Gravity Duals of Lifshitz-like Fixed Points

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arXiv:0808.1725

This talk will be rather strange. It will be divided into three sections:

I. Introduction -- effective field theory of conductors, and the trouble with high T_C (50% of talk)

II. The "party line" today, and a small problem it suggests for us to attack (10% of talk)

III. A summary of our progress on this problem, so far: gravity duals of critical theories with dynamical scaling (40% of talk)

There are several interesting hierarchy problems that have preyed on the minds of theorists for the past few decades.

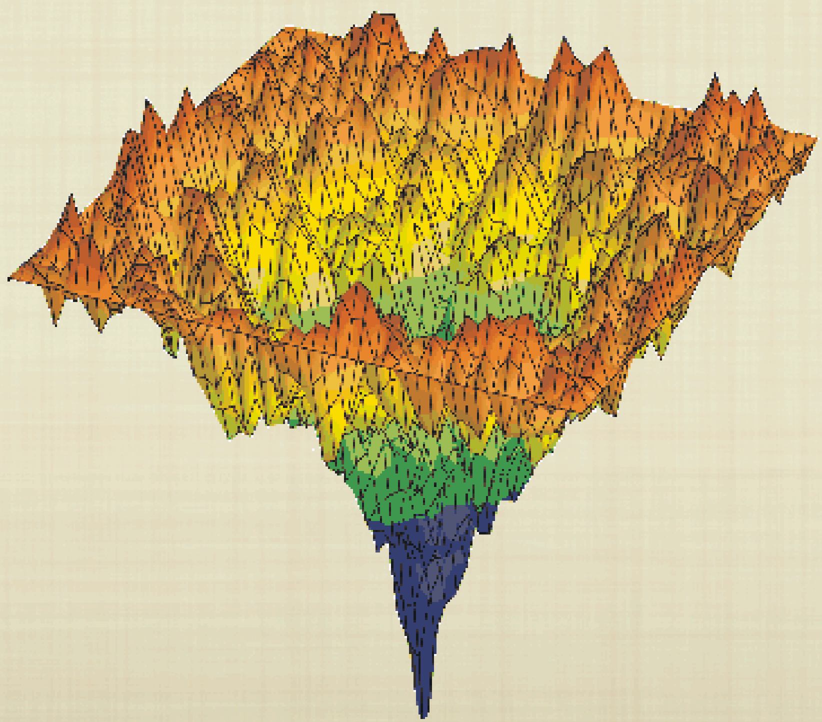
Two you know well; the standard model allows two relevant operators

$$\int d^4x\ \sqrt{-g}\Lambda$$

$$\int d^4x\sqrt{-g}\ m_H^2 H^\dagger H$$

natural question: what prevents these parameters from moving to the UV cut-off of our theory?

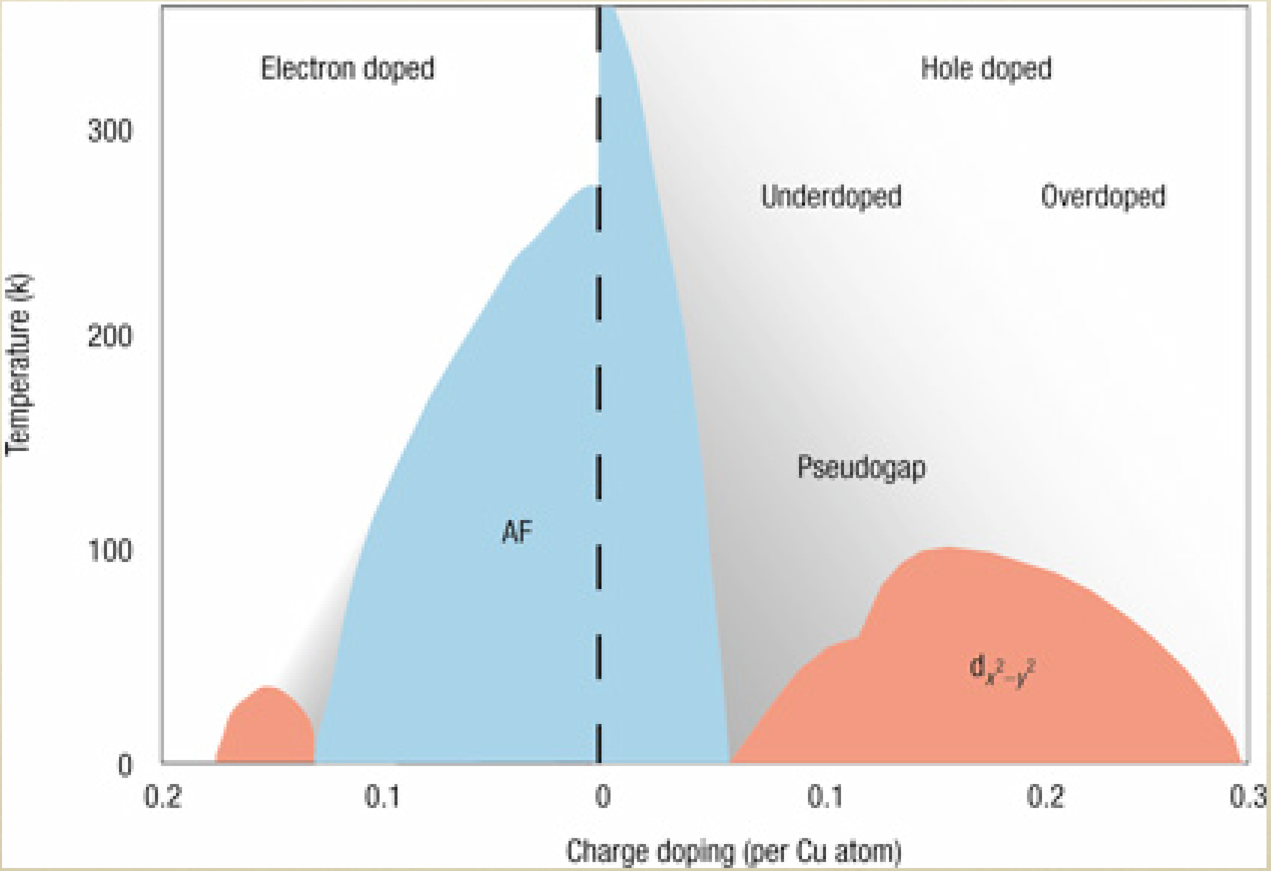
For the cosmological term, the only plausible answer i've heard so far is environmental selection in a landscape.



