# Quantum Criticality Beyond the Standard Model and Ultra Unification

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2111.10369, 2106.16248, w/Yi-Zhuang You (UCSD),
1910.14668 [JHEP], w/Zheyan Wan (YMSC)

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# Outline

1. **Ultra Unification** : Quantum Field Theory /Matter.
- Anomalies, Cobordisms.
- Logic: Three (two+one) Assumptions.
- Consequences: Gapped Topological/Gapless CFT sectors, Extra dim.
- (Gauge-) symmetry-breaking, -preserving, -extension mass or energy gap.
- Dirac/Majorana mass vs Interaction induced vs Topological mass/energy gap.
- $\Omega^5_{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4} = \Omega^4_{\mathrm{Pin}^+} = \mathbb{Z}_{16}$ - Path Integral.

2. **Quantum Criticality Beyond the Standard Model**
- $\Omega^5_{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathrm{Spin}(10)} = \mathbb{Z}_2$ and $w_2 w_3$ anomaly. - 4d boundary or 5d bulk criticality.
- GUT-Higgs fragmentary fermionic parton + Dark Gauge sector.

3. **Combined Synthesis - Conclusion**
- 15n vs 16n Weyl fermion scenarios (SM or GG vs other GUTs).
- Topological criticality or phase transition (topological order).
- Non-invertible Categorical Higher symmetries retraction.

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Standard Model (SM) with (15+1)n Weyl fermions coupled to Yang-Mills gauge $su(3)_c \times su(2)_{\mathrm{L}} \times u(1)_{\tilde{Y}}$ in representation (rep):

| | $\mathfrak{su}(3)$ | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)^{Y_1}$ | $\mathfrak{u}(1)^{X_1}$ |
|---|---|---|---|---|
| $u_L$, $d_L$ | **3** | **2** | 1 | 1 |
| $\bar{u}_R$ | $\bar{\mathbf{3}}$ | **1** | −4 | 1 |
| $\bar{d}_R$ | $\bar{\mathbf{3}}$ | **1** | 2 | −3 |

| | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)^{Y_1}$ | $\mathfrak{u}(1)^{X_1}$ |
|---|---|---|---|
| $\nu_{eL}$, $e_L$ | **2** | −3 | −3 |
| $\bar{\nu}_{eR}$ (?) | **1** | 0 | 5 |
| $\bar{e}_R$ | **1** | 6 | 1 |

(each Weyl fermion $\mathbf{2}_L$ of Spin(1,3))

$$\bar{d}_R \oplus l_L \oplus q_L \oplus \bar{u}_R \oplus \bar{e}_R \oplus \bar{\nu}_R$$
$$= (\overline{\mathbf{3}}, \mathbf{1})_{2,L} \oplus (\mathbf{1}, \mathbf{2})_{-3,L} \oplus (\mathbf{3}, \mathbf{2})_{1,L} \oplus (\overline{\mathbf{3}}, \mathbf{1})_{-4,L} \oplus (\mathbf{1}, \mathbf{1})_{6,L} \oplus (\mathbf{1}, \mathbf{1})_{0,L}.$$

SM gauge group $G_{\mathrm{SM}_q} \equiv \frac{\mathrm{SU}(3)_c \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\tilde{Y}}}{\mathbb{Z}_q}$ with $q = 1, 2, 3, 6$.

**Open Issues**: Neutrino $\bar{\nu}_R$ exists or not? How many $\bar{\nu}_R$? How do $\nu_L$ (of $l_L$) and $\bar{\nu}_R$ get masses? Dirac or Majorana masses?

**Other Resolution**: Ultra Unification. Other ways to get "mass," or replace neutrinos by 4d TQFT, CFT, or 5d invertible TQFT, etc.

**(Conventional) Quadratic mass**:
Dirac mass with some Higgs: $(\bar{\nu}_R \phi_H^\dagger \nu_L + \bar{\nu}_L \phi_H \nu_R)$.

Majorana mass: $\frac{\mathrm{i}\, m_{\text{Maj}}}{2}(\chi^{\mathrm{T}}\sigma^2\chi + \chi^\dagger\sigma^2\chi^*)$.

Both Dirac and Majorana masses:

$$\frac{1}{2}\Bigg( \Big( \Big(l_{L\nu_e}, l_{L\nu_\mu}, l_{L\nu_\tau}\Big)\frac{\langle\phi_H\rangle}{|\langle\phi_H\rangle|}, \chi^\dagger_{\nu_e}, \chi^\dagger_{\nu_\mu}, \chi^\dagger_{\nu_\tau} \Big) \begin{array}{c} \\ 3 \\ 3 \end{array}\!\! \begin{pmatrix} \overset{3}{0} & \overset{3}{M_{\text{Dirac}}} \\ M_{\text{Dirac}} & M_{\text{S}} \end{pmatrix} \begin{pmatrix} l_{L\nu_e}\frac{\langle\phi_H\rangle}{|\langle\phi_H\rangle|} \\ l_{L\nu_\mu}\frac{\langle\phi_H\rangle}{|\langle\phi_H\rangle|} \\ l_{L\nu_\tau}\frac{\langle\phi_H\rangle}{|\langle\phi_H\rangle|} \\ \chi^\dagger_{\nu_e} \\ \chi^\dagger_{\nu_\mu} \\ \chi^\dagger_{\nu_\tau} \end{pmatrix} + h.c. \Bigg).$$

Seesaw mechanism: 3 mass eigenstates have small mass
$\simeq \frac{|M_{\text{Dirac}}|^2}{|M_{\text{S}}|} \ll |M_{\text{Dirac}}|$.

**Other Resolution**: Ultra Unification. Other ways to get "mass," or replace neutrinos by 4d TQFT, CFT, or 5d invertible TQFT, etc. Insight: discrete $\mathbf{B}-\mathbf{L}$ symmetries, global anomalies, cobordism.

# Anomalies (invertible, local $\mathbb{Z}$ or global $\mathbb{Z}_n$ classes)

1. **HEP**: $(d-1)$dim 't Hooft anomalies — obstruction to gauge a global symmetry $G$ ($G$-bundle with $G$-connection). Lead to ill-defined $G$-gauge theory. But useful $G$ probe background field $A$ to constrain quantum dynamics. $\mathbf{Z}[A,\dots] \to \mathbf{Z}[A',\dots] = \mathrm{e}^{\mathrm{i}\Theta}\,\mathbf{Z}[A,\dots]$. Section of determinant line bundle.

Spacetime-internal sym
$G \equiv \left(\frac{G_{\text{spacetime}} \ltimes G_{\text{internal}}}{N_{\text{shared}}}\right) \equiv G_{\text{spacetime}} \ltimes_{N_{\text{shared}}} G_{\text{internal}}$.

2. **CondMat**: The internal part of $G$-symmetry cannot be local onsite symmetry at the deep UV (Planck or lattice cutoff scale). Anomalous non-onsite internal symmetry. Obstruction to gauge $G$.

3. **Math**: specified by $d$dim invertible TQFT known as cobordism invariant of cobordism group $\Omega^d_G \equiv \mathrm{TP}_d(G)$.

Dynamical fates with anomaly: No $G$-symmetric trivial gapped phase of $(d-1)$dim on the boundary of $d$dim nontrivial iTQFT vacuum.

# Classify all invertible anomalies for SM and GUT gauge groups:

(Quantum) Anomaly in Physics: **Boundary** Phenomenon vs **Bulk**.

**adjective** for anomalies:

- (1) invertible vs noninvertible.
- (2) $\mathbb{Z}$ vs $\mathbb{Z}_n$-class: **perturbative local** vs **nonperturbative global** anomaly.
- (3) probes: *gauge* anomaly vs mixed *gauge-grav.* vs *gravitational* anomaly.

- chiral or axial anomaly: chiral internal symmetry.

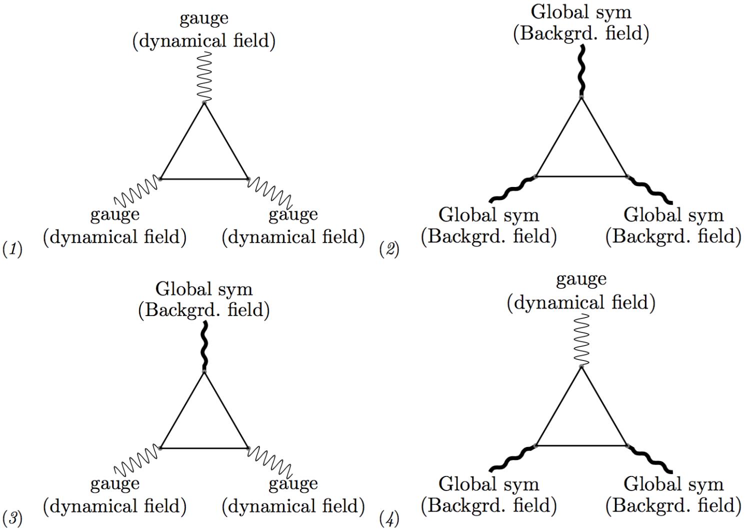
- (4) **bosonic** (SO/O/E) vs **fermionic** (Spin/Pin$^{\pm}$/DPin/EPin-structure).
- (5) **background** fields ('t Hooft anomalies) or **dynamical** fields

R.B.Laughlin '81, Witten '83-85, Callan-Harvey '84-'85, Dai-Freed '94, etc.

