Dualities and Dimensional Reduction

in Topological Quantum Order and Processing of Quantum Information

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Introduction and Overview

- A vague phylosophy,

  "Interactions are more important than elementary degrees of freedom,"

  and its technical implementation: **BOND ALGEBRAS**.

- **Exact solvability** (Lie bond algebras)

  Nussinov and Ortiz Phys. Rev. B 79, 214440 (2009)

- **Dualities**
  - Perturbation theory for strongly coupled systems
  - Symmetries, transition points and boundaries of phase diagrams

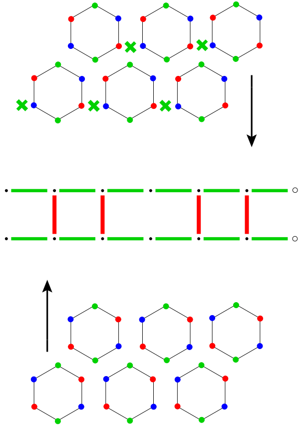
  1. **Unified, generalized theory of quantum and classical dualities**
  2. **Systematic derivation of topological degrees of freedom**
  3. **Fermionization as a duality: derivation of the JW mapping**
  4. **Gauge theories and TQO**
  5. **Numerical applications: simplified STL for quantum Monte Carlo, dual boundary conditions,...**

Cobanera et. al., PRL 104, 020402 (2010), Adv. Phys. 60, 679 (2011)

Introduction and Overview

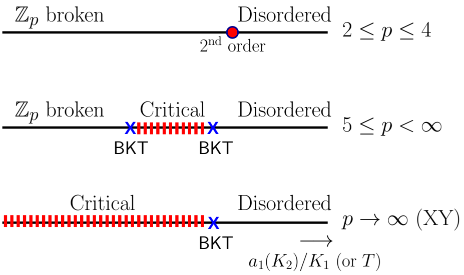
- **Exact and Effective Dimensional reduction (holographic correspondences)**
  1. Exact dimensional reduction as a duality
  2. Tensor networks (DMRG) for two and three dimensional systems ?
  3. When is a system "quasi" lower dimensional? Dim-red inequalities symmetry principles for dimensional reduction

Cobanera et. al. arXiv:1110.2179v1 [cond-mat.stat-mech]



Introduction and Overview

- **Non-Abelian dualities**
  1. The character of a duality is not determined by the group of symmetries
  2. New dualities for the $S = 1/2$ Heisenberg model *in any number of dimensions*
  3. Self-duality, Non-abelian and emergent symmetries, and novel topological excitation in the p-clock model, **Nuc. Phys. B 854 (2012), 780**



Bonds

Model Building in Quantum Mechanics

$$\textbf{EDFs} \;\Rightarrow\; \textbf{basic interactions } \{h_\Gamma\}_\Gamma \Rightarrow$$

$$\Rightarrow \quad H = \sum_\Gamma \lambda_\Gamma h_\Gamma \;\Rightarrow\; \textbf{Emergent EDFs}$$

The **BONDS** h_Γ are the "atomic constituents" of the Hamiltonian.

Example:

$$\sigma_i^x, \sigma_i^z \;\Rightarrow\; \{\sigma_i^x, \sigma_i^z \sigma_{i+1}^z\}_i \;\Rightarrow\; H_{\rm I} = \sum_i [h\sigma_i^x + J\sigma_i^z\sigma_{i+1}^z] \;\Rightarrow\; \textbf{Kinks}$$

Bonds are SPARSE: $\qquad [h_\Gamma, h_{\Gamma'}] = 0 \qquad$ **for most** Γ'

Typically a consecquence of *LOCALITY*

Bond Algebra = Algebra of Interactions

Our Philosophy: Interactions are more important than elementary degrees of freedom.

What are the EDFs? $\leftrightarrow$ What is the algebra of the EDFs?

1. fermionic or bosonic algebra?
2. $SU(N)$ spins, "Hopf spins"? etc. etc. etc. ...

What are the interactions? $\leftrightarrow$ What is the algebra of interactions?

Definition

The **bond algebra** of $H = \sum_\Gamma \lambda_\Gamma h_\Gamma$ is the von Neumann algebra of operators $\mathcal{A}_H$ generated by the set of bonds $\{h_\Gamma\}_\Gamma$.

(Cobanera et. al., PRL 104, 020402 (2010))

$$\mathcal{A}_H = Linear\ Span\ \{\mathbb{1}, h_\Gamma, h_\Gamma^\dagger, h_\Gamma h_{\Gamma'}, h_\Gamma^\dagger h_{\Gamma'}, h_{\Gamma'}^\dagger h_\Gamma, h_{\Gamma'}^\dagger h_\Gamma^\dagger, h_\Gamma h_{\Gamma'} h_{\Gamma''}, \cdots\}$$

Bond Algebras and Dualities

Idea: Use bond algebras to *compare* Hamiltonians

$\Phi : A_{H_1} \to A_{H_2}$ one-to-one and onto

$$\begin{aligned} \Phi(\mathbb{1}) &= \mathbb{1}, & \Phi(\mathcal{O}^\dagger) = \Phi(\mathcal{O})^\dagger, \\ \Phi(\mathcal{O}_1\mathcal{O}_2) &= \Phi(\mathcal{O}_1)\Phi(\mathcal{O}_2), & \Phi(\mathcal{O}_1 + \lambda\mathcal{O}_2) = \Phi(\mathcal{O}_1) + \lambda\Phi(\mathcal{O}_2). \end{aligned}$$

