Random Planar Curves and Conformal Field Theory

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Outline

- recall some facts about 2d CFT
- what kind of field theories are being described, and their lattice versions
- the basics of SLE
- conformal restriction measures
- identification of the stress tensor and derivation of the Ward identities of $c = 0$ CFT
- extension to $c > 0$: conformal loop ensemble (CLE)

Conformal Field Theory

- massless, renormalised 2d euclidean QFT
- local operators which transform simply under conformal transformations $z \to f(z)$:

$$\phi(z,\bar{z}) \to f'(z)^{\Delta_\phi}\,\overline{f'(z)}^{\overline{\Delta}_\phi}\;\phi\big(f(z),\overline{f(z)}\,\big)$$

- stress tensor $T(z)$ generates infinitesimal conformal transformations $z \to z+\alpha(z)$ via insertion of

$$\int_C \frac{dz}{2\pi i}\alpha(z)T(z) + \text{c.c.}$$

  into correlation functions
- equivalent to OPEs

$$T(z)\cdot\phi(z_1,\bar{z}_1) = \frac{\Delta_\phi}{(z-z_1)^2}\phi(z_1,\bar{z}_1) + \frac{1}{z-z_1}\partial_{z_1}\phi(z_1,\bar{z}_1) + \cdots$$

$$T(z)\cdot T(z_1) = \frac{c/2}{(z-z_1)^4} + \frac{2}{(z-z_1)^2}T(z_1) + \frac{1}{z-z_1}\partial_{z_1}T(z_1) + \cdots$$

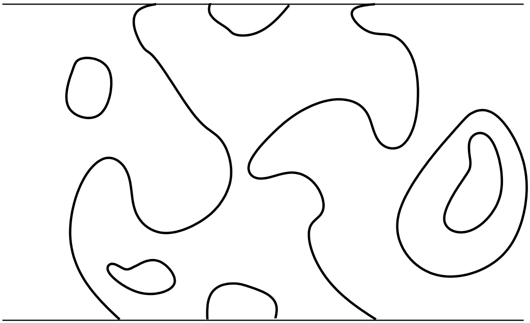
Example: $O(n)$ scalar field theory

- n-component field Φ_j, bare action

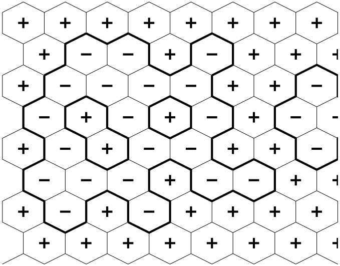
$$S = \int \Big[\sum_{j=1}^{n} \big((\partial \Phi_j)^2 + m_0^2 \Phi_j^2 \big) + \lambda_0 \big(\sum_{j=1}^{n} \Phi_j^2 \big)^2 \Big] d^2 r$$

- critical point at $m_R^2 = 0$ for $n \leq 2$
- RG fixed point at $\lambda_0 \to \infty$
- world-lines of particles do not cross

Space-imaginary time picture

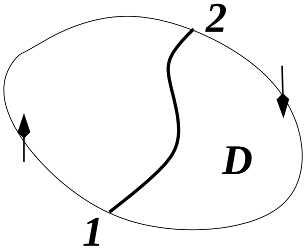


Lattice version

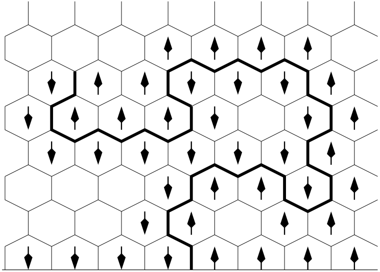


- gas of non-intersecting loops and open curves weighted by their total length, factor n for each closed loop, eg
  - $n = 1$: Ising model
  - $n = 2$: dual to Kosterlitz-Thouless transition
  - $n = 0$: self-avoiding walks (“quenched approximation”)

- in the continuum limit (at critical point) these loops become fractal curves - what is the measure on these?
  - or, what is the measure on just one of them?
  - specify conditions on the boundary of a simple connected domain $\mathcal{D}$ such that there is always a single open curve from r_1 to r_2:



- such curves can be ‘grown’ on the lattice by a discrete exploration process:



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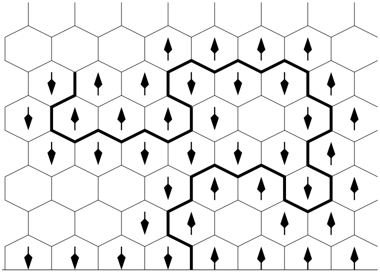
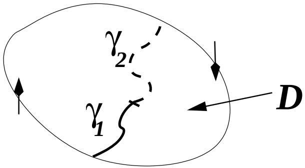