Cobordism: Kapustin'14, Kapustin-Thorngren-Turzillo-Wang'14 (proposed), Freed-Hopkins'16 (systematic), Wan-JW'18 arXiv:1812.11967: Encode higher-sym/classifying space. Wan-JW-Zheng'19 arXiv:1912.13504

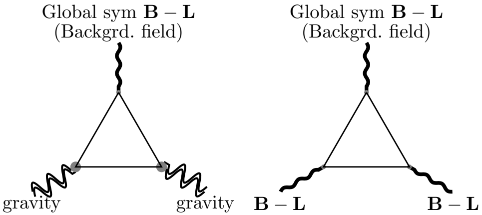
Application to SM: Garcia-Etxebarria-Montero'18, JW-Wen'18, Davighi-Gripaios-Lohitsiri'19, Wan-JW'19 arXiv:1910.14668

# The Use of Anomalies (perturbative local anomalies)




(*1*) Dynamical gauge anomaly. (*2*) 't Hooft anomaly of background (Backgrd.) fields. (*3*) Adler-Bell-Jackiw (ABJ) type of anomalies. (*4*) Anomaly that involves two background fields of global symmetries and one dynamical gauge field.

## $(\mathbf{B}-\mathbf{L})$-(gravity)$^2$ and $(\mathbf{B}-\mathbf{L})^3$ as $\mathbb{Z}$-class ABJ anomaly or 't Hooft anomaly




$$(\mathbf{B}-\mathbf{L})\text{-(gravity)}^2 \Rightarrow j_{\mathbf{B}} \quad : \quad N_{\text{generation}} \cdot (N_c/3) \cdot \Big(2-1-1\Big) = 0.$$

$$j_{\mathbf{L}} \quad : \quad N_{\text{generation}} \cdot \Big(2-1-n_{\nu_R}\Big) = N_{\text{generation}} \cdot (1-n_{\nu_R}).$$

$$(\mathbf{B}-\mathbf{L})^3 \Rightarrow j_{\mathbf{B}} \quad : \quad N_{\text{generation}} \cdot N_c \cdot (1/3)^3 \cdot \Big(2-1-1\Big) = 0.$$

$$j_{\mathbf{L}} \quad : \quad N_{\text{generation}} \cdot (1)^3 \cdot \Big(2-1-n_{\nu_R}\Big) = N_{\text{generation}} \cdot (1-n_{\nu_R})$$

$\mathrm{d} \star j_{\mathbf{B}} = 0$ but $\mathrm{d} \star (j_{\mathbf{B}} - j_{\mathbf{L}}) = 0$ only when $n_{\nu_R} = 1$.
Require the 16th Weyl fermion, or break $(\mathbf{B}-\mathbf{L})$, or?

Baryon - Lepton $\mathbf{B}-\mathbf{L}$ and a variant $X$ (Wilczek-Zee '79)

$$X \equiv X_1 = 5(\mathbf{B}-\mathbf{L}) - 4Y = 5(\mathbf{B}-\mathbf{L}) - \frac{2}{3}Y_1.$$

The $\mathrm{U}(1)_{\mathbf{B}-\mathbf{L}}$-grav$^2$ or $\mathrm{U}(1)_X$-grav$^2$ has a perturbative local (gauge-gravity) anomaly of $\mathbb{Z}$ class. Standard Lore: With gravity, SM with 15n Weyl fermions do not cancel anomaly. Some consequences:

- $\mathrm{U}(1)_{\mathbf{B}-\mathbf{L}}$ or $\mathrm{U}(1)_X$ global symmetry current is not conserved. (ABJ anomaly.)
- $\mathrm{U}(1)_{\mathbf{B}-\mathbf{L}}$ or $\mathrm{U}(1)_X$ is broken (to $\mathbb{Z}_{\text{even},X}$ or $\mathbb{Z}_2^F$).
- Add right-handed neutrinos.
- Neutrino massive? SM extension includes gravity.

Do these anomaly remain when we break $X$ to discrete? $\mathbb{Z}_{\text{even},X}$ or $\mathbb{Z}_2^F \subset \mathbb{Z}_{4,X} = Z(\mathrm{Spin}(10)) \subset \mathrm{U}(1)_X$, so $X^2 = (-1)^F$. Do we gain nonperturbative global anomalies?

Spoiler: In particular, we will find that the $\mathbb{Z}_{4,X}$-grav$^2$ has a nonperturbative global (gauge-gravity) anomaly of $\mathbb{Z}_{16}$ class.

Tachikawa-Yonekura'18, Garcia-Etxebarria-Montero'18, Hsieh'18, Guo-Ohmori-Putrov-Wan-JW'18, Wan-JW'19

## Anomalies of SM and GUT via cobordism:

hep-th/0607134: Freed. generalized cohomology.

Check the bordism group $\Omega_G^d \equiv \mathrm{TP}_d(G)$ —

1808.00009: Inaki Garcia-Etxebarria, Miguel Montero.
$G = \mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4,\ \mathrm{Spin} \times \mathrm{SU}(n),\ \mathrm{Spin} \times \mathrm{Spin}(n)$.

1809.11171: JW-Wen. $G = \frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^F},\ \mathrm{Spin} \times \mathrm{SU}(5)$.

1910.11277: Joe Davighi, Ben Gripaios, Nakarin Lohitsiri.
$G = \mathrm{Spin} \times G_{\mathrm{SM}_q}$, $\mathrm{Spin} \times \mathrm{Spin}(n)$, other GUTs

1910.14668: Wan-JW. (Subtle twisted cases.)
$G = \mathrm{Spin} \times G_{\mathrm{SM}_q}$, $\mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times G_{\mathrm{SM}_q}$, $\mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \mathrm{SU}(5)$,
$\mathrm{Spin} \times \mathrm{Spin}(n)$, $\frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F}$ e.g., $n = 10, 18$, other GUTs

Conservative view vs Optimistic view (New Arena Beyond the SM).

# Classify 4d Anomalies by 5d iTQFT/SPTs via Cobordism

## $G = \mathrm{Spin} \times G_{\mathrm{SM}_{q=6}}$, $\mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times G_{\mathrm{SM}_{q=6}}$ and $G = \mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \mathrm{SU}(5)$:

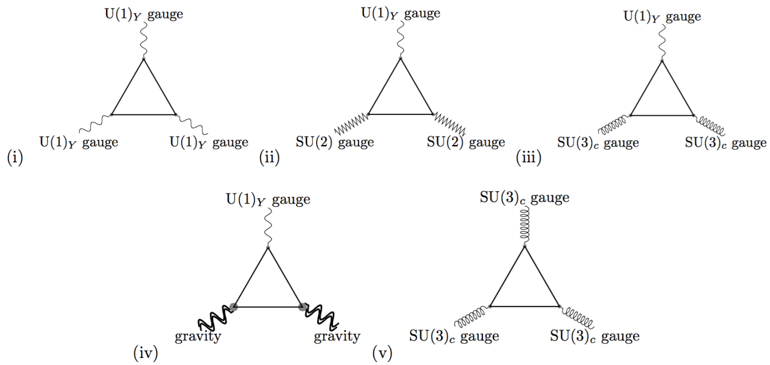
| Cobordism group $\mathrm{TP}_d(G)$ with $G_{\mathrm{SM}_q} \equiv (\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_q$ with $q = 1, 2, 3, 6$ | | |
|---|---|---|
| | classes | cobordism invariants |
| | | $G = \mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times G_{\mathrm{SM}_6}$ |
| 5d | $\mathbb{Z}^5 \times \mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}$ | $\mu(\mathrm{PD}(c_1(\mathrm{U}(3))))$, $c_1(\mathrm{U}(3))^2 \mathrm{CS}_1^{\mathrm{U}(3)}$, $\frac{(\mathcal{A}_{\mathbb{Z}_2})^2 \mathrm{CS}_3^{\mathrm{U}(2)} + \mathrm{CS}_1^{\mathrm{U}(3)} c_2(\mathrm{U}(2))}{2} \sim \frac{(\mathcal{A}_{\mathbb{Z}_2})^2 \mathrm{CS}_3^{\mathrm{U}(2)} + c_1(\mathrm{U}(3)) \mathrm{CS}_3^{\mathrm{U}(2)}}{2}$ $\frac{(\mathcal{A}_{\mathbb{Z}_2})^2 \mathrm{CS}_3^{\mathrm{U}(3)} + \mathrm{CS}_1^{\mathrm{U}(3)} c_2(\mathrm{U}(3)) + \mathrm{CS}_5^{\mathrm{U}(3)}}{2} \sim \frac{(\mathcal{A}_{\mathbb{Z}_2})^2 \mathrm{CS}_3^{\mathrm{U}(3)} + c_1(\mathrm{U}(3)) \mathrm{CS}_3^{\mathrm{U}(3)} + \mathrm{CS}_5^{\mathrm{U}(3)}}{2}$, $\mathrm{CS}_5^{\mathrm{U}(3)}$, $(\mathcal{A}_{\mathbb{Z}_2}) c_2(\mathrm{U}(3))$, $(\mathcal{A}_{\mathbb{Z}_2}) c_2(\mathrm{U}(2))$, $c_1(\mathrm{U}(3))^2 \eta'$, $\eta(\mathrm{PD}(\mathcal{A}_{\mathbb{Z}_2}))$ |
| | | $G = \mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \mathrm{SU}(5)$ |
| 5d | $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_{16}$ | $\frac{(\mathcal{A}_{\mathbb{Z}_2})^2 \mathrm{CS}_3^{\mathrm{SU}(5)} + \mathrm{CS}_5^{\mathrm{SU}(5)}}{2}$, $(\mathcal{A}_{\mathbb{Z}_2}) c_2(\mathrm{SU}(5))$, $\eta(\mathrm{PD}(\mathcal{A}_{\mathbb{Z}_2}))$ |

## $G = \mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{Spin}(10)$:

| $G = \mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{Spin}(N)$ for $N \geq 7$, e.g. $\mathrm{Spin}(N) = \mathrm{Spin}(10)$ or $\mathrm{Spin}(18)$ for SO(10) or SO(18) GUT | | |
|---|---|---|
| 5d | $\mathbb{Z}_2$ | $w_2(TM) w_3(TM) = w_2(V_{\mathrm{SO}(N)}) w_3(V_{\mathrm{SO}(N)})$ |

JW-Wen '18 1809.11171, Wan-JW'19 1910.14668 uses Adams spectral sequence.
Other related work uses Atiyah-Hirzebruch spectral sequence.