For the higgs mass term, there are many beautiful proposals that involve dynamical stabilization (technicolor, supersymmetry). LHC will test these.

There have also been serious proposals that the higgs, too, is simply fine tuned. The accidental symmetries of the standard model protect against various disasters that happen in almost all extensions, and require slightly epicyclic model building.

Here, I want to discuss a (very preliminary) line of investigation into a third hierarchy problem that has proven equally vexing. Unlike the other two problems, in this case **we know already that the solution does not involve fine tuning; there is interesting dynamics at play.**



I will now repeat a (well known) argument that we should think of the existence of the high temperature superconductors as a hierarchy problem. I heavily rely on Polchinski's discussion in the following.

Benfatto and Gallavotti; Shankar; Polchinski

Landau's Theory of Fermi Liquids

Let's consider the effective field theory governing excitations of a conductor. The relevant energy scale (say a typical width of a conduction electron band) is

$$E_0 \sim 1 - 10 \text{ eV}$$

In a conductor, current is excited by any tiny electric field, so there must be gapless charged excitations. We can try to write down an effective theory valid at scales

$$E << E_0$$

At the high scale we encounter the messy theory of electrons with their Coulomb interactions. Lets try to write down a simpler theory of low energy modes, just using symmetries and guesswork.

Guess: light fields are spin 1/2 fermions. (Lets call them "electrons").

It is important to emphasize that while this is justified in very dilute or weakly interacting systems, in a strongly interacting system the truth could be very different.

Lets begin by studying the free action:

$$S = \int dt \; d^3p \; [\; i\psi^\dagger_\sigma(p)\partial_t\psi_\sigma(p)$$

$$-(\epsilon(p) - \epsilon_F) \; \psi^\dagger_\sigma(p)\psi_\sigma(p) \;]$$

ϵ_F is the Fermi energy. In the ground state, all states with $\epsilon < \epsilon_F$ are filled. Low-lying excitations are obtained by adding an electron just above the Fermi surface, or removing one (leaving a hole) just below.

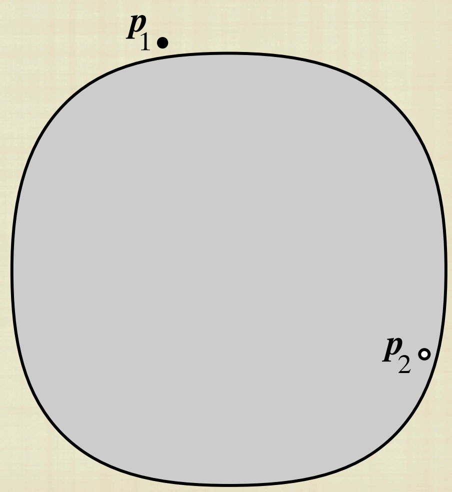


Figure 1: Fermi sea (shaded) with two low-lying excitations, an electron at $\mathbf{p}_1$ and a hole at $\mathbf{p}_2$.

Now, let us ask how the fields behave if we scale all energies down by a factor of

$$s < 1.$$

In a relativistic theory, we would scale momenta down with a factor of s. Here, on the other hand, we should scale the momenta toward the Fermi surface.

So if

$$\mathbf{p} \ = \ \mathbf{k} + \mathbf{l}$$

WITH K ON THE FERMI SURFACE AND L ORTHOGONAL TO IT, THEN ONE SHOULD SCALE

$$E \to sE,\ \mathbf{k} \to \mathbf{k},\ \mathbf{l} \to s\mathbf{l}$$

EXPAND THE SINGLE-PARTICLE ENERGY

$$\epsilon(\mathbf{p}) - \epsilon_F = l v_F(\mathbf{k}) + \mathcal{O}(l^2)$$

THEN UNDER THE COMBINED SCALING

$$dt \to s^{-1} dt,\ d\mathbf{k} \to d\mathbf{k},\ d\mathbf{l} \to s d\mathbf{l},\ \partial_t \to s\partial_t, l \to sl$$

EACH TERM IN OUR ACTION

$$S = \int dt\, d^3p \, [\, i\psi^\dagger_\sigma(p)\partial_t\psi_\sigma(p)$$

$$-(\epsilon(p) - \epsilon_F)\ \psi^\dagger_\sigma(p)\psi_\sigma(p)\ ]$$

SCALES LIKE

$$s\psi^\dagger\psi$$

WE CONCLUDE THAT WE SHOULD SCALE

$$\psi \to s^{-1/2}\psi$$

Now, lets ask if this theory is natural or not. Are there relevant operators we haven't included in our action?