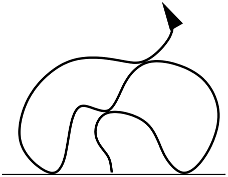
- SLE describes the continuous version of this

The postulates of SLE



- Denote the curve by γ, and divide it into two disjoint parts.
- conditional measure on γ_2 given γ_1 is the same as the unconditional measure on γ_2 in $\mathcal{D} \setminus \gamma_1$
- moreover this is conformally related to the measure on γ in $\mathcal{D}$

- choose $\mathcal{D}$ = upper half plane $\mathbf{H}$
- let K_t be the curve + all the regions enclosed by it at time t



- let $g_t(z)$ be the conformal mapping which sends $\mathbf{H} \setminus K_t$ to $\mathbf{H}$, normalised so that

$$g_t(z) \sim z + 0 + \frac{2t}{z} + \cdots \qquad (\text{as } z \to \infty)$$

- g_t sends the growing tip into a_t on the real axis
- the evolution of g_t satisfies the Loewner equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t}$$

- if curve is continuous so is a_t
- so instead of thinking about a measure on curves we can think about a measure on continuous functions a_t

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Theorem. [Schramm] *If above postulates hold then a_t is proportional to a standard Brownian motion.*
That is

$$a_t = \sqrt{\kappa}\, B_t$$

so that $\langle a_t \rangle = 0$, $\langle \left(a_{t_1} - a_{t_2}\right)^2 \rangle = \kappa |t_1 - t_2|$.

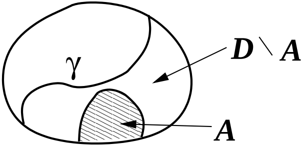
- one-parameter family of conformally invariant measures on curves labelled by κ
- many boundary and bulk scaling dimensions can be derived rigorously from the postulates of SLE [Lawler-Schramm-Werner]
- stochastic process $\Rightarrow$ Fokker-Planck equations (2nd order PDEs) $\rightleftharpoons$ BPZ differential equations of CFT following from condition that boundary field Φ_j satisfies $L_{-2}\Phi_j \propto L_{-1}^2\Phi_j$ [Bauer-Bernard]

$$n = -2\cos(4\pi/\kappa) \qquad (2 \leq \kappa \leq 8)$$

$$\text{central charge} \quad c = \frac{(3\kappa - 8)(6-\kappa)}{2\kappa}$$

- how we identify the stress tensor T for these random curves?
- can we derive the conformal Ward identities?

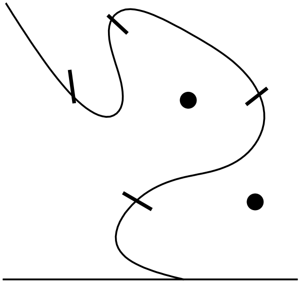
- start with the simplest case $n = 0$ ("quenched approximation")
- satisfies *conformal restriction*



- measure on γ restricted not to lie in A is the same as the measure we get by conformally mapping $\mathcal{D} \to \mathcal{D} \setminus A$
- expect this to be true for $n = 0$ but not in general, because 'vacuum processes' are sensitive to A.
- Theorems (1) `[L-S-W]` SLE satisfies this only for $\kappa = \frac{8}{3}$; (2) `[Werner]` there is a unique measure on single self-avoiding *loops* which satisfies restriction

What is the stress tensor?

- in Minkowski space $T_{\mu\nu}dS^{\nu}$ gives the energy-momentum flow across dS^{μ}
- its trace measures response to a dilatation (so vanishes at an RG fixed point)
- in Euclidean space its non-zero components measure the response of the medium to a local anisotropic shear
- in 2d it has two independent components $(T, \overline{T})$ which have 'spin' ± 2: under $z \to ze^{i\theta}$, $T \to e^{-2i\theta}T$, $\overline{T} \to e^{2i\theta}\overline{T}$
- leads to the following guess:



- slits of lengths $\{\epsilon_j\}$, at angles $\{\theta_j\}$, centred on points $\{z_j\}$
- let

$$P\big(\{\epsilon_j\},\{\theta_j\},\{z_j\}\big) = \mathrm{Pr}\big(\gamma \text{ intersects every slit}\big)$$

and let

$$Q\big(\{z_j\}\big) = \lim_{\epsilon_j\to 0}\prod_j (8/\pi\epsilon_j^2)\prod_j \int \frac{d\theta_j}{2\pi} e^{-2i\theta_j} P(\ldots)$$

Theorem. `[Doyon-Riva-JC]`: the limit exists and if we identify

$$Q\big(\{z_j\}\big) = \frac{\langle\phi(0)T(z_1)T(z_1)\ldots\phi(\infty)\rangle}{\langle\phi(0)\phi(\infty)\rangle}$$

then the RHS satisfies the conformal Ward identities with $c = 0$.