# I. (Local) Anomalies of $\mathrm{Spin}(d) \times G_{\mathrm{SM}_q}|_{(\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1))/\mathbb{Z}_q}$




1. $\mathrm{U}(1)_Y^3$: 4d anomaly from 5d $\mathrm{CS}_1^{\mathrm{U}(1)} c_1(\mathrm{U}(1))^2$ and 6d $c_1(\mathrm{U}(1))^3$
2. $\mathrm{U}(1)_Y$-$\mathrm{SU}(2)^2$: 4d anomaly from 5d $\mathrm{CS}_1^{\mathrm{U}(1)} c_2(\mathrm{SU}(2))$, 6d $c_1(\mathrm{U}(1))c_2(\mathrm{SU}(2))$
3. $\mathrm{U}(1)_Y$-$\mathrm{SU}(3)_c^2$: 4d anomaly from 5d $\mathrm{CS}_1^{\mathrm{U}(1)} c_2(\mathrm{SU}(3))$, 6d $c_1(\mathrm{U}(1))c_2(\mathrm{SU}(3))$
4. $\mathrm{U}(1)_Y$-(gravity)$^2$: 4d anomaly from 5d $\mu(\mathrm{PD}(c_1(\mathrm{U}(1))))$, 6d $\frac{c_1(\mathrm{U}(1))(\sigma - \mathrm{F}\cdot\mathrm{F})}{8}$
5. $\mathrm{SU}(3)_c^3$: 4d anomaly from 5d $\frac{1}{2}\mathrm{CS}_5^{\mathrm{SU}(3)}$, 6d $\frac{1}{2}c_3(\mathrm{SU}(3))$
6. 4d global Witten SU(2) anomaly from 5d $c_2(\mathrm{SU}(2))\tilde{\eta}$, 6d $c_2(\mathrm{SU}(2))\mathrm{Arf}$.
   It becomes part of local anomaly in $\mathbb{Z}$ when $q = 2, 6$.

Davighi-Gripaios-Lohitsiri1910.11277, Wan-JW 1910.14668, Davighi-Lohitsiri 2001.07731 JW 2006.16996.

## II. (Local+Global) Anomalies: $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\text{internal/gauge}}$

Focus on $\mathbb{Z}_{4,X} = Z(\mathrm{Spin}(10)) \subset \mathrm{U}(1)_X$ where $X = 5(\mathbf{B}-\mathbf{L}) - 4Y$.

$G = \mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\mathrm{SM}_q}$ and $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \mathrm{SU}(5)$:

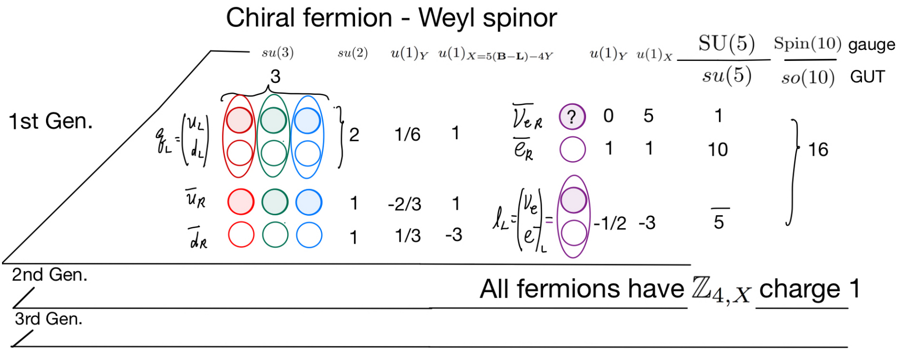
$$\mathrm{TP}_{d=5}(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\mathrm{SM}_q}) = \begin{cases} \mathbb{Z}^5 \times \mathbb{Z}_2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_{16}, & q = 1,3. \\ \mathbb{Z}^5 \times \mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}, & q = 2,6. \end{cases}$$

$$\mathrm{TP}_{d=5}(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \mathrm{SU}(5)) = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_{16}.$$

$\mathcal{A}_{\mathbb{Z}_2} = (\mathcal{A}_{\mathbb{Z}_4} \bmod 2) \in \mathrm{H}^1(M, \mathbb{Z}_2)$ is a generator $\mathrm{H}^1(\mathrm{B}(\mathbb{Z}_{4,X}/\mathbb{Z}_2^F), \mathbb{Z}_2)$ of $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$.

1. Mutated Witten SU(2) anomaly $c_2(\mathrm{SU}(2))\tilde{\eta}$:
   4d $\mathbb{Z}_2$ to $\mathbb{Z}_4$ global anomaly free ($q = 1,3$): $c_2(\mathrm{SU}(2))\eta'$.
   4d $\mathbb{Z}_2$ to $\mathbb{Z}$ local anomaly free ($q = 2,6$): $\frac{1}{2}\mathrm{CS}_1^{\mathrm{U}(2)} c_2(\mathrm{U}(2)) \sim \frac{1}{2} c_1(\mathrm{U}(2))\mathrm{CS}_3^{\mathrm{U}(2)}$.
2. $(\mathcal{A}_{\mathbb{Z}_2})c_2(\mathrm{SU}(2))$: 4d $\mathbb{Z}_2$ global anomaly free ($q = 2,6$)
3. $(\mathcal{A}_{\mathbb{Z}_2})c_2(\mathrm{SU}(3))$: 4d $\mathbb{Z}_2$ global anomaly free
4. $c_1(\mathrm{U}(1))^2\eta'$: 4d $\mathbb{Z}_4$ global anomaly free
5. $(\mathcal{A}_{\mathbb{Z}_2})c_2(\mathrm{SU}(5))$: 4d $\mathbb{Z}_2$ global anomaly free
6. $\eta(\mathrm{PD}(\mathcal{A}_{\mathbb{Z}_2}))$: $\Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4} \cong \Omega_4^{\mathrm{Pin}^+} = \mathbb{Z}_{16}$.
   4d $\mathbb{Z}_{16}$ global anomaly not canceled for $15 N_{\text{gen}}$ Weyl fermions. Alternative stories?

# Standard Model and GUT anomaly cancellation




| SM particle | SU(3) | SU(2) | $U(1)_Y$ | $U(1)_{\mathbf{B-L}}$ | $U(1)_X$ | $\mathbb{Z}_{4,X}$ | $\mathbb{Z}_2^F$ |
|---|---|---|---|---|---|---|---|
| $\bar{d}_R$ | $\bar{\mathbf{3}}$ | **1** | 1/3 | −1/3 | −3 | 1 | 1 |
| $l_L$ | **1** | **2** | −1/2 | −1 | −3 | 1 | 1 |
| $q_L$ | **3** | **2** | 1/6 | 1/3 | 1 | 1 | 1 |
| $\bar{u}_R$ | $\bar{\mathbf{3}}$ | **1** | −2/3 | −1/3 | 1 | 1 | 1 |
| $\bar{e}_R = e_L^+$ | **1** | **1** | 1 | 1 | 1 | 1 | 1 |
| $\bar{\nu}_R = \nu_L$ | **1** | **1** | 0 | 1 | 5 | 1 | 1 |
| $\phi_H$ | **1** | **2** | 1/2 | 0 | -2 | 2 | 0 |

# **Logic** to Ultra Unification

4d $\mathbb{Z}_{16}$ global anomaly not cancelled for $15 N_{\text{gen}}$ Weyl fermions.
Alternative stories for including or not
the 16th Weyl fermion ("sterile"/right-handed neutrinos)?

# Logic to Ultra Unification

**Assumptions:**

1. Standard Model (SM) $G_{\text{internal}}$: Lie algebra $su(3) \times su(2) \times u(1)$.
   $G_{\text{SM}_q} \equiv \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_q}, \quad q = 1, 2, 3, 6.$
2. $15 \times (N_{\text{gen}} = 3)$ Weyl fermions (spacetime Weyl spinors) observed, applicable to both SM and SU(5) GUT.
3. Discrete Baryon$-$Lepton number preserved (or not) at high energy: $\mathbb{Z}_{4,X\equiv 5(\mathbf{B}-\mathbf{L})-4Y} \supset \mathbb{Z}_2^F$, so $X^2 = (-1)^F$, also dynamically gauged at higher energy (if we embed the theory into quantum gravity).

**Check:** Perturbative local & nonperturbative global anomalies via cobordism.

# Logic to Ultra Unification

**Consequences:** $\mathbb{Z}_{16}$ anomaly index as total $(N_{\text{gen}} = 3) \cdot (15 = -1 \mod 16)$.

$$(-(N_{\text{gen}} = 3) + n_{\nu_{e,R}} + n_{\nu_{\mu,R}} + n_{\nu_{\tau,R}} + \text{new hidden sectors}) = 0 \mod 16.$$

Anomaly-cancellation?

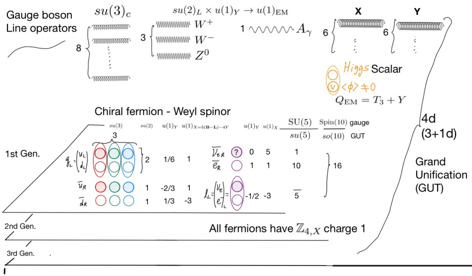
(1) **Standard Lore:** $R$-handed neutrino (16th Weyl) $n_{\nu_R} = 1$.
$\mathbb{Z}_{4,X}$ preserved (gapless, or **Dirac** mass) vs broken (gap) by **Majorana** mass.