The symmetries we get to impose on allowed operators include:

1. Electron number

2. Translation invariance [really discrete lattice symmetries; this leads to band theory but for now we will ignore this complication].

3. Spin SU(2)

Lets start with terms quadratic in the fields:

$$\int dt\ d^2\mathbf{k}\ d\mathbf{l}\ \mu(\mathbf{k})\psi^\dagger_\sigma(\mathbf{p})\psi_\sigma(\mathbf{p})$$

This scales as 1/s. Have we already lost? It looks relevant!

But the answer is no. We had an unknown function $\epsilon(\mathbf{p})$ in our action, and this can be absorbed into the definition of that function. This means that the existence of a fermi surface is natural, but it is unnatural to assume that it has any particular precise shape.

* Adding one time derivative, or one factor of l, makes the operator marginal; these terms are already included in our action.

* Additional time derivatives or factors of l just make irrelevant operators.

How about quartic interactions? Consider:

$$\int dt \ \left(\Pi_{i=1}^{4} \ d^2\mathbf{k}_i \ d\mathbf{l}_i\right) \ V(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}, \mathbf{k_4})$$

$$\psi^\dagger_\sigma(\mathbf{p}_1)\psi_\sigma(\mathbf{p}_3)\psi^\dagger_{\sigma'}(\mathbf{p}_2)\psi_{\sigma'}(\mathbf{p}_4) \ \delta^3(\mathbf{p_1} + \mathbf{p_2} - \mathbf{p_3} - \mathbf{p_4})$$

This scales as s times the scaling of the momentum-conserving delta function.

Glib (wrong) argument:

$$\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \sim \delta^3(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k_3} - \mathbf{k}_4)$$

(Ignore the L factors because they are scaling to zero in the infrared).

Then, the overall scaling of the quartic is like s, and this operator is irrelevant.

And of course higher order operators are even more irrelevant.

However, the assertion that

$$\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \sim \delta^3(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k_3} - \mathbf{k}_4)$$

admits an important subtlety. Let us write (without loss of generality):

$$\mathbf{p}_3 = \mathbf{p}_1 + \delta\mathbf{k}_3 + \delta\mathbf{l}_3, \quad \mathbf{p}_4 = \mathbf{p}_2 + \delta\mathbf{k}_4 + \delta\mathbf{l}_4$$

then the delta function is:

$$\delta^3(\delta\mathbf{k}_3 + \delta\mathbf{k}_4 + \delta\mathbf{l}_3 + \delta\mathbf{l}_4)$$

* For generic momenta, the delta ks are linearly independent and we can indeed ignore the ls in the delta function.

* But if $\mathbf{p}_1 = -\mathbf{p}_2$ then in the argument of $\delta^3(\delta\mathbf{k}_3 + \delta\mathbf{k}_4)$, one component vanishes automatically!

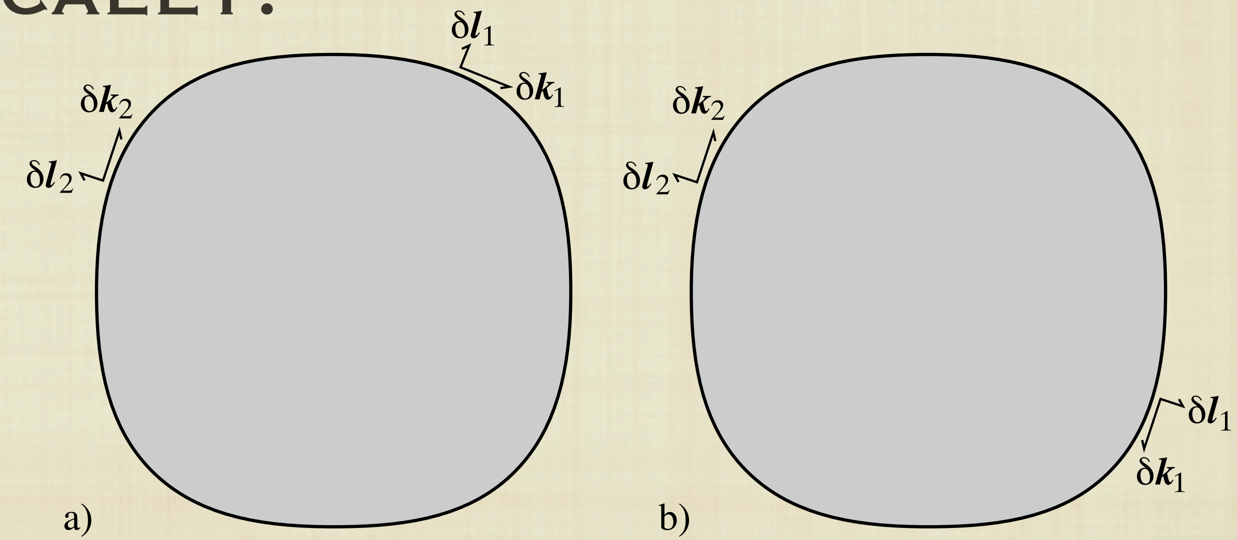


Figure 3: a) For two generic points near a two-dimensional Fermi surface, the tangents $\delta\mathbf{k}_i$ are linearly independent. b) For diametrically opposite points on a parity-symmetric Fermi surface, the tangents are parallel.

