integral identities.

Conclusion

Conclusion

- We have proposed duality domain wall connecting two dual SU(N) gauge theories and carried out various tests.

- Enhanced global symmetry in the UV CFT can be seen even in IR gauge theory through the duality wall action and 4d duality at the interface.

- New duality between Sp(N) and SU(N+1) gauge theories and the duality wall between them have been proposed.

Future directions :

- Study on boundary conditions in 5d gauge theories.
- Other duality walls or other type of domain walls.
- Defects in the presence of domain walls.