(2) **My proposal:** New hidden sectors beyond SM:

1. $\mathbb{Z}_{4,X}$-symmetry-preserving anomalous gapped 4d TQFT (**Topological** Mass). (Topo.Green-Schwarz mechanism. Boundary topological order [2+1d Vishwanath-Senthil'12].)
2. $\mathbb{Z}_{4,X}$-5d invertible TQFT (SPTs) by cobordism invariant $\eta(\text{PD}(\mathcal{A}_{\mathbb{Z}_2}))$.
3. $\mathbb{Z}_{4,X}$-gauged-5d-(Symmetry)-Enriched Topological state (SETs) + gravity.
4. $\mathbb{Z}_{4,X}$-symmetry-breaking gapped phase (e.g. Landau phase or 4d TQFT).
5. $\mathbb{Z}_{4,X}$-symmetry-preserving gapless or breaking gapless (e.g., extra CFT).

HEP-PH **Gapped Extended Excitation/Objects beyond Particle Physics**.
HEP-PH **Gapless Unparticle CFT Physics**.

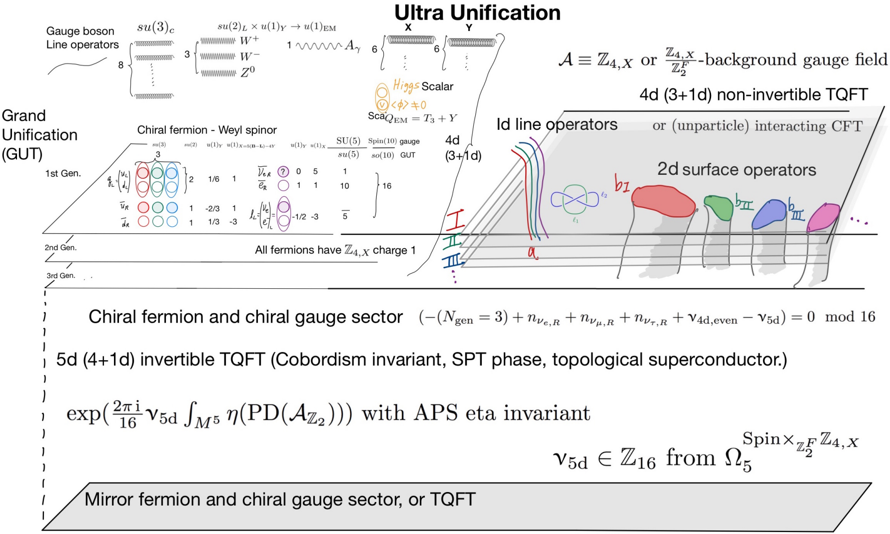
# Extra dimension 5d (4+1d)




Chiral fermion and chiral gauge sector

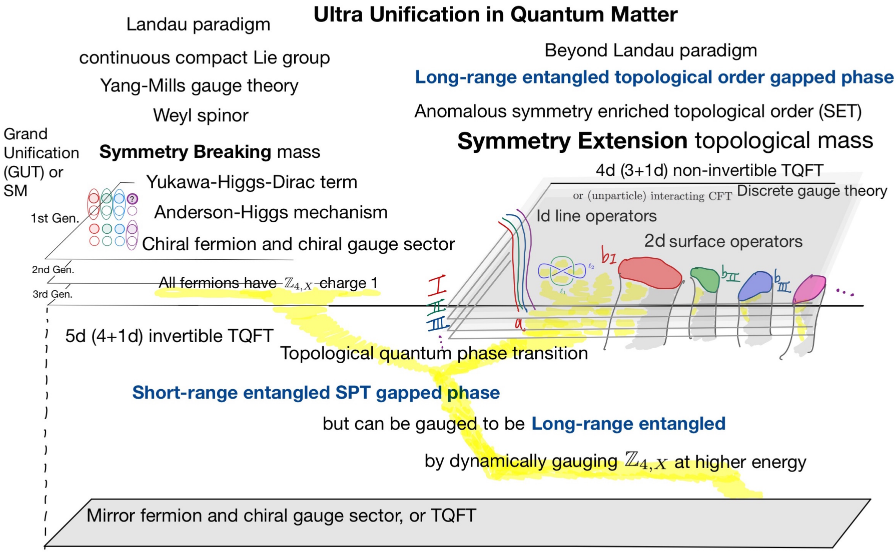
Extra Dimension 5d (4+1d)

# Ultra Unification 4d and 5d coupled quantum system




Application to Beyond SM: Neutrino Physics and Dark Matter.

# Ultra Unification 4d and 5d coupled quantum system



# Logic to Ultra Unification

$$(-(N_{\text{gen}}=3)+n_{\nu_{e,R}}+n_{\nu_{\mu,R}}+n_{\nu_{\tau,R}}+\text{new hidden sectors})=0 \mod 16$$

$$\boxed{(-(N_{\text{gen}}=3)+n_{\nu_{e,R}}+n_{\nu_{\mu,R}}+n_{\nu_{\tau,R}}+\nu_{\text{4d,even}}-\nu_{\text{5d}})=0 \mod 16.}$$

• $\nu_{\text{4d,odd}}=1,3,5,7,\dots\in\mathbb{Z}_{16}\Rightarrow$ Obstruction to symmetry-preserving gapped phase. No 4d TQFTs constructible.

Cordova-Ohmori'19 1912.13069.

• $\nu_{\text{4d,even}}=2,4,6,8,\dots\in\mathbb{Z}_{16}\Rightarrow$ Symmetry-preserving gapped phase. 4d TQFTs constructible.

Based on generalization of symmetry-extension method. JW-Wen-Witten'17 1705.06728. Hsieh'18 1808.02881, JW-Wan-Wang 1912.13504, JW 2006.16996, 2012.15860.

Possible implications to HEP-PH:
A HEP frontier beyond the conventional 0d particle physics relies on the TQFT and gapped extended objects (gapped 1d line and 2d surface operators or defects, etc., whose open ends carry deconfined fractionalized particle or anyonic string excitations), or (unparticle) CFT.

# More on $\mathbb{Z}_{16}$ global anomaly cancellation with $\nu_{\text{4d,even}}$

$\exp(\frac{2\pi\,\mathrm{i}}{16}\cdot\nu\cdot\eta(\mathrm{PD}(\mathcal{A}_{\mathbb{Z}_2})))$ for $\Omega_5^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4}\cong\Omega_4^{\mathrm{Pin}^+}=\mathbb{Z}_{16}$.

4d $\mathbb{Z}_{16}$ global anomaly not canceled for $15(N_{\text{gen}}=3)$ Weyl fermions.
New hidden gapped sector 5d iTQFT/SPTs/cobordism invariant (with $\frac{\nu_{\text{even}}}{2}\in\mathbb{Z}_8$):

$$\mathbf{Z}_{\text{5d-iTQFT}}^{(\nu_{\text{even}})}[\mathcal{A}_{\mathbb{Z}_4}]=\exp(\frac{2\pi\,\mathrm{i}}{8}\cdot(\frac{\nu_{\text{even}}}{2})\cdot\Big(\mathrm{ABK}(\mathrm{PD}((\mathcal{A}_{\mathbb{Z}_2})^3))\Big)\Big|_{M^5})$$
$$=\exp(\frac{2\pi\,\mathrm{i}}{8}\cdot(\frac{\nu_{\text{even}}}{2})\cdot\Big(4\cdot\mathrm{Arf}(\mathrm{PD}((\mathcal{A}_{\mathbb{Z}_2})^3))+2\cdot\tilde{\eta}(\mathrm{PD}((\mathcal{A}_{\mathbb{Z}_2})^4))+(\mathcal{A}_{\mathbb{Z}_2})^5\Big)\Big|_{M^5}).$$

$\mathcal{A}_{\mathbb{Z}_2}=(\mathcal{A}_{\mathbb{Z}_4}\text{ mod }2)\in\mathrm{H}^1(M,\mathbb{Z}_2)$ is a generator $\mathrm{H}^1(\mathrm{B}(\mathbb{Z}_{4,X}/\mathbb{Z}_2^F),\mathbb{Z}_2)$ of $\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathbb{Z}_{4,X}$.
Arf-Brown-Kervaire (ABK) for $\Omega_2^{\mathrm{Pin}^-}=\mathbb{Z}_8$. Arf for $\Omega_2^{\mathrm{Spin}}=\mathbb{Z}_2$. $\tilde{\eta}$ for $\Omega_1^{\mathrm{Spin}}=\mathbb{Z}_2$.

- $\nu_{\text{4d,even}}=2,4,8,\dots\in\mathbb{Z}_{16}\Rightarrow$ Symmetry-preserving gapped 4d TQFTs constructible.

**Mechanisms that can generate (symmetry-preserving) gapped phases?**
Gap (Similar to confinement) without $G$-symmetry (chiral $\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathbb{Z}_{4,X}$) breaking.

$$1\to\mathbb{Z}_2^K\to\mathbb{Z}_4^{\hat{G}}\xrightarrow{r}\mathbb{Z}_2^G=(\frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F})^G\to 1.$$
$$1\to[\mathbb{Z}_2]\to\mathrm{Spin}\times\mathbb{Z}_{4,X}\times G_{\mathrm{SM/GUT}}\to\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathbb{Z}_{4,X}\times G_{\mathrm{SM/GUT}}\to 1.$$

May require *additional symmetry extension*. *Non-abelian* fermionic $[\mathbb{Z}_2]$ TQFT.
**Ultra Unification** Functional Path Integral — YouTube video available.