Then, one component of the delta function actually does constrain the transverse L fluctuations, and so scales inversely to L, as s^{-1}

At such momenta, the four-fermi interaction becomes **marginal**.

Computing the one-loop beta function for this marginal interaction, one finds:

$$V(E) = \frac{V}{1+NV\log(E_0/E)}$$

with N the density of states at the fermi energy.

Then we see that a repulsive interaction grows weaker at low-energy, while an attractive interaction grows stronger.

We expect two different contributions to the quartic interaction between electrons in a generic conductor:

1 -- Some sort of screened Coulomb interaction Vc

2 -- An attractive contribution that comes from integrating out phonons, Vp [strictly speaking, should have included these as goldstones in our effective theory].

What happens at low energies depends on the sign of Vc-Vp (at the natural energy scale where we "integrate out" phonon-electron interactions).

Case 1: $$V_C - V_P > 0$$

The low-energy theory looks free and boring.

Case 2: $$V_C - V_P < 0$$

We added a marginally relevant operator: at low-energies it produces a Cooper-pair condensate that breaks U(1) (like a technicolor model). Superconductor!

After this long review of things you already know (presented here in an awkward language), it is easy to describe the trouble with the high temperature superconductors.

The resistivity as a function of temperature in a normal superconductor, obeys a formula:

$$\rho(T) \sim A + BT^5$$

The first term arises from impurity scattering; the second from phonons (very suppressed due to phase space and momentum suppressions in the vertex coupling them to electrons).

In contrast, in the high temperature superconductors, one finds:

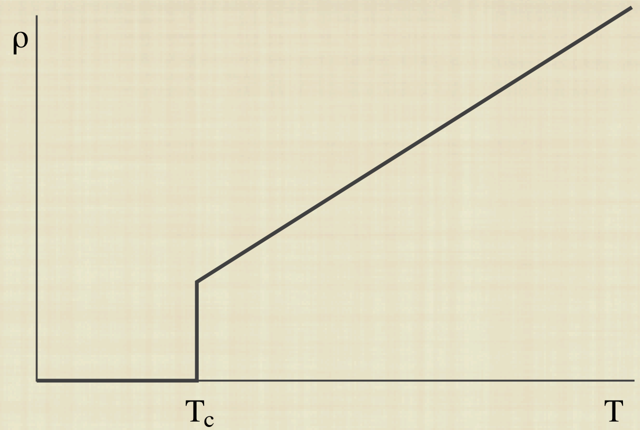


Figure 6: Resistivity versus temperature in a typical high-T_c material: zero below T_c, and linear above.

$$\rho(T) \sim A + CT$$

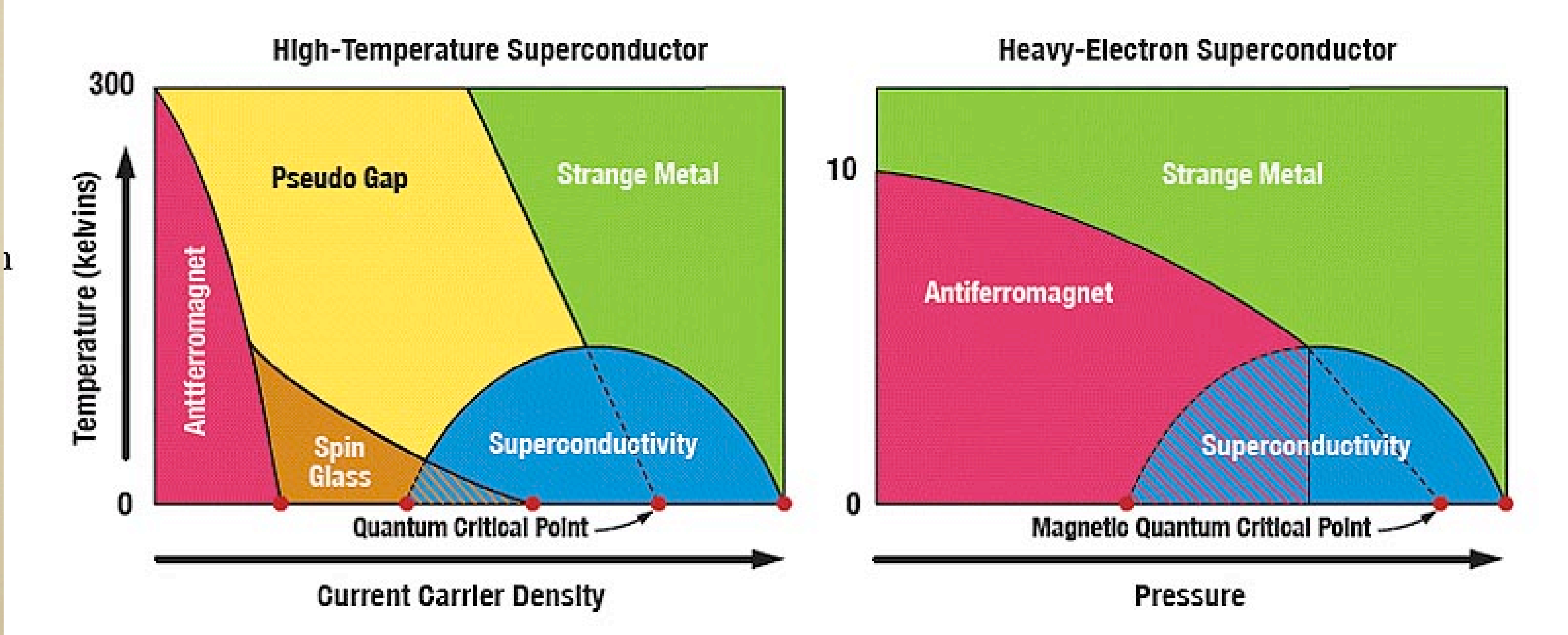
We should be able to write down some low-dimension operator to generate this in our effective theory.