Definition

Φ is a duality, and H_1 is **dual** to H_2, if $\Phi(H_1) = H_2$

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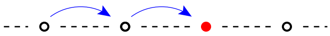
Theorem

$\Phi(\mathcal{O}) = \mathcal{U}\mathcal{O}\mathcal{U}^\dagger$ **Dualities are unitary equivalences!!!**

Either $\boxed{\mathcal{U}\mathcal{U}^\dagger = \mathcal{U}^\dagger\mathcal{U} = \mathbb{1}}$, or $\boxed{\mathcal{U}\mathcal{U}^\dagger = \mathbb{1} \quad \text{and} \quad \mathcal{U}^\dagger\mathcal{U} = P = P^2}$.

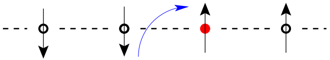
Transmutation of statistics I

$$H_{\mathrm{F}} = \sum_{i=1}^{N-1} \lambda(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i)$$



Bonds:
$\{c_{i+1}^\dagger c_i \,|\, i = 1, \cdots, N-1\}$

$$H_{\mathrm{XY}} = \sum_{i=1}^{N-1} \lambda(\sigma_i^+ \sigma_{i+1}^- + \sigma_{i+1}^+ \sigma_i^-)$$



Bonds:
$\{\sigma_{i+1}^+ \sigma_i^- \,|\, i = 1, \cdots, N-1\}$

Very different EDFs, but isomorphic bond algebras:

$$\boxed{c_{i+1}^\dagger c_i \xrightarrow{\Phi_{\mathrm{d}}} \sigma_{i+1}^+ \sigma_i^-}$$

H_{F} is **dual** (unitarily equivalent!) to H_{XY}

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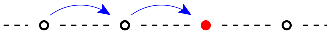

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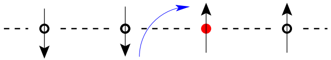

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Dual Fermions: $\boxed{c_i \xrightarrow{\Phi_{\mathrm{d}}} ???}$

Transmutation of statistics II: Fermions as dual topological collective modes

1. **Enlarge** $\mathcal{A}_F$ by adding c_1 to the set of bonds

$$\Rightarrow \quad c_2 = [c_1, c_1^\dagger c_2], \quad c_3 = [c_2, c_2^\dagger c_3], \quad \cdots, \quad c_N = [c_{N-1}, c_{N-1}^\dagger c_N]$$

2. **Extend** Φ_d so that all algebraic relations are preserved

$$\Rightarrow \quad c_1 \xrightarrow{\Phi_\mathrm{d}} \sigma_1^- . \qquad \text{Then, for } i = 2, \cdots, N$$

3. $\Phi_\mathrm{d}(c_2) = [\Phi_\mathrm{d}(c_1), \Phi_\mathrm{d}(c_1^\dagger c_2)] = [\sigma_1^-, \sigma_1^+ \sigma_2^-] = -\sigma_1^z \sigma_2^-$, and so on...

4. $$\boxed{c_i \xrightarrow{\Phi_\mathrm{d}} \prod_{j=1}^{i-1} (-\sigma_j^z)\, \sigma_i^- \equiv \hat{c}_i}$$

JW transformation = dual fermions

Dualities and Fermionization

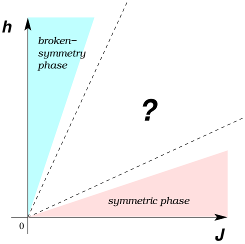
- Fermionization can be understood as a duality in any number of dimensions, and
- the corresponding JW transformation can be derived as a fermionic topological excitation
- Bond algebras can be used to
  1. show that fermionization is not possible under certain conditions
  2. look for dual representations of a model that are better suited for fermionization. Example: Two-dimensional Ising model in a transverse field

(Cobanera et. al., Adv. Phys 60, 679 (2011))

Are we really talking of dualities here? The quantum Ising chain

$$H_{\mathsf{I}}[h, J] = \sum_i \left[h\sigma_i^x + J\sigma_i^z\sigma_{i+1}^z\right]$$

An infinite quantum Ising Chain



Bond	anticommutes with		Bond^2
σ_i^x	$\sigma_{i-1}^z\sigma_i^z$	$\sigma_i^z\sigma_{i+1}^z$	$\mathbb{1}$
$\sigma_i^z\sigma_{i+1}^z$	σ_i^x	σ_{i+1}^x	$\mathbb{1}$

$$\sigma_i^x \xrightarrow{\Phi_{\mathrm{d}}} \sigma_i^z\sigma_{i+1}^z$$
$$\sigma_i^z\sigma_{i+1}^z \xrightarrow{\Phi_{\mathrm{d}}} \sigma_{i+1}^x$$