## Math Physics Equations: **Ultra Unification** Path (Functional) Integral example

$$\mathbf{Z}_{\rm UU}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \mathbf{Z}_{\substack{\text{5d-iTQFT}/\\ \text{4d-SM+TQFT}}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \mathbf{Z}^{(-\nu_{5\rm d})}_{\text{5d-iTQFT}}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}^{(\nu_{4\rm d})}_{\text{4d-TQFT}}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}^{(n_{\nu_{e,R}}, n_{\nu_{\mu,R}}, n_{\nu_{\tau,R}})}_{\rm SM}[\mathcal{A}_{\mathbb{Z}_4}].$$

$$\mathbf{Z}_{\rm SM}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}][\mathcal{D}A][\mathcal{D}\phi] \dots \exp(\mathrm{i}\ S_{\rm SM}[\psi, \bar{\psi}, A, \phi, \dots, \mathcal{A}_{\mathbb{Z}_4}]\big|_{M^4})$$

$$S_{\rm SM} = \int_{M^4} \Big(\mathrm{Tr}(F_I \wedge \star F_I) - \frac{\theta_I}{8\pi^2} g_I^2 \mathrm{Tr}(F_I \wedge F_I)\Big) + \int_{M^4} \Big(\bar{\psi}(\mathrm{i}\,\not{D}_{A,\mathcal{A}_{\mathbb{Z}_4}})\psi$$

*(Gauge) Symmetry breaking* $\quad + |D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}\phi|^2 - \mathrm{U}(\phi) - (\psi_L^\dagger \phi(\mathrm{i}\sigma^2 \psi_L'^*) + \mathrm{h.c.})\Big)\ \mathrm{d}^4x$

$$(-(N_{\rm gen} = 3) + n_{\nu_{e,R}} + n_{\nu_{\mu,R}} + n_{\nu_{\tau,R}} + \nu_{4\rm d} - \nu_{5\rm d}) = 0 \mod 16.$$

$$\mathbf{Z}^{(-\nu_{5\rm d}=-2)}_{\text{5d-iTQFT}}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}^{(\nu_{4\rm d}=2)}_{\text{4d-TQFT}}[\mathcal{A}_{\mathbb{Z}_4}] = \sum_{c \in \partial'^{-1}(\partial[\mathrm{PD}(\mathcal{A}^3)])} \mathrm{e}^{\frac{2\pi\,\mathrm{i}}{8}\mathrm{ABK}(c \cup \mathrm{PD}(\mathcal{A}^3))}$$

$$\cdot \frac{1}{2^{|\pi_0(M^4)|}} \sum_{\substack{a \in C^1(M^4,\mathbb{Z}_2),\\ b \in C^2(M^4,\mathbb{Z}_2)}} (-1)^{\int_{M^4} a(\delta b + \mathcal{A}^3)} \cdot \mathrm{e}^{\frac{2\pi\,\mathrm{i}}{8}\mathrm{ABK}(c \cup \mathrm{PD}'(b))}.$$

$$1 \to [\mathbb{Z}_2] \to \mathrm{Spin} \times \mathbb{Z}_{4,X} \times G_{\rm SM/GUT} \to \mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\rm SM/GUT} \to 1.$$

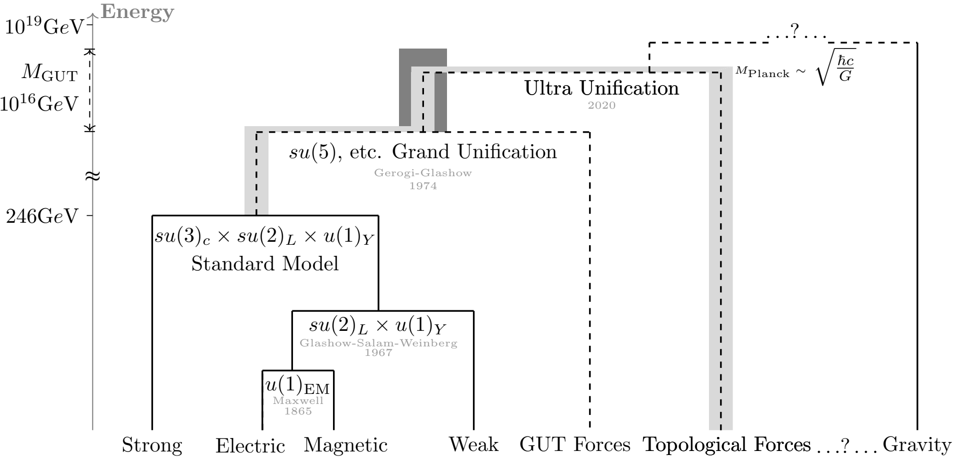
*Symmetry extension trivialize anomaly* (JW-Wen-Witten'17 1705.06728). Fermionic *non-abelian* TQFT.

# Other ways to give mass to "neutrinos"

**What is mass?** correlation function (of the corresponding operators/excitations/states) decaying exponentially.

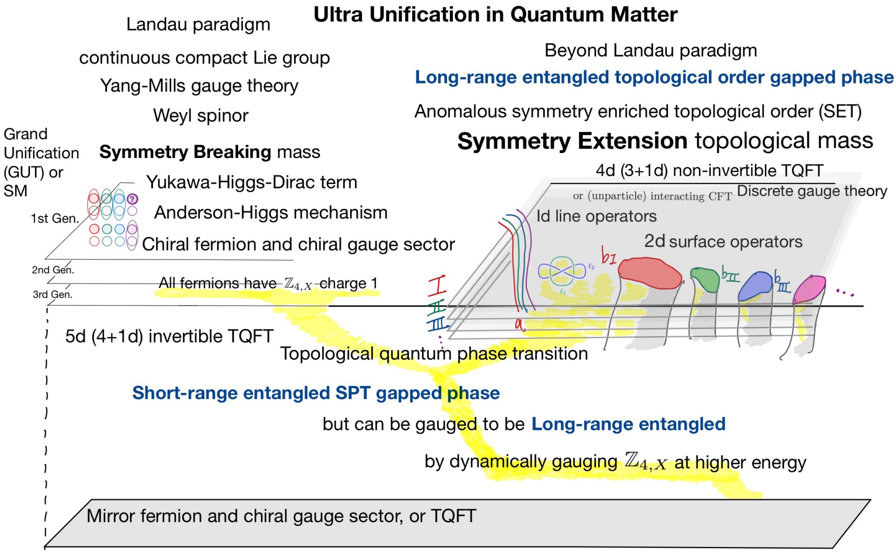
| Mass mechanism | Symmetry Property | Topological Order with low energy TQFT | Description: |
|---|---|---|---|
| (1) Anderson-Higgs | Symmetry Breaking | ✗ | Mean-Field |
| (2) Confinement: Chiral SB | Symmetry Breaking | ✗ | Mean-Field |
| (3) Confinement: s confinement | Symmetry Preserving | ✗ | Many-Body or Interacting |
| (4) Symmetric Mass Generation (Anomaly-Free) | Symmetry Preserving | ✗ | Many-Body or Interacting |
| (5) Symmetric Gapped Topological Order (Anomalous) | Symmetry Preserving | ✓ | Many-Body or Interacting |
| (6) Symmetry Extension Gapped (Anomaly Trivialized) | Symmetry Extension $K \to \tilde{G} \xrightarrow{\iota} G$ | ✓: TO/TQFT if $K$ is gauged, and if spacetime dim $d \geq 3$<br>✗: no TO/TQFT if $\tilde{G}$ remains ungauged. | Many-Body or Interacting |

# **Fundamental Physics** embodies Topological Matter




**HEP-phenomenology**: beyond 0d particle physics (to extended objects or unparticle CFT). New Direction:**Quantum Matter in Math/Physics**.

# Ultra Unification 4d and 5d coupled quantum system



# Outline

1. **Ultra Unification** : Quantum Field Theory /Matter.
- Anomalies, Cobordisms.
- Logic: Three (two+one) Assumptions.
- Consequences: Gapped Topological/Gapless CFT sectors, Extra dim.
- (Gauge-) symmetry-breaking, -preserving, -extension mass or energy gap.
- Dirac/Majorana mass vs Interaction induced vs Topological mass/energy gap.
- $\Omega^5_{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4} = \Omega^4_{\mathrm{Pin}^+} = \mathbb{Z}_{16}$ - Path Integral.

2. **Quantum Criticality Beyond the Standard Model**
- $\Omega^5_{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathrm{Spin}(10)} = \mathbb{Z}_2$ and $w_2 w_3$ anomaly. - 4d boundary or 5d bulk criticality.
- GUT-Higgs fragmentary fermionic parton + Dark Gauge sector.

3. **Combined Synthesis - Conclusion**
- 15n vs 16n Weyl fermion scenarios (SM or GG vs other GUTs).
- Topological criticality or phase transition (topological order).
- Non-invertible categorical Higher symmetries retraction.

Wilczek et al: “Gauge group” of GUT is a key issue.

Seiberg et al: “Gauge group” is not a physical description of a theory. Many dualities.

Let us provide a resolution.
JW-YZYou, arXiv:2106.16248, 2111.10369.

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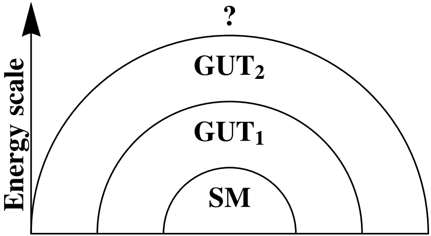
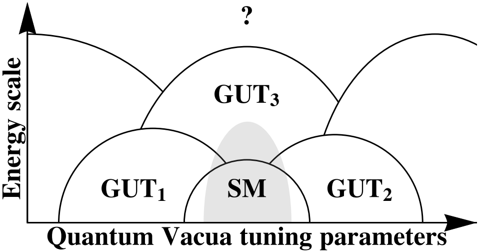
Seiberg et al: “Gauge group” is not a physical description of a theory. Many dualities.

Let us provide a resolution.
JW-YZYou, arXiv:2106.16248, 2111.10369.

• Standard lore: our vacuum governed by one of the candidate SMs, while lifting towards one of Grand Unifications (GUTs) at higher energy scales.

• In contrast, we introduce an alternative viewpoint that the SM is a low energy quantum vacuum arising from various neighbor GUT vacua competition in an immense quantum phase diagram.




Deformation class of QFT (Seiberg. etc.),
Deformation class of Quantum Gravity (McNamara-Vafa, etc.)