But we can't. Write down the most general possible effective Lagrangian and there is no operator that would give this power of T.

* Caveat: If the fermi surface takes very special shapes, one can generate infrared singularities that cancel powers of T and could conceivably lead to such behaviors. But the shape of the fermi surface is a relevant parameter. The phenomenon is seen in many materials, and is stable under changes in doping of several percent -- suggesting that the correct low-energy theory has no relevant parameters at all.

So even in the normal state of high temperature superconductors, something interesting is going on: Landau's Fermi Liquid Theory has broken down!

The Modern Dogma



"Its the quantum critical point's fault!"

Phenomena where breakdown of Fermi Liquid Theory happen seem to be related to the existence of interacting, zero-temperature critical points. These separate phases are best characterized by distinct types of order, with different order parameters. The fluctuations at the critical point are driven by quantum effects at zero temperature (perhaps even in an unphysical region of the parameter space).

c.f. S. Sachdev,
"Quantum Phase Transitions"

Interestingly, many of these critical points exhibit the phenomenon of "dynamical scaling." That is, they involve critical theories where there is a scale invariance under which

$$t \to \lambda^z t, \quad x \to \lambda x, \quad z \neq 1$$

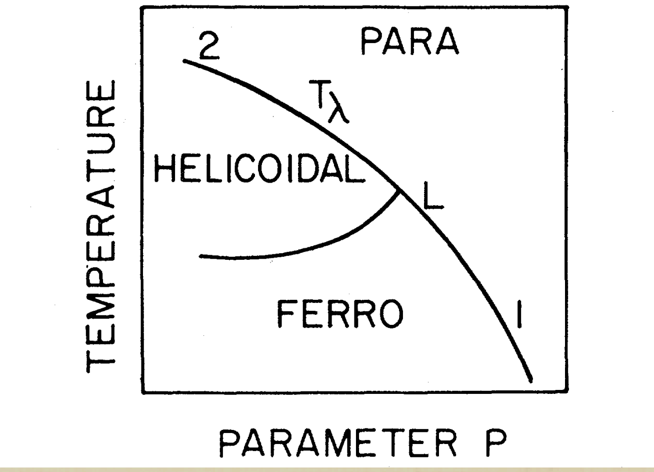
in contrast with the scaling symmetry embedded in the usual conformal group

$$t \to \lambda t, \quad x \to \lambda x$$

A toy model to keep in mind (analogous to the conformal field theory of a free boson, but with nontrivial dynamical critical exponent z) is given by the Lagrangian:

$$\mathcal{L} \;=\; \int d^2x\; dt\;\; \left((\partial_t M)^2 - (\nabla^2 M)^2 \right)$$

This "Lifshitz fixed point" has z = 2 and is actually known to govern the behavior of some magnetic materials and liquid crystals:



Hornreich, Lubin, Shtrikman

More generally, such fixed points arise in the phase diagram of various toy models of the high temperature superconductors (e.g. the Rokhsar-Kivelson dimer model). They have been studied somewhat extensively in recent work by e.g. Ardonne, Fendley, and Fradkin; Freedman, Nayak, and Shtengel (who found some analogous fixed points in gauge theories); and Balents, Vishwanath and Senthil.

Gravity Duals of Fixed Points with $z \neq 1$

It is of interest to develop new tools to study these critical theories, perhaps for applications to condensed matter problems arising in the study of strongly correlated electrons. For strongly coupled conformal field theories, a new tool is offered by the AdS/CFT correspondence. Quantum gravity in the background:

$$ds^2 = -r^2 \; dt^2 + r^2 \; (dx^2 + dy^2) + \frac{dr^2}{r^2}$$

is dual to a 2+1 dimensional CFT.

Maldacena;
Gubser, Klebanov, Polyakov;
Witten

* The extra dimension geometrizes the energy scale in the dual field theory

* Scale transformations act via

$$(t, x, y) \to \lambda\ (t, x, y),\ \ r \to \frac{r}{\lambda}$$

which leave the AdS metric invariant.