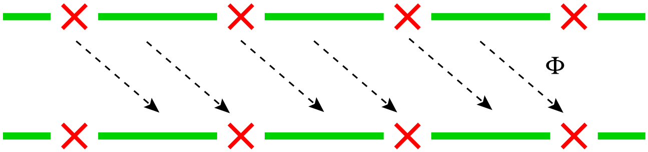
$H_{\mathsf{I}}[h, J]$ is dual (unitarily equivalent!) to $H_{\mathsf{I}}[J, h]$

$\Rightarrow E(J, h) = E(h, J) \Rightarrow \boxed{J = h}$ transition line

Self-duality and kinks

Duality Mapping

$$\sigma_i^x \xrightarrow{\Phi_{\rm d}} \sigma_i^z \sigma_{i+1}^z \qquad \sigma_i^z \sigma_{i+1}^z \xrightarrow{\Phi_{\rm d}} \sigma_{i+1}^x$$



A **duality** is a mapping of bonds that preserves the algebra of interactions

$$\mu_i^x \equiv \Phi_{\rm d}(\sigma_i^x) = \sigma_i^z \sigma_{i+1}^z$$

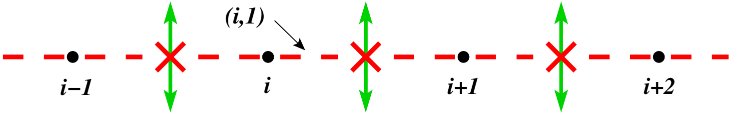
$$\mu_i^z \equiv \Phi_{\rm d}(\sigma_i^z) = \Phi_{\rm d}(\sigma_i^z \sigma_{i+1}^z \times \sigma_{i+1}^z \sigma_{i+2}^z \times \cdots) = \sigma_{i+1}^x \sigma_{i+1}^x \sigma_{i+2}^x \cdots$$

(Fradkin and Susskind, Phys. Rev. D 17 (1978) 2637)

Dualities and TQO: The one-dimensional extended toric code

$(i,1) =$ **link connecting site** i **and** $i+1$

$$H_{\rm ETC} = \sum_i \left[h_z \sigma^z_{(i,1)} + h_x \sigma^x_{(i,1)} + J_x \sigma^x_{(i,1)} \sigma^x_{(i+1,1)} \right] \qquad \sigma^x_{(i,1)} \sigma^x_{(i+1,1)} \equiv A_{i+1}$$



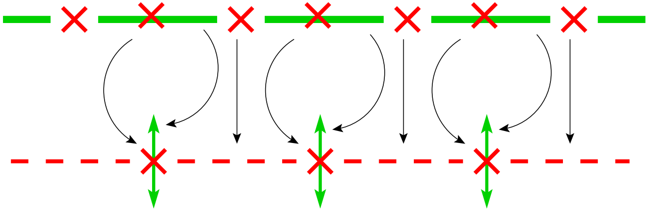
Bond	anticommutes with			Bond^2
$\sigma^z_{(i,1)}$	$\sigma^x_{(i,1)}$	$\sigma^x_{(i-1,1)}\sigma^x_{(i,1)}$	$\sigma^x_{(i,1)}\sigma^x_{(i+1,1)}$	$\mathbb{1}$
$\sigma^x_{(i,1)}\sigma^x_{(i+1,1)}$		$\sigma^z_{(i,1)}$	$\sigma^z_{(i+1,1)}$	$\mathbb{1}$
$\sigma^z_{(i,1)}$	$\sigma^x_{(i,1)}$			$\mathbb{1}$

(Tupitsyn et. al., Phy. Rev. B 82, 8 (2012); two dimensions)

$$H^D_{\rm ETC} = \sum_i \left[J_x \sigma^x_i + h_z \sigma^z_i \sigma^z_{(i,1)} \sigma^z_{i+1} + h_x \sigma^x_{(i,1)} \right]$$

Duality Mapping:

$$\sigma^x_i \xrightarrow{\Phi_{\rm d}} \sigma^x_{(i-1,1)} \sigma^x_{(i,1)} \equiv A_i, \qquad \sigma^z_i \sigma^z_{(i,1)} \sigma^z_{i+1} \xrightarrow{\Phi_{\rm d}} \sigma^z_{(i,1)} \qquad \sigma^x_{(i,1)} \xrightarrow{\Phi_{\rm d}} \sigma^x_{(i,1)}$$



Have we lost degrees of freedom???

Dualities and Gauge Symmetries

$$H^{D}_{\mathrm{ETC}} = \sum_i \left[J_x \sigma^x_i + h_z \sigma^z_i \sigma^z_{(i,1)} \sigma^z_{i+1} + h_x \sigma^x_{(i,1)} \right] \qquad \mathbb{Z}_2 \text{ Higgs model}$$

(Fradkin and Shenker, Phys. Rev. D 19, 3682 (1979))

Gauge Symmetries: $\sigma^x_i A_i = \sigma^x_{(i-1,1)} \sigma^x_i \sigma^x_{(i,1)}$

A state ρ is physical if and only if $[\rho, \sigma^x_i A_i] = 0$

NOTICE: $\boxed{\sigma^x_i A_i \xrightarrow{\Phi_{\mathrm{d}}} A_i A_i = \mathbb{1}}$

The duality changes the number of EDFs because it eliminates all the gauge symmetries.