- Spacetime-Internal Symmetries. $G \equiv (\frac{G_{\text{spacetime}} \ltimes G_{\text{internal}}}{N_{\text{shared}}})$
- Anomalies.
- Cobordism class.
- Deformable on the same Hilbert space.

We will first treat the internal symmetry as a global symmetry. $G_{\text{internal}}$ is physical. In this case, we focus on 0-symmetry. We will dynamically gauge $G_{\text{internal}}$ later. Then, there will also be considerations of:
◦ higher-symmetries,
◦ invertible symmetries or non-invertible (categorical) symmetries.

# Deformation class of QFT: cobordism class

For 4d SM/BSM physics, we propose such a 5d cobordism invariant:

$$\mathbf{Z}_{\text{5d-iTQFT}}^{(\mathrm{p},\nu)} \equiv \boxed{\exp(\frac{2\pi \mathrm{i}}{16} \cdot \nu \cdot \eta(\mathrm{PD}(\mathcal{A}_{\mathbb{Z}_4} \mod 2))\Big|_{M^5})} \cdot \boxed{\exp(\mathrm{i}\pi \cdot \mathrm{p} \cdot \int_{M^5} w_2 w_3)},$$

with $\nu \in \mathbb{Z}_{16}$, $\mathrm{p} \in \mathbb{Z}_2$, .

- The $\mathcal{A}_{\mathbb{Z}_4} \in \mathrm{H}^1(M, \mathbb{Z}_{4,X})$ is a cohomology class discrete gauge field of the $\mathbb{Z}_{4,X}$-symmetry.

- A 4d Atiyah-Patodi-Singer (APS) $\eta$ invariant $\equiv \eta_{\mathrm{Pin}^+} \in \mathbb{Z}_{16}$.
A 4d (3+1d) topological superconductor, protected by time-reversal $\mathrm{T}^2 = (-1)^F$.

- Stiefel-Whitney (SW) characteristic class: $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(10)})$.

• In general, we can regard the SM arising near the quantum criticality (critical regions) between the competing neighbor vacua.

• In particular detail, we demonstrate how the $su(3) \times su(2) \times u(1)$ SM with 16n Weyl fermions arisen near the quantum criticality between the competition of Georgi-Glashow $su(5)$ model and Pati-Salam $su(4) \times su(2) \times su(2)$ model.

• Internal symmetry as a global symmetry: Deconfined quantum criticality (Senthil-Vishwanath-Balents-Sachdev-Fisher 2003) generalized to 3+1d.

• Internal symmetry is dynamically gauged then as in our vacuum.

# What is criticality? What is a phase transition?

• **Criticality**: Gapless excitations (e.g., massless, conformal) and with an infinite correlation length, it can be either

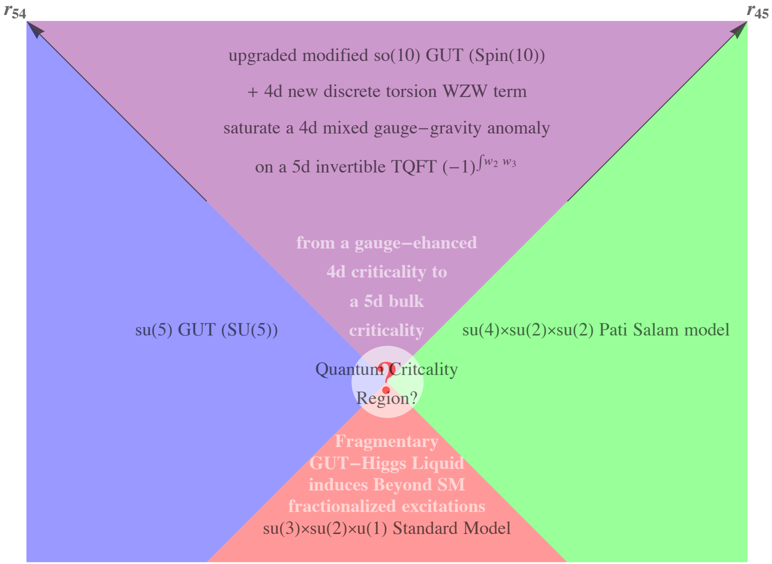
(i) a **continuous phase transition** as an unstable critical point/line/etc. as an unstable renormalization group (RG) fixed point which has at least one relevant perturbation in the phase diagram,

(ii) a **critical phase** as a stable critical region controlled by a stable RG fixed point which does not have any relevant perturbation in the phase diagram.

• **Phase transition**:

**continuous phase transition** (second or higher order, gapless modes).

**discontinuous phase transition** (first-order, without gapless modes, and with a finite correlation length).




# We propose a BSM Landscape (not Swampland)

$$\mathrm{U}(\Phi_{\mathbf{R}}) = \Big(r_{45}(\Phi_{45})^2 + \lambda_{45}(\Phi_{45})^4\Big) + \Big(r_{54}(\Phi_{54})^2 + \lambda_{54}(\Phi_{54})^4\Big) + \dots .$$

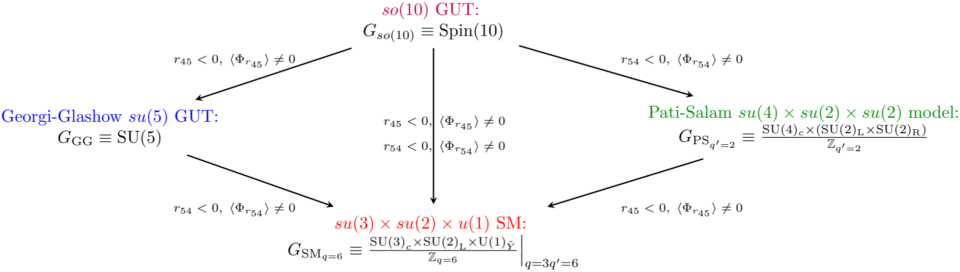
• To manifest a Beyond-the-Standard-Model (BSM) and Beyond-Landau-Ginzburg quantum criticality between Georgi-Glashow and Pati-Salam models, we introduce a parent effective field theory of a modified *so*(10) GUT (with a Spin(10) gauge group) plus a new 4d discrete torsion class of Wess-Zumino-Witten-like term that saturates a nonperturbative global mixed gauge-gravity anomaly captured by a 5d invertible topological field theory $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(10)})$.

• We show an analogous gapless 4d deconfined quantum criticality with new BSM fractionalized fragmentary excitations of Color-Flavor separation, and gauge enhancement including a Dark Gauge force sector.

• If the internal symmetries are dynamically gauged (as they are in our quantum vacuum), we show the 4d criticality as a boundary criticality such that only appropriately gauge enhanced dynamical GUT gauge fields can propagate into an extra-dimensional 5d bulk.

# Lie Group Embedding and Symmetry Breaking

$$
\begin{array}{ccc}
\bar{G}_{\mathrm{SM}_6} \equiv \mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \frac{\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y}{\mathbb{Z}_6} & \hookrightarrow & \bar{G}_{\mathrm{GG}} \equiv \mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \mathrm{SU}(5) \\
\downarrow & & \downarrow \\
\bar{G}_{\mathrm{PS}_2} \equiv \mathrm{Spin} \times_{\mathbb{Z}_2^F} \frac{\mathrm{Spin}(6) \times \mathrm{Spin}(4)}{\mathbb{Z}_2} & \hookrightarrow & \bar{G}_{so(10)} \equiv \mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)
\end{array}
$$



$S^{\text{WZW}}_{\text{GUT}}$ action: Modified $so(10)$ GUT (Spin(10)) plus a new 4d discrete torsion class of WZW-like term that saturates $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(10)})$ anomaly from $\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) = \mathbb{Z}_2$, not from $\mathrm{TP}_5(\mathrm{Spin} \times \mathrm{Spin}(10)) = 0$.

$$
\begin{aligned}
S_{\text{YM-Weyl}} &= \int_{M^4} \mathrm{Tr}(F \wedge \star F) + \mathrm{d}^4 x \big(\psi_L^\dagger (\mathrm{i}\bar{\sigma}^\mu D_{\mu,A}) \psi_L\big), \\
S_{\text{Higgs}} &= \int_{M^4} \mathrm{d}^4 x \big(|D_{\mu,A} \Phi_{\mathbf{R}}|^2 - \mathrm{U}(\Phi_{\mathbf{R}})\big), \\
S_{\text{Yukawa}} &= \int_{M^4} \mathrm{d}^4 x \Big(\frac{1}{2}\phi^{\intercal} \Phi^{\mathrm{bi}} \phi + \frac{1}{2}\sum_{a=1}^{5} \big(\psi_L^{\intercal}\, \mathrm{i}\sigma^2 (\phi_{2a-1}\Gamma_{2a-1} - \mathrm{i}\phi_{2a}\Gamma_{2a})\psi_L + \text{h.c.}\big)\Big), \\
S^{\text{WZW}} &= \frac{1}{\pi}\int_{M^5} \mathcal{B}(\Phi_{\mathbf{54}}) \wedge \mathrm{d}\mathcal{C}(\Phi_{\mathbf{45}})\Big|_{M^4 = \partial M^5} = \pi \int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}}) \smile \delta C(\hat{\Phi}^{\mathrm{bi}}))\Big|_{M^4 = \partial M^5}.
\end{aligned}
$$

Spin(10) rep: $\psi_L$ in **16**, $\phi$ in **10**, $\Phi^{\mathrm{bi}}$ in **100**, $\tilde{\Phi}^{\mathrm{bi}} = \Phi_{\mathbf{54}}$ in **54**, $\hat{\Phi}^{\mathrm{bi}} = \Phi_{\mathbf{45}}$ in **45**.

$S^{\text{WZW}}$ on a closed $M^5$ is a 5d invertible TQFT $w_2 w_3 = w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(10)})$.
Wang-Potter-Senthil 1306.3238, Kravec-McGreevy-Swingle 1409.8339, JW-You 2106.16248.