- *Proof:* based on conformal restriction applied to the probabilities that γ avoids subsets of the slits.

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- *Proof:* based on conformal restriction applied to the probabilities that γ avoids subsets of the slits.
- by conditioning γ also to pass around given points $\{\zeta_j\}$ and taking limits as they coincide, can generate a whole set of local fields which form a closed operator algebra
- $\Rightarrow$ complete and rigorous construction of a whole sector of the CFT

How do we make CFTs with $c > 0$?

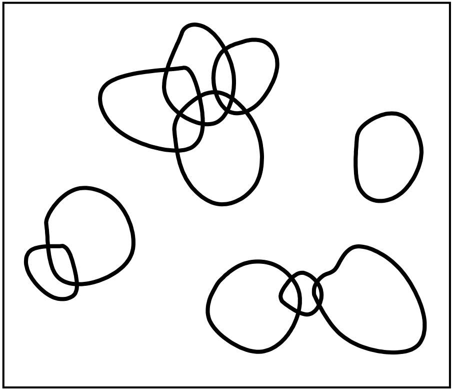
Conformal Loop Ensemble

`[Werner-Lawler-Sheffield]:`

- start with the (unique) measure on single self-avoiding loops
- partition function

$$Z \propto \int^{L} \frac{dR}{R} \sim \ln L$$

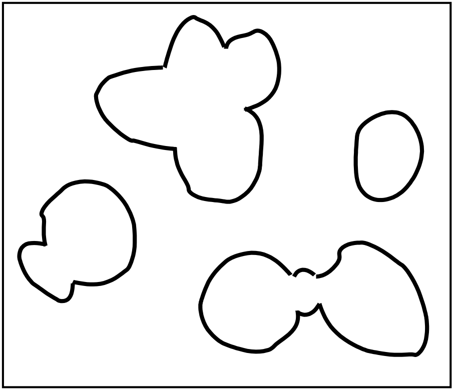
- let them rain down independently and uniformly for a 'time' τ



$$Z \sim e^{\text{const.}\tau \ln L}$$

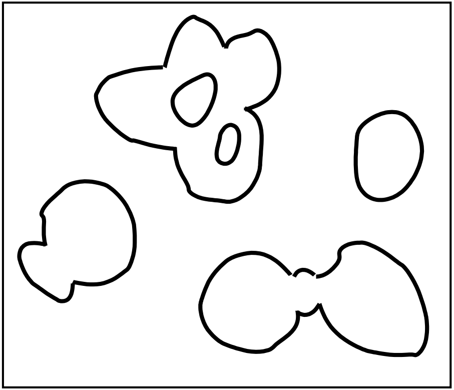
- for small enough τ these form disjoint clusters

- look only at the outermost boundaries:



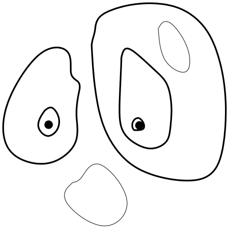
- these are conjectured to be the same as the outermost set of loops in the $O(n)$ model for $n > 0$

- to get the full nested set, fill them iteratively



- none of this changes Z, so central charge $c = \text{const.}\ \tau$
- if τ too large, get one big cluster ($\Rightarrow c > 1$)

The stress tensor for the conformal loop ensemble



- let $N(1,2)=$ number of loops separating z_1 and z_2
- this has a divergence $\propto \log\big((z_1-z_2)/a\big)$ from small loops, so subtract this and define

$$T(z) \propto \lim_{z_1 \to z_2} \left(\partial_{z_1} \partial_{z_2} N(1,2) - \frac{\text{const.}}{(z_1-z_2)^2} \right)$$

- $N(1,2)$ is conformally invariant, so the first term transforms with conformal weight 2
- subtraction leads to the conformal anomaly $c \neq 0$
- we can use restriction property to show that T satisfies conformal Ward identities as before

Summary

- SLE and its extensions give a (rigorous) geometrical picture of the continuum limit of systems which should also be described by CFT
- conformal invariance is manifest
- in the simplest case of conformal restriction we can identify the stress tensor and derive the Ward identities of a $c = 0$ CFT
- we can define a complete set of local correlation functions and show they satisfy expected OPEs
- by using the CLE we can extend this to theories with $c > 0$