$$\Phi_{\mathrm{d}}(\mathcal{O}) = U_{\mathrm{d}} \mathcal{O} U^{\dagger}_{\mathrm{d}} \qquad U_{\mathrm{d}} U^{\dagger}_{\mathrm{d}} = \mathbb{1} \qquad U^{\dagger}_{\mathrm{d}} U_{\mathrm{d}} = P_{GI}$$

Topological quantum order in the Higgs model: Generalizations

- Both the $\mathbb{Z}_2$ Higgs model and (extended) toric code model have natural (canonical) generalizations to any number of dimensions and arbitrary **Abelian** group G. If the group if continous we may be able to take the continuum limit.
- **They are always dual**, and the phase diagrams of some of these generalizations are under investigation. In two dimensions, the continuum limit of the ETC model with group $\mathbb{R}$ is the Stückelberg model of mass generation.
- In two dimension, the duality still holds on more general lattices like the honeycomb lattice. **It suggests some interesting questions on the stability of some string-net topological phases.**
- **The Big Challenge:** What if G is **non-Abelian**?

STL decomposition/Feynmann's path integral

$$\mathcal{Z}_E = \sum_{\{\phi_1\},\cdots,\{\phi_N\}} \langle\phi_1|e^{\frac{-1}{N}H}|\phi_2\rangle\langle\phi_2|e^{\frac{-1}{N}H}|\phi_3\rangle\cdots\langle\phi_N|e^{\frac{-1}{N}H}|\phi_1\rangle = \mathrm{Tr}\ (e^{\frac{-1}{N}H})^N$$

$$\mathcal{Z}_E = \mathrm{Tr}\ (e^{\frac{-1}{N}H})^N = \mathrm{Tr}\ (e^{\frac{-1}{N}H^D})^N = \mathcal{Z}_E^D$$

Bond-Algebraic Classical Dualities

$$\mathcal{Z} = \mathrm{Tr}\ (T_1\cdots T_s)^N \qquad\qquad T_i = \textstyle\prod_\Gamma t_{i\Gamma}$$

Bond algebra $\mathcal{A}_\mathcal{Z}$: algebra generated by the $\{t_{i\Gamma}\}$

Strong Coupling/Weak Coupling dualities are "classical" descendants of quantum dualities

$$\mathcal{Z}_{\mathsf{I}}(K,\tilde{h}) = \sum_{\{\sigma_i\}} \exp\left[\sum_{i=1}^{N}(K\sigma_i\sigma_{i+1} + \tilde{h}\sigma_i)\right] = \mathrm{Tr}\,\left(T_1T_2\right)^N$$

$$T_1 = e^K + e^{-K}\ \sigma^x, \qquad T_2 = e^{\tilde{h}\ \sigma^z} = \cosh(\tilde{h}) + \sinh(\tilde{h})\ \sigma^z,$$

$$T_1^D = e^K + e^{-K}\ \sigma^z = A\ e^{\tilde{h}^*\sigma^z}, \qquad T_2^D = e^{\tilde{h}\ \sigma^x} = B(e^{K^*} + e^{-K^*}\ \sigma^x),$$

$$\boxed{\sinh(2K)\sinh(2\tilde{h}^*) = 1, \qquad \sinh(2K^*)\sinh(2\tilde{h}) = 1}$$

Bond-algebraic dualities are unitary transformations

$$\frac{\mathcal{Z}_{\mathrm{I}}(K,\tilde{h})}{(2\sinh(2\tilde{h}))^{N/2}} = \frac{\mathrm{Tr}\ (T_1 T_2)^N}{(2\sinh(2\tilde{h}))^{N/2}} = \frac{\mathrm{Tr}\ (T_2^D T_1^D)^N}{(2\sinh(2\tilde{h}))^{N/2}} = \frac{\mathcal{Z}_{\mathrm{I}}(K^*,\tilde{h}^*)}{(2\sinh(2\tilde{h}^*))^{N/2}}$$

Classical self-dual line defined by $\tilde{h}^* = \tilde{h}$ and $K^* = K$

$$\sinh(2K)\sinh(2\tilde{h}) = 1$$

Can only be *critical* if $\tilde{h} = 0 \quad \Rightarrow \quad K \to \infty$, i.e., at zero temperature.

Strong Coupling/Weak Coupling

- Quantum dualities are "mapped" to strong coupling/weak coupling dualities of partition functions/Euclidean path integrals
- Many possible dualities! Only one corresponds to the standard classical duality based on the Fourier transform

Holographies and dualities

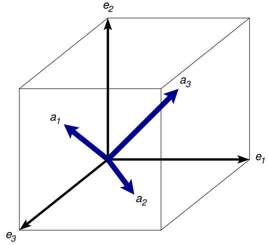
- Dimensional reduction and holographic correspondences qualify those situations when the "apparent", geometric dimension of a system is not the dimension that best characterizes its response to probes and information-theoretic aspects.
  1. **Restricted dynamics from conservation laws** (sliding dynamics)
  2. **Restricted dynamics from special couplings and interactions** (layered systems)
  3. **Kaluza-Klein compactification** (string theory)
  4. **Gauge-gravity dualities** (AdS-CFT correspondence)