Happily, AdS arises as a solution of gravity with a negative cosmological constant, which can be found as the low-energy effective theory in many string compactifications. So many examples are known.

We, in contrast, would like gravitational duals to systems that enjoy the modified scale invariance (say for z=2):

$$(x, y) \to \lambda\ (x, y),\ t \to \lambda^2\ t$$

A natural guess, then, is that the gravity dual metric could take the form:

$$ds^2 = -r^4\ dt^2 + r^2\ (dx^2 + dy^2) + \frac{dr^2}{r^2}$$

This is designed to be invariant under the modified scale transformation.

Now unlike the AdS metric, this weird "Lifshitz metric" is not a famous solution of supergravity or string theory. So our program will be to:

a) Exhibit gravity + "matter" systems that plausibly arise in M-theory, and that can support this kind of spacetime metric.

b) Develop the analogue of the AdS/CFT map for this spacetime, and see if e.g. correlators and phase structure match (qualitatively, for now) with those found in real 2+1 dimensional systems.

Gravity Solutions

We expect perturbations of our fixed point can flow to normal CFTs. So we should begin with a gravity theory that could give rise to AdS space: say 4d gravity with a negative cosmological term:

$$\mathcal{S}_{\mathrm{grav}} = \int_X d^4x \, \sqrt{-g} \, (R - 2\Lambda)$$

M or String theory actions typically also have various p-form gauge fields.

A reasonably modest choice of additional content that can support our desired metric is:

$$\mathcal{S}_{\text{gauge}} = -\frac{1}{2}\int_X d^4x\sqrt{-g}\,(F_2 \wedge *F_2 + F_3 \wedge *F_3)$$

$$\mathcal{S}_{\text{topological}} = c\int_X B_2 \wedge F_2$$

where

$$dB_2 = F_3$$

We can make an obvious scale invariant ansatz for the fluxes which preserves time and space translation symmetries as well as spatial rotations:

$$F_2 \;=\; Ar\; dr \wedge dt, \quad F_3 = \; Br\; dr \wedge dx \wedge dy$$

We now have the parameters

$$\Lambda,\; c,\; A,\; B$$

and we can solve for the necessary values to get various dynamical scaling exponents, including z=2. So the desired gravity solutions exist in the metric + 2 form + 3 form system.

Modified AdS/CFT Map:

In standard AdS/CFT, there is a map between bulk fields and boundary operators. For instance, for the AdS dual of a 3D conformal field theory, a boundary scaling operator of dimension Δ is dual to a bulk field with Lagrangian:

$$\mathcal{S}_{\rm bulk} = \int d^4x \sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)$$

$$m^2 = \Delta\left(\Delta - 3\right)$$

The differential equation that phi satisfies in AdS space is:

$$\partial_z^2\phi - \frac{2}{z}\partial_z\phi + (\partial_t^2 + \partial_x^2 + \partial_y^2)\phi \;=\; \frac{m^2}{z^2}\phi$$

where $z = 1/r$. For modes which are only a function of z, one then finds fall off at small z (the boundary) that goes like:

$$\phi \;=\; c_1\; z^{\Delta} \;+\; c_2\; z^{3-\Delta}$$

The two-point function of phi computed using the GKPW prescription is as expected for a field of dimension Δ.

In contrast, in our space-time, the differential equation satisfied by a massive scalar field will be:

$$\partial_z^2\phi - \frac{3}{z}\partial_z\phi + z^2\partial_t^2\phi + (\partial_x^2 + \partial_y^2)\phi = \frac{m^2}{z^2}\phi$$

As a result, solutions fall off near the boundary like

$$\phi \ = \ c_1 \ z^{\Delta} \ + \ c_2 \ z^{4-\Delta}$$

with

$$m^2 \ = \ \Delta \ (\Delta - 4)$$

So e.g. a marginal operator in the boundary theory, which maps to a massless bulk scalar field (no potential, to reproduce the boundary moduli space), would have $\Delta = 3$ in ordinary AdS/CFT, but has $\Delta = 4$ in our theories with dynamical exponent z=2.

Correlation Functions

The correlation functions in these theories show some interesting differences from those in CFTs. Here we illustrate with the two-point function of a marginal operator (dual to a massless bulk scalar).

Note that the two-point function is **not determined** by symmetries in these theories; it contains an unknown function $f(x^2/t)$

To find the two-point function, we evaluate the on-shell bulk action and differentiate twice with respect to ϕ :

$$S(\phi) = \int d^3x \int_{\epsilon}^{\infty} dz(-\phi\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu\phi + \partial_\mu(\sqrt{g}g^{\mu\nu}\phi\partial_\nu\phi))$$

$$= \int d^3x \; [\sqrt{g}g^{zz}\phi\partial_z\phi|_{\epsilon}^{\infty}$$

$$= \int d^2\mathbf{k}d\omega \; \phi(0,\mathbf{k},\omega)\mathcal{F}(\mathbf{k},\omega)\phi(0,-\mathbf{k},-\omega)$$