Fractionalization (by PW Anderson et al.: Emergence):
Quantum Matter (Iso)Spin phases vs Higgs field:
- Order (Symmetry breaking).
- Disorder (Symmetry preserving).
- Fractionalization (partons, emergent gauge fields).

Long-range entangled Resonating Valence Bond (RVB).

## Bosonic construction of WZW term

Composite GUT-Higgs

$$\Phi_{ab}^{\rm bi} = \phi_a \phi_b \text{ contain} \begin{cases} \mathrm{Tr}\Phi^{\rm bi} = \sum_a \Phi_{aa}^{\rm bi} \text{ gives } \Phi_{\mathbf{R}} = \Phi_{\mathbf{1}} \text{ in } \mathbf{1}_{\rm S}. \\ \hat{\Phi}^{\rm bi} \equiv \Phi_{[a,b]}^{\rm bi} = \frac{1}{2}(\Phi_{ab}^{\rm bi} - \Phi_{ba}^{\rm bi}) = \frac{1}{2}(\phi_a\phi_b - \phi_b\phi_a) = \frac{1}{2}[\phi_a, \phi_b] = \Phi_{\mathbf{45}}. \\ \tilde{\Phi}^{\rm bi} \equiv \Phi_{\{a,b\}}^{\rm bi} = \frac{1}{2}(\Phi_{ab}^{\rm bi} + \Phi_{ba}^{\rm bi}) = \frac{1}{2}(\phi_a\phi_b + \phi_b\phi_a) = \frac{1}{2}\{\phi_a, \phi_b\} = \Phi_{\mathbf{54}}. \end{cases}$$

## Fermionic parton construction of WZW term

Fragmentary GUT-Higgs with emergent gauge field

$$\Phi_{ab}(x) \sim \xi_a^\dagger(x) \exp(\mathrm{i} \int_x^x a_{\mu,\text{gauge}}^{\text{dark}} \, \mathrm{d} x^\mu) \xi_b(x)$$

# Bosonic construction of WZW term

Homotopy group

| | | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ |
|---|---|---|---|---|---|---|---|
| GG | $\frac{\mathrm{O}(10)}{\mathrm{U}(5)}$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| PS | $\frac{\mathrm{O}(10)}{\mathrm{O}(6)\times\mathrm{O}(4)}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}^2$ | $\mathbb{Z}_2^2$ |

Cohomology group:
GG $C(\hat{\Phi}^{\mathrm{bi}}) = B'(\hat{\Phi}^{\mathrm{bi}}) \in \mathrm{H}^2(\mathrm{O}(10)/\mathrm{U}(5), \mathbb{Z}_2) = \mathbb{Z}_2^2$ and $\mathrm{H}^2(\mathrm{SO}(10)/\mathrm{U}(5), \mathbb{Z}_2) = \mathbb{Z}_2$.
PS $B(\tilde{\Phi}^{\mathrm{bi}}) \in \mathrm{H}^2(\mathrm{O}(10)/(\mathrm{O}(6) \times \mathrm{O}(4)), \mathbb{Z}_2) = \mathbb{Z}_2^2$.

$$\exp(\mathrm{i}\, S^{\mathrm{WZW}}[\Phi]) = \exp(\mathrm{i}\pi \int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}}) \smile \delta B'(\hat{\Phi}^{\mathrm{bi}})) = \exp(\mathrm{i}\, 2\pi \int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}}) \smile \mathrm{Sq}^1 B'(\hat{\Phi}^{\mathrm{bi}}))\Big|_{M^4=\partial M^5}$$

(This is a bosonic construction. We will show next the fermion parton construction.)

Match 4d anomaly of 5d invertible TQFT:
$\exp(\mathrm{i}\pi \int_{M^5} w_2(TM) w_3(TM)) = \exp(\mathrm{i}\pi \int_{M^5} w_2(V_{\mathrm{SO}(10)}) w_3(V_{\mathrm{SO}(10)}))$

# Fermionic parton construction of WZW term

$S^{\text{WZW}}_{\text{GUT}}$ action: Modified $so(10)$ GUT (Spin(10)) plus a new 4d discrete torsion class of WZW-like term that saturates $w_2 w_3(TM) = w_2 w_3(V_{\text{SO}(10)})$ anomaly.

$$
\begin{aligned}
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S^{\text{WZW}}_{\text{QED}'_4} &= \int_{M^4} \mathrm{d}^4 x\, \bar{\xi}(\mathrm{i}\gamma^\mu D'_\mu - \tilde{\Phi}^{\text{bi}} - \mathrm{i}\gamma^{\text{FIVE}}\hat{\Phi}^{\text{bi}})\xi.\\
S^{\text{WZW}}_{\text{QED}'_5} &= \int_{M^5} \mathrm{d}^5 x\, \bar{\xi}(\mathrm{i}\tilde{\gamma}^\mu D'_\mu - m - \tilde{\gamma}^5\tilde{\Phi}^{\text{bi}} - \tilde{\gamma}^6 \mathrm{i}\hat{\Phi}^{\text{bi}} - \mathrm{i}\tilde{\gamma}^5\tilde{\gamma}^\mu\tilde{\gamma}^\nu\mathcal{B}_{\mu\nu} - \mathrm{i}\tilde{\gamma}^6\tilde{\gamma}^\mu\tilde{\gamma}^\nu\mathcal{C}_{\mu\nu})\xi.
\end{aligned}
$$

Spin(10) rep: $\psi_L$ in $\mathbf{16}$, $\xi$ in $\mathbf{10}$, $\phi$ in $\mathbf{10}$, $\Phi^{\mathrm{bi}}$ in $\mathbf{100}$, $\tilde{\Phi}^{\mathrm{bi}} = \Phi_{\mathbf{54}}$ in $\mathbf{54}$, $\hat{\Phi}^{\mathrm{bi}} = \Phi_{\mathbf{45}}$ in $\mathbf{45}$.
$D'_\mu = \nabla_\mu - \mathrm{i}\, a^{\text{dark}}_{\mu,\text{gauge}} - \mathrm{i}\, g A_\mu$ of $\mathrm{U}(1)'^{\text{dark}}_{\text{gauge}}$ and $\mathrm{Spin}(10)$.

4d $\xi$ in $\mathbf{2}_L \oplus \mathbf{2}_R$ of $\mathrm{Spin}(1,3)$ and 5d $\xi$ in $2 \times \mathbf{4}$ of $\mathrm{Spin}(1,4)$.

# Check QED-WZW term produces $w_2 w_3$ anomaly

• Dirac fermion $\xi$ in $\mathbf{2}_L \oplus \mathbf{2}_R$ of Spin(3,1) and $(1, \mathbf{10})$ of $\mathrm{U}(1)' \times \mathrm{SO}(10)$.

$\xi$ Rep is incompatible with $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$, we need to introduce a new fermion parity $\mathbb{Z}_2^{F'}$ for a $\mathrm{DSpin} \equiv (\mathbb{Z}_2^F \times \mathbb{Z}_2^{F'}) \rtimes \mathrm{SO}$ structure.

$G_{\text{QED}_4'} \equiv \mathrm{Spin}' \times_{[\mathbb{Z}_2^{F'}]} [\mathrm{U}(1)'] \times \mathrm{SO}(10) \equiv \mathrm{Spin}^{c'} \times \mathrm{SO}(10)$.

$G^{\text{modified}}_{so(10)\text{-GUT}} \equiv (\mathrm{DSpin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) \times_{[\mathbb{Z}_2^{F'}]} [\mathrm{U}(1)']$.

∘ Additional (emergent and gauged) symmetry to forbid quadratic mass:

$\mathrm{U}(1)' : \xi \to e^{\mathrm{i}\theta}\xi$ forbids $\xi^{\mathrm{T}}_{\mathrm{L/R}}\,\mathrm{i}\sigma^2 \xi_{\mathrm{L/R}}$.

$\mathbb{Z}_2^{\mathrm{CP}'} : \xi(t, \vec{x}) \to \gamma^0 \gamma^{\text{FIVE}} \xi^*(t, -\vec{x})$ forbids $\bar{\xi}\xi$.

$\mathbb{Z}_2^{\mathrm{T}'} : \xi(t, \vec{x}) \to \mathcal{K}\gamma^0 \gamma^{\text{FIVE}} \xi(-t, \vec{x})$ forbids $\mathrm{i}\bar{\xi}\gamma^{\text{FIVE}}\xi$.

∘ Use the new SU(2) anomaly to check $w_2 w_3$ anomaly:

• $\xi$ viewed as in Weyl $\mathbf{2}_L$ of Spin(3,1) and $(\mathbf{2}, \mathbf{10})$ of $\mathrm{SU}(2)' \times \mathrm{SO}(10)$.

$G_{\text{QCD}_4'} = \mathrm{Spin}' \times_{[\mathbb{Z}_2^{F'}]} [\mathrm{SU}(2)'] \times \mathrm{SO}(10) \equiv \mathrm{Spin}^{h'} \times \mathrm{SO}(10)$.