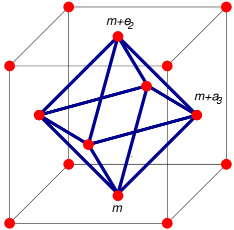
- Bond algebras display an *internal connectivity* that may or may not reflect the apparent geometric connectivity of the model.
- **Bond-algebraic dualities can change the dimension of a system.**

Topological quantum order and dimensional reduction

An fcc lattice has exactly one octahedron per lattice site. Define the "octahedron operator"

$$O_{\mathbf{m}} = \sigma^x_{\mathbf{m}+\mathbf{a_1}-\mathbf{a_2}} \sigma^x_{\mathbf{m}+\mathbf{a_3}} \sigma^y_{\mathbf{m}} \sigma^y_{\mathbf{m}+\mathbf{e_2}} \sigma^z_{\mathbf{m}+\mathbf{a_3}-\mathbf{a_2}} \sigma^z_{\mathbf{m}+\mathbf{a_1}}$$





$H_{xyz} = -J \sum_{\mathbf{m}} O_{\mathbf{m}}$ Chamon, PRL 94, 040402 (2005)

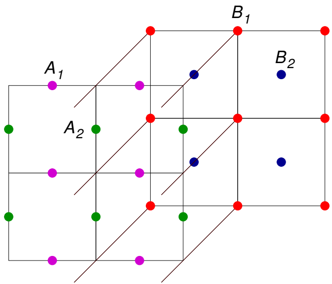
displays topological quantum order. **Its bond algebra is commutative.**

The XYZ model

The fcc lattice is quadripartite. If it satisfies periodic boundary conditions, then ($i = 1, 2$)

$$\prod_{\mathbf{m}\in A_i} O_m = \prod_{\mathbf{m}\in B_i} O_{\mathbf{m}} = \mathbb{1} \ .$$

These constraints further structure the commutative bond algebra of the model.



The XYZ model is dual to **four decoupled, periodic, Ising chains.** (Cobanera et. al. arXiv:1110.2179v1 [cond-mat.stat-mech])

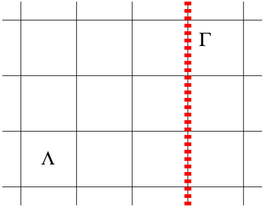
Some Consequences for the Storage of Quantum Information

- Many models of TQO are dual to one dimensional models. This is not because their bond algebra is commutative, but rather because the constraints are simple. We can add non-commutativity and preserve dimensional reduction.
- **Thermal fragility:** a periodic Ising chains display short autocorrelation times at any finite temperature, *regardless of its size*. Models of TQO that display this type of dimensional reduction may not be good quantum memories. Most famously,
  1. The Toric Code, Honeycomb toric code, topological color codes, and
  2. the XYZ model just discussed
- But, exact dimensional reduction is a rare. How can we quantify and exploit **approximate or effective** dimensional reduction?

Cobanera et. al., arXiv:1110.2179v1 [cond-mat.stat-mech]

Effective Dimensional Reduction in Classical Systems

(Batista and Nussinov, Phys. Rev. B 72, 045137 (2005))



$$\phi(\mathbf{x}) = \begin{cases} \phi_0(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \Gamma \\ \psi(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \bar{\Lambda} \end{cases} .$$

$f[\phi] = f[\phi_0]$ localized observable

$$\langle f \rangle^D = \sum_{\{\psi\}} \sum_{\{\phi_0\}} f[\phi_0] \frac{e^{-\beta E[\phi_0,\psi]}}{\mathcal{Z}} = \sum_{\{\psi\}} \frac{z[\psi]}{\mathcal{Z}} \frac{\sum_{\{\phi_0\}} f(\phi_0) e^{-\beta E[\phi_0,\psi]}}{z[\psi]}$$

$$\langle f \rangle_l^d \equiv \min_\psi \langle f \rangle^d[\psi] = \langle f \rangle^d[\psi_{\min}], \qquad \langle f \rangle_u^d \equiv \max_\psi \langle f \rangle^d[\psi] = \langle f \rangle^d[\psi_{\max}]$$

$$\boxed{\langle f \rangle_l^d \leq \langle f \rangle^D \leq \langle f \rangle_u^d}$$

$\langle f \rangle_l^d \; : \; E_l[\phi_0, \psi_{\min}]$ and $\langle f \rangle_u^d \; : \; E_u[\phi_0, \psi_{\max}]$ LOCAL effective theories

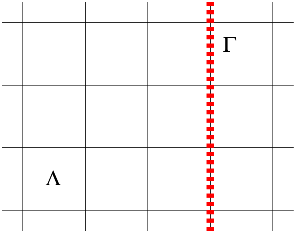
Consider a system on a volume Λ with distinguishable bulk $\bar{\Lambda}$ and boundary Γ:

$$\mathcal{H}_\Lambda = \mathcal{H}_\Gamma \otimes \mathcal{H}_{\bar{\Lambda}}$$

We can write an arbitrary state as

$$\rho = \sum_i \lambda_i \, \rho_{\Gamma i} \otimes \rho_{\bar{\Lambda} i}, \qquad \lambda_i \in \mathbb{R}, \qquad \sum_i \lambda_i = 1.$$