The "flux factor" is given by:

$$\mathcal{F}(\mathbf{k},\omega) = [\tilde{G}(z,-\mathbf{k},-\omega)\sqrt{g}g^{zz}\partial_z\tilde{G}(z,\mathbf{k},\omega)]|_{\epsilon}^{\infty}$$

where the propagator satisfies

$$\partial_z^2\tilde{G} - \frac{3}{z}\tilde{G} - (\omega^2 z^2 + \mathbf{k}^2)\tilde{G} = 0$$

with boundary conditions:

$$\tilde{G}(0,\omega,\mathbf{k}) = 1$$

$$\tilde{G} \text{ finite as z} \to \infty$$

These conditions uniquely determine the propagator:

$$\tilde{G}(z,k) = e^{-|\omega| z^2/2}\ \Gamma(\frac{|\mathbf{k}|^2}{4|\omega|} + \frac{3}{2})\ U(\frac{|\mathbf{k}|^2}{4|\omega|} - \frac{1}{2}, -1, |\omega| z^2)$$

with U the confluent hypergeometric function of second kind. This vanishes at infinity, so the only contributions to the correlator come from the boundary as expected.

So without further ado we can plug this into the flux factor and read off the two-point function of the dual scaling operator:

$$\langle \mathcal{O}_\phi(\mathbf{k},\omega)\mathcal{O}_\phi(-\mathbf{k},-\omega)\rangle = -\frac{1}{2}\mathbf{k}^2|\omega| - \frac{1}{8}(4\omega^2 - \mathbf{k}^4)\log|\omega|$$

$$-\frac{1}{8}(4\omega^2 - \mathbf{k}^4)\psi(\frac{3}{2} + \frac{\mathbf{k}^2}{4|\omega|})$$

here we subtracted some divergent terms via adding local boundary terms, and also dropped uninteresting contact terms.

Notice that the first two contributions are completely localized in space.

such terms are of course forbidden in Lorentz invariant theories.

They may well be related to the kind of ultralocal behavior found for correlation functions of certain scaling operators in the Lifshitz theory by Ghaemi, Vishwanath and Senthil (2004).

These authors proposed that such behavior could be relevant in explaining experiments which show clear autocorrelation functions (at fixed spatial location with time separation), but no spatial correlaton, in certain materials.

Their results hold at finite temperature (but not zero temperature). It would be interesting to see if the correlators in the black hole solution in our spacetime exhibit complete ultralocality.

Phase Structure

One goal is to work out the different phases one can obtain by perturbing these kinds of gravity duals in various ways.

The simplest interesting perturbation of the free toy model

$$\mathcal{L} \;=\; (\partial_t M)^2 - (\nabla^2 M)^2$$

is to add a perturbation by

$$\delta\mathcal{L} \;\sim\; \epsilon\,(\nabla M)^2$$

Depending on the sign of the perturbation, this either has the effect of:

a) inducing renormalisation group flow to a normal CFT (in this case, a free theory).

b) causing an instability to condensation of low-momentum modes of M, leading to some kind of spatially modulated structures.

Do we see analogous phenomena in our strongly coupled theory?

We can see the renormalisation group flow in a) as follows (I will be brief; the equations may not be so illuminating).

To see the flow, we should generalize our ansatz for the background fields:

$$ds^2 \;=\; -\,r^4\;f(r)^2\;dt^2 + r^2\;(dx^2 + dy^2) + g(r)^2\;\frac{dr^2}{r^2}$$

$$F_0 \;=\; A$$

$$F_2 = Bh(r)\;(rg(r)f(r))\;dr \wedge dt$$

$$F_3 = Cj(r)\;(rg(r))\;dr \wedge dx \wedge dy$$

The equations of motion that the new functions f,g,h,j must satisfy can easily be derived, and are:

$$2r\frac{f'}{f} = (5 - h^2 + j^2)g^2 - 5$$

$$rj' = 2gh + \frac{1}{2}j + \frac{1}{2}jg^2(h^2 - j^2 - 5)$$

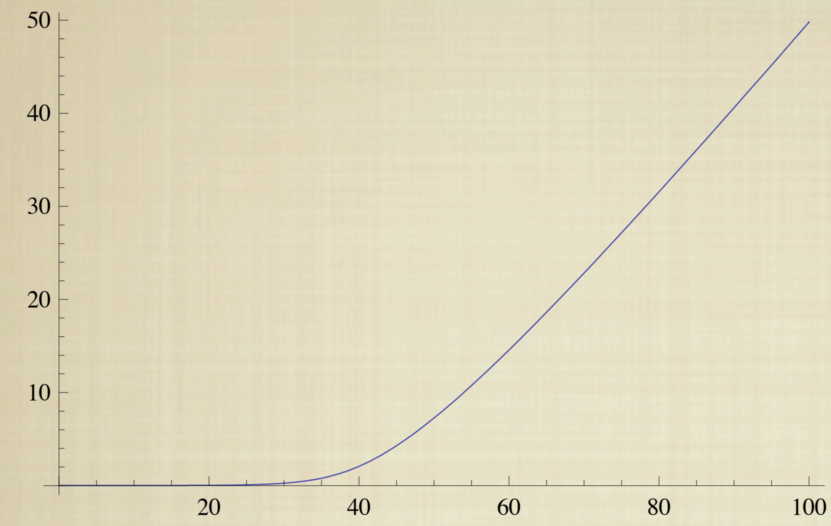
$$rh' = 2gj - 2h$$

$$rg' = \frac{1}{2}g^3(h^2 + j^2 - 5) + \frac{3}{2}g$$

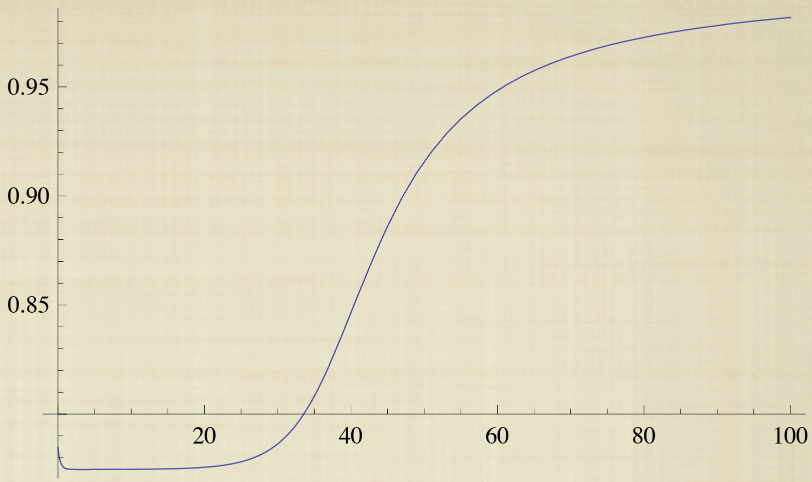
We solve for the RG flow by "shooting." It is hard to hit the AdS fixed point on the nose by perturbing the UV theory, because the AdS fixed point has relevant operators.

But our UV theory (in this approximation) has only one marginally irrelevant operator in addition to the relevant perturbations. So it is easy to fix "almost AdS" boundary conditions near r=0, and calculate the RG flow back "up" to the Lifshitz point. A representative trajectory of f,g,h,j as a function of r is shown on the next page.

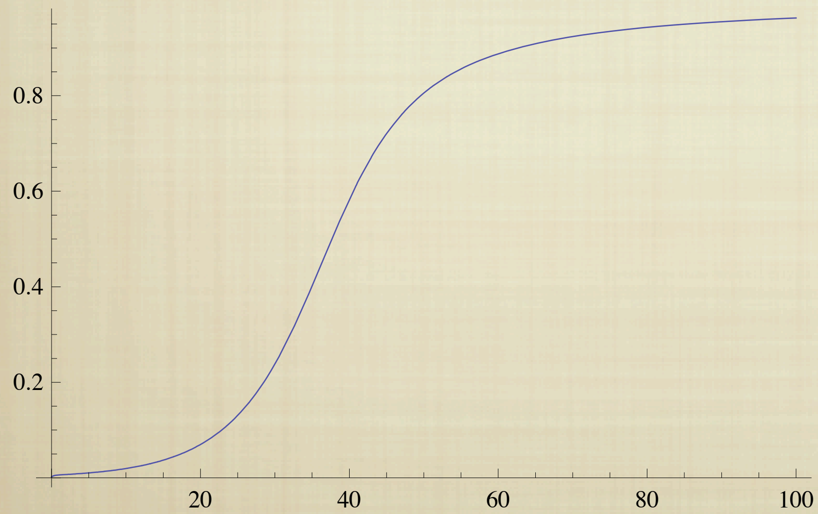
plot of rf(r) :



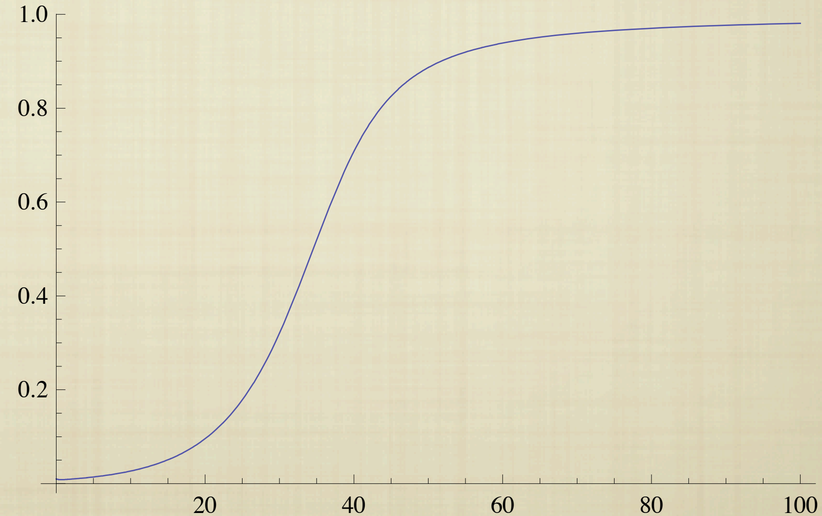
plot of g(r) :



plot of h(r) :



plot of j(r) :



Our current goals:

* Find the black hole solutions in these spacetimes to see if interesting new features emerge in the correlators.

* Directly study z=2 Non-Abelian gauge theories (c.f. Freedman, Nayak, Shtengel) and try to find fixed points. These could morally be the gauge duals of our backgrounds (and would have a principled, large N, reason to have a string description).