If the λ_i are all positive, the state is separable (unentangled).



$$f = f_\Gamma \otimes \mathbb{1}_{\bar{\Lambda}}$$

Localized observable

Entanglement-based Effective Dimensional Reduction

Arbitrary state $\rho = \sum_i \lambda_i \, \rho_{\Gamma i} \otimes \rho_{\bar{\Lambda} i}, \qquad \lambda_i \in \mathbb{R}, \qquad \sum_i \lambda_i = 1.$

$f = f_\Gamma \otimes \mathbb{1}_{\bar{\Lambda}}$ **localized on the boundary**

Theorem

$$L_+ \langle f \rangle_l^+ - L_- \langle f \rangle_l^- \leq \mathrm{Tr}_{\,\Lambda}(\rho f) \leq L_+ \langle f \rangle_u^+ - L_- \langle f \rangle_u^- .$$

Where $L_+ = \sum_{i_+} \lambda_{i_+}, \quad L_- = \sum_{i_-} |\lambda_{i_-}|$ are both positive,

$$\langle f \rangle_u^+ \equiv \max_{i_+} \mathrm{Tr}_{\,\Gamma}(\rho_{\Gamma i_+} f_\Gamma), \qquad \langle f \rangle_u^- \equiv \min_{i_-} \mathrm{Tr}_{\,\Gamma}(\rho_{\Gamma i_-} f_\Gamma),$$

$$\langle f \rangle_l^+ \equiv \min_{i_+} \mathrm{Tr}_{\,\Gamma}(\rho_{\Gamma i_+} f_\Gamma), \qquad \langle f \rangle_l^- \equiv \max_{i_-} \mathrm{Tr}_{\,\Gamma}(\rho_{\Gamma i_-} f_\Gamma).$$

If state ρ is unentangled, then $L_- = 0$ and $L_+ = 1$:

$$\langle f \rangle_l^+ \leq \mathrm{Tr}_{\,\Lambda}(\rho f) \leq \langle f \rangle_u^+$$

Cobanera et. al., arXiv:1110.2179v1 [cond-mat.stat-mech]

Effective Dimensional Reduction

- Entanglement-based inequalities are ideal to establish a connection to classical notions of effective dimensional reduction
- There are other inequalities that are better suited to purely quantum-mechanical investigations (Cobanera et. al. arXiv:1110.2179v1 [cond-mat.stat-mech]).
- Effective dimensional reduction combined with low dimensional gauge like symmetries and results like Elitzur's or Mermin-Wagner-Coleman theorem can put strong constraints on symmetry breakdown in higher dimensions.
- Effective dimensional reduction may help to asses the viability of realistic proposals for topological quantum memories.

Summary and conclusions

1. Bond algebras are useful!!!
2. Bond-Algebraic dualities are one of the best developed applications bond algebras. They work well with TQO because they can handle gauge symmetries easily.
3. Bond algebras encode the "true" dimensionality of a system as witnessed by its interactions, and a duality can then unveil exact dimensional reduction
4. Exact dimensional reduction is rare, so we propose a set inequalitiesto quantify *effective dimensional reduction*. They may may be of consequence to quantum information processing.

Acknowledgments, and Thank you!

Gerardo Ortiz, Indiana University, Bloomington IN

Zohar Nussinov, Washington University, St Louis MO

Emanuel Knill, NIST, Boulder CO

Thank you!

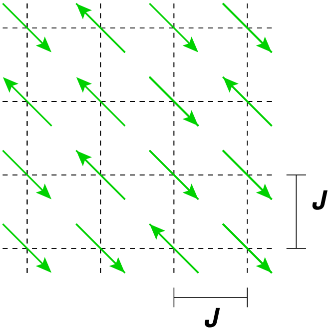
$$\mathcal{Z}_{\mathrm{I}}[K] = \sum_{\sigma_{\mathbf{r}}} \exp\left[K \sum_{\mathbf{r}} \sum_{\nu=1,2} \sigma_{\mathbf{r}+\mathbf{e}_\nu}\sigma_{\mathbf{r}}\right]$$

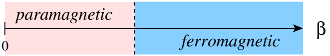
$K = -\beta J = -J/k_B T \geq 0$, *ferromagnetic*

One discrete global symmetry

$$\sigma_{\mathbf{r}} \mapsto -\sigma_{\mathbf{r}}$$

that **can** be broken. What is the critical temperature?





Bond algebras for classical dualities

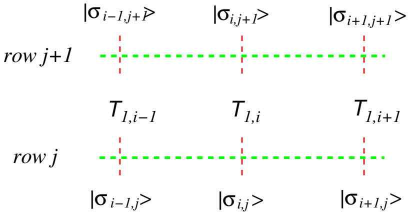
If we introduce the row-to-row transfer matrices

$$T_0 = \prod_i exp[K\sigma^z_i\sigma^z_{i+1}], \qquad T_1 = \prod_i \left(e^K + e^{-K}\sigma^x_i\right)$$

then we can write

$$\mathcal{Z}_{\mathrm{I}}[K] = \mathrm{Tr}\ [T_1 T_0 T_1 T_0 \cdots T_1 T_0] = \mathrm{Tr}\ [(T_1 T_0)^N]$$

provided we agree to compute the trace in the diagonal basis for the σ^z_i*.* N determines the height (number of rows) of the system. **The bond algebra is the same as before!**



$$= \prod_i \mathrm{exp}[-\beta J \sigma_{i,j}\sigma_{i,j+1}]$$

The self-duality of Kramers and Wannier and the critical temperature of the Ising model

$$T_0 \xrightarrow{\Phi_{\mathrm{d}}} T_0^D = \prod_i \exp[K\sigma_i^x], \qquad T_1 \xrightarrow{\Phi_{\mathrm{d}}} T_1^D = \prod_i \left(e^K + e^{-K}\sigma_i^z\sigma_{i+1}^z\right)$$

This is a **UNITARY TRANSFORMATION**. Hence

$$\mathcal{Z}_{\mathrm{I}}[K] = \mathrm{Tr}\,[(T_1 T_0)]^N] = \mathrm{Tr}\,[(T_1^D T_0^D)] \equiv \mathcal{Z}_{\mathrm{I}}^D$$

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$$\mathcal{Z}_{\mathrm{I}}[K] = \mathcal{Z}_{\mathrm{I}}^D \propto \mathcal{Z}_{\mathrm{I}}[K^*], \qquad K^* = -\frac{1}{2}\ln\tanh(K)$$

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A weak coupling-strong coupling transformation has emerged! If there is only one critical point, then its value must be

$$K_c = \frac{1}{2}\ln(1+\sqrt{2})$$

Symmetries and dualities

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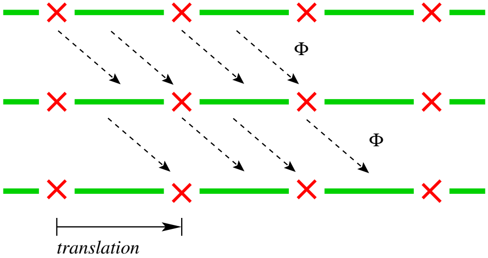
3. **Self-dualities**

$$\mathcal{U}_d H[\lambda_1, \lambda_2 \cdots] \mathcal{U}_{\rm d}^\dagger = H[\lambda_1^*, \lambda_2^*, \cdots]$$

become **extra, discrete, non-trivial** symmetries at self-dual points where $\lambda_i = \lambda_i^*$.

Symmetries and Dualities II: An example

The self-duality of the Ising model is the square-root of a translation by one unit to the right, $\mathcal{U}_{\rm d}^2 = T(1)$:



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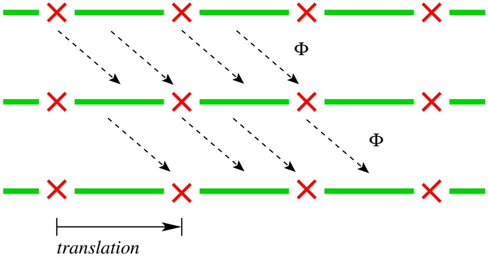


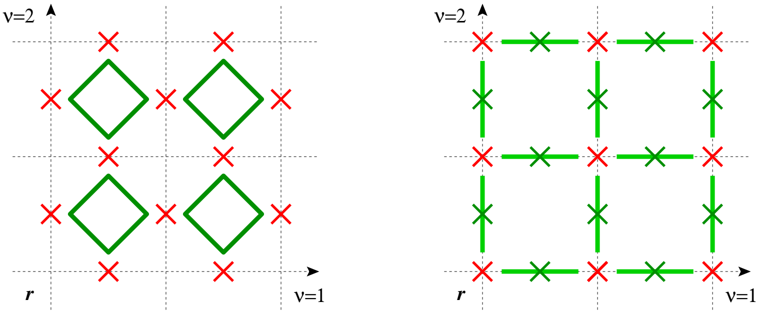
It becomes an **extra symmetry** of the model's **self-dual point**

$$\boxed{\mathcal{U}_{\rm d} H_{\rm I}[h, J = h]\mathcal{U}_{\rm d}^{\dagger} = H_{\rm I}[h, J = h]}$$

where the phase transition occurs.

The $\mathbb{Z}_2$ Higgs model $\quad (B_{(\mathbf{r},3)} \equiv \sigma^z_{(\mathbf{r},1)}\sigma^z_{(\mathbf{r}+\mathbf{e_1},2)}\sigma^z_{(\mathbf{r}+\mathbf{e_2},1)}\sigma^z_{(\mathbf{r},2)})$:

$$H_{\mathrm{AH}} = \sum_{\mathbf{r}} \left(J_x \sigma^x_{\mathbf{r}} + J_z B_{(\mathbf{r},3)}\right) + \sum_{\mathbf{r}} \sum_{\nu=1,2} \left(h_z \sigma^z_{\mathbf{r}} \sigma^z_{(\mathbf{r},\nu)} \sigma^z_{\mathbf{r}+\mathbf{e}_\nu} + h_x \sigma^x_{(\mathbf{r},\nu)}\right)$$

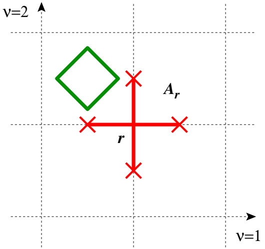


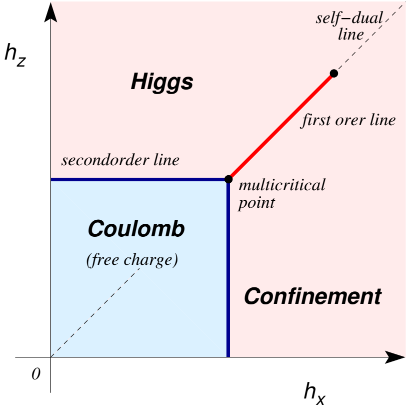
Symmetries and phase diagram of the $\mathbb{Z}_2$ Higgs model

The gauge symmetries are
$G_\mathbf{r} \equiv \sigma^x_r A_\mathbf{r}$, with

$$A_\mathbf{r} \equiv \sigma^x_{(\mathbf{r},1)} \sigma^x_{(\mathbf{r},2)} \sigma^x_{(\mathbf{r}-\mathbf{e}_1,1)} \sigma^x_{(\mathbf{r}-\mathbf{e}_2,2)}$$

the *star operator*.





There can be no spontaneous breakdown of gauge symmetries (Elitzur's theorem). But we can try to get rid of them to have easier access to the model's phase diagram. Dualities!

Topological quantum order in the Higgs model

The **bond algebra of the $\mathbb{Z}_2$ Higgs model** has at least one **dual representation** that "leaps to the eye:"

$$\sigma^x_{\mathbf{r}} \xrightarrow{\Phi_{\rm d}} A_{\mathbf{r}} \qquad\qquad B_{(\mathbf{r},3)} \xrightarrow{\Phi_{\rm d}} B_{(\mathbf{r},3)}$$

$$\sigma^z_{\mathbf{r}}\sigma^z_{(\mathbf{r},\nu)}\sigma^z_{\mathbf{r}+\mathbf{e}_\nu} \xrightarrow{\Phi_{\rm d}} \sigma^z_{(\mathbf{r},\nu)} \qquad\qquad \sigma^x_{(\mathbf{r},\nu)} \xrightarrow{\Phi_{\rm d}} \sigma^x_{(\mathbf{r},\nu)}$$

The Dual Hamiltonian

$$H_{\rm AH} \xrightarrow{\Phi_{\rm d}} \boxed{H_{\rm ETC} = \sum_{\mathbf{r}} \Big[\big(J_x A_{\mathbf{r}} + J_z B_{(\mathbf{r},3)}\big) + \sum_{\nu=1,2} \big(h_z \sigma^z_{(\mathbf{r},\nu)} + h_x \sigma^x_{(\mathbf{r},\nu)}\big)\Big]}$$

The Higgs model is dual to he **Toric Code** model in a magnetic field. But this extended toric code has **no gauge symmetries !!!!** Where did they go?

$$G_{\mathbf{r}} = \sigma^x_{\mathbf{r}} A_{\mathbf{r}} \xrightarrow{\Phi_{\rm d}} A_{\mathbf{r}} A_{\mathbf{r}} = \mathbb{1}$$

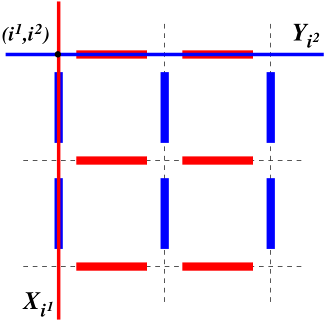
The duality has solved completely the gauge constraints!!!!

A Symmetry Principle for Dimensional Reduction and TQO

- What is the link between TQO and dimensional reduction?
- Models of TQO tipically display *d*-**dimensional gauge-like symmetries**, that combined with dimensional-reduction techniques can yield important information about phase diagrams.

$$H_{\text{POC}} = -\sum_{\mathbf{r}} (J_1 \; \sigma^x_{\mathbf{r}} \sigma^x_{\mathbf{r}+\mathbf{e_1}} + J_2 \; \sigma^y_{\mathbf{r}} \sigma^y_{\mathbf{r}+\mathbf{e_2}})$$

$$X_{i^1} = \prod_{i^2} \sigma^x_{i^1, i^2} \qquad Y_{i^2} = \prod_{i^1} \sigma^y_{i^1, i^2}$$



- *d* gauge-like symmetries have been proposed to be the symmetry principle underlying both TQO and dimensional reduction. (Cobanera et. al. arXiv:1110.2179v1 [cond-mat.stat-mech])

Dualities in numerical simulations: Dual boundary conditions for finite systems

- **Exact dualities for finite systems require special boundary condision**, called dual boundary conditions.

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- Dual boundary conditions are **model-specific**, and can be computed on a case-by-case basis straight from the bond algebra of the finite systems under consideration.

$$H_{\mathrm{I}}^{N} = h\sigma_i^x + \sum_{i=2}^{N}[J\sigma_{i-1}^z\sigma_i^z + h\sigma_i^x]$$

H_{I}^{N}

NOT Self-Dual, $E(h, J) \neq E(J, h)$

$H_{\mathrm{I}}^{N} + J\sigma_1^z$

Self-Dual, $E(h, J) = E(J, h)$