$G^{\text{modified}}_{so(10)\text{-GUT}} \equiv (\mathrm{DSpin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) \times_{[\mathbb{Z}_2^{F'}]} [\mathrm{SU}(2)']$.

$$\begin{array}{ccccccccc} \mathrm{U}(1)' \times \mathrm{SO}(10) \hookrightarrow \mathrm{SU}(2)' \times \mathrm{SO}(10) & \hookrightarrow & \mathrm{Sp}(10) & \hookleftarrow & \mathrm{Sp}(2)\times\mathrm{Sp}(8) & \hookleftarrow & \mathrm{SU}(2)''\times \mathrm{Sp}(8) \\ \mathbf{10}_1 \quad (\mathbf{2},\mathbf{10}) & \sim & \mathbf{20} & \sim & (\mathbf{4},\mathbf{1})\oplus(\mathbf{1},\mathbf{16}) & \sim & (\mathbf{4},\mathbf{1})\oplus(\mathbf{1},\mathbf{16}). \end{array}$$

| $d$ | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|
| $\mathrm{TP}_d(\mathrm{Spin}\times \mathrm{SU}(2))$ | $\mathbb{Z}_2$ | $\mathbb{Z}^2$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | ... |
| $\mathrm{TP}_d(\mathrm{Spin}\times_{\mathbb{Z}_2} \mathrm{SU}(2))$ | 0 | $\mathbb{Z}^2$ | 0 | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | ... |
| $\mathrm{TP}_d(\mathrm{Spin}\times_{\mathbb{Z}_2} \mathrm{Spin}(10))$ | 0 | $\mathbb{Z}^2$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | ... |

bulk $d$d and boundary $(d-1)$d. $d=5$:

1st $\mathbb{Z}_2$: Witten anomaly. iTQFT as $c_2(\mathrm{SU}(2))\tilde{\eta} \equiv \tilde{\eta}\mathrm{PD}(c_2(\mathrm{SU}(2)))$.
2nd $\mathbb{Z}_2$ detected on non-spin $M^5$: $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(3)})$.

also $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(10)})$

JW-Wen-Witten'18 1810.00844

| SU(2) isospin | 0 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | 2 | $\frac{5}{2}$ | 3 | $\frac{7}{2}$ | mod 4 | $2r+\frac{1}{2}$ | $4r+\frac{3}{2}$ | mod 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| SU(2) Rep **R** (dim) | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | mod 8 | $\mathbf{4}r+\mathbf{2}$ | $\mathbf{8}r+\mathbf{4}$ | mod 8 |
| Witten SU(2) anomaly | | ✓ | | | | ✓ | | | | ✓ | | |
| New SU(2) anomaly | | | | ✓ | | | | | | | ✓ | |

$\mathbb{Z}_n$ class: torsion part, or **nonperturbative global anomaly**.

JW-Wen arXiv:1809.11171, Wan-JW arXiv:1812.11967: Encode higher-symmetry/classifying space.

Works on $w_2 w_3$: Kapustin, Thorngren, Wen, Fidkowski-Haah-Hastings, Chen-Hsin, etc

# Fermionic parton construction of WZW term

$S^{\text{WZW}}_{\text{GUT}}$ action: Modified $so(10)$ GUT (Spin(10)) plus a new 4d discrete torsion class of WZW-like term that saturates $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(10)})$ anomaly.

$$
\begin{aligned}
S_{\text{YM-Weyl}} &= \int_{M^4} \mathrm{Tr}(F \wedge \star F) + \mathrm{d}^4 x \big(\psi_L^\dagger (\mathrm{i}\bar{\sigma}^\mu D_{\mu,A})\psi_L\big),\\
S_{\text{Higgs}} &= \int_{M^4} \mathrm{d}^4 x \big(|D_{\mu,A}\Phi_{\mathbf{R}}|^2 - \mathrm{U}(\Phi_{\mathbf{R}})\big),\\
S_{\text{Yukawa}} &= \int_{M^4} \mathrm{d}^4 x \Big(\frac{1}{2}\phi^{\intercal}\Phi^{\mathrm{bi}}\phi + \frac{1}{2}\sum_{a=1}^{5}\big(\psi_L^{\intercal}\,\mathrm{i}\sigma^2(\phi_{2a-1}\Gamma_{2a-1} - \mathrm{i}\phi_{2a}\Gamma_{2a})\psi_L + \text{h.c.}\big)\Big),\\
S^{\text{WZW}} &= \frac{1}{\pi}\int_{M^5} \mathcal{B}(\Phi_{\mathbf{54}}) \wedge \mathrm{d}\mathcal{C}(\Phi_{\mathbf{45}})\Big|_{M^4=\partial M^5} = \pi \int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}}) \smile \delta C(\hat{\Phi}^{\mathrm{bi}}))\Big|_{M^4=\partial M^5}.\\
S^{\text{WZW}}_{\text{QED}'_4} &= \int_{M^4} \mathrm{d}^4 x\, \bar{\xi}(\mathrm{i}\gamma^\mu D'_\mu - \tilde{\Phi}^{\text{bi}} - \mathrm{i}\gamma^{\text{FIVE}}\hat{\Phi}^{\text{bi}})\xi.\\
S^{\text{WZW}}_{\text{QED}'_5} &= \int_{M^5} \mathrm{d}^5 x\, \bar{\xi}(\mathrm{i}\tilde{\gamma}^\mu D'_\mu - m - \tilde{\gamma}^5\tilde{\Phi}^{\text{bi}} - \tilde{\gamma}^6\,\mathrm{i}\,\hat{\Phi}^{\text{bi}} - \mathrm{i}\tilde{\gamma}^5\tilde{\gamma}^\mu\tilde{\gamma}^\nu\mathcal{B}_{\mu\nu} - \mathrm{i}\tilde{\gamma}^6\tilde{\gamma}^\mu\tilde{\gamma}^\nu\mathcal{C}_{\mu\nu})\xi.
\end{aligned}
$$

Spin(10) rep: $\psi_L$ in **16**, $\xi$ in **10**, $\phi$ in **10**, $\Phi^{\mathrm{bi}}$ in **100**, $\tilde{\Phi}^{\mathrm{bi}} = \Phi_{\mathbf{54}}$ in **54**, $\hat{\Phi}^{\mathrm{bi}} = \Phi_{\mathbf{45}}$ in **45**.
$D'_\mu = \nabla_\mu - \mathrm{i}\, a^{\text{dark}}_{\mu,\text{gauge}} - \mathrm{i} g A_\mu$ of $\mathrm{U}(1)'^{\text{dark}}_{\text{gauge}}$ and Spin(10).

4d $\xi$ in $\mathbf{2}_L \oplus \mathbf{2}_R$ of $\mathrm{Spin}(1,3)$ and 5d $\xi$ in $2 \times \mathbf{4}$ of $\mathrm{Spin}(1,4)$.

Yang-Mills $G$ gauge group theory with Weyl spinor. $so(10)$, GG, flipped, PS, SM. New Parton.

**$so(10)$:**

| | $\mathfrak{su}(3)$ | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)^{Y_1}$ | $\mathfrak{u}(1)^{X_1}$ | | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)^{Y_1}$ | $\mathfrak{u}(1)^{X_1}$ | $\mathfrak{so}(10)$ |
|---|---|---|---|---|---|---|---|---|---|
| $u_L$, $d_L$ | $\mathbf{3}$ | $\mathbf{2}$ | 1 | 1 | $\nu_{eL}$, $e_L$ | $\mathbf{2}$ | −3 | −3 | $\mathbf{16}$ |
| $\bar{u}_R$ | | $\mathbf{1}$ | −4 | 1 | $\bar{\nu}_{eR}$ | $\mathbf{1}$ | 0 | 5 | |
| $\bar{d}_R$ | | $\mathbf{1}$ | 2 | −3 | $\bar{e}_R$ | $\mathbf{1}$ | 6 | 1 | |

**$su(5)$:**

| | $\mathfrak{su}(3)$ | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)^{Y_1}$ | $\mathfrak{u}(1)^{X_1}$ | | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)^{Y_1}$ | $\mathfrak{u}(1)^{X_1}$ | $\mathfrak{su}(5)$ |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\bar{\nu}_{eR}$ | $\mathbf{1}$ | 0 | 5 | $\mathbf{1}$ |
| $u_L$, $d_L$ | $\mathbf{3}$ | $\mathbf{2}$ | 1 | 1 | $\bar{e}_R$ | $\mathbf{1}$ | 6 | 1 | $\mathbf{10}$ |
| $\bar{u}_R$ | | $\mathbf{1}$ | −4 | 1 | | | | | |
| $\bar{d}_R$ | | $\mathbf{1}$ | 2 | −3 | $\nu_{eL}$, $e_L$ | $\mathbf{2}$ | −3 | −3 | $\bar{\mathbf{5}}$ |

**$su(5)'$ (flipped):**

| | $\mathfrak{su}(3)$ | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)^{Y_1}$ | $\mathfrak{u}(1)^{X_1}$ | | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)^{Y_1}$ | $\mathfrak{u}(1)^{X_1}$ | $\mathfrak{su}(5)'$ |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\bar{e}_R$ | $\mathbf{1}$ | 6 | 1 | $\mathbf{1}$ |
| $u_L$, $d_L$ | $\mathbf{3}$ | $\mathbf{2}$ | 1 | 1 | $\bar{\nu}_{eR}$ | $\mathbf{1}$ | 0 | 5 | $\mathbf{10}$ |
| $\bar{d}_R$ | | $\mathbf{1}$ | 2 | −3 | | | | | |
| $\bar{u}_R$ | | $\mathbf{1}$ | −4 | 1 | $\nu_{eL}$, $e_L$ | $\mathbf{2}$ | −3 | −3 | $\bar{\mathbf{5}}$ |

**Pati-Salam:**

| | $\mathfrak{su}(3)$ | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)^{Y_1}$ | $\mathfrak{u}(1)^{X_1}$ | | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)^{Y_1}$ | $\mathfrak{u}(1)^{X_1}$ | $\mathfrak{su}(4)\times\mathfrak{su}(2)_L \times\mathfrak{su}(2)_R$ |
|---|---|---|---|---|---|---|---|---|---|
| $u_L$, $d_L$ | $\mathbf{3}$ | $\mathbf{2}$ | 1 | 1 | $\nu_{eL}$, $e_L$ | $\mathbf{2}$ | −3 | −3 | $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ |
| $\bar{u}_R$ | | $\mathbf{1}$ | −4 | 1 | $\bar{\nu}_{eR}$ | $\mathbf{1}$ | 0 | 5 | $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ |
| $\bar{d}_R$ | | $\mathbf{1}$ | 2 | −3 | $\bar{e}_R$ | $\mathbf{1}$ | 6 | 1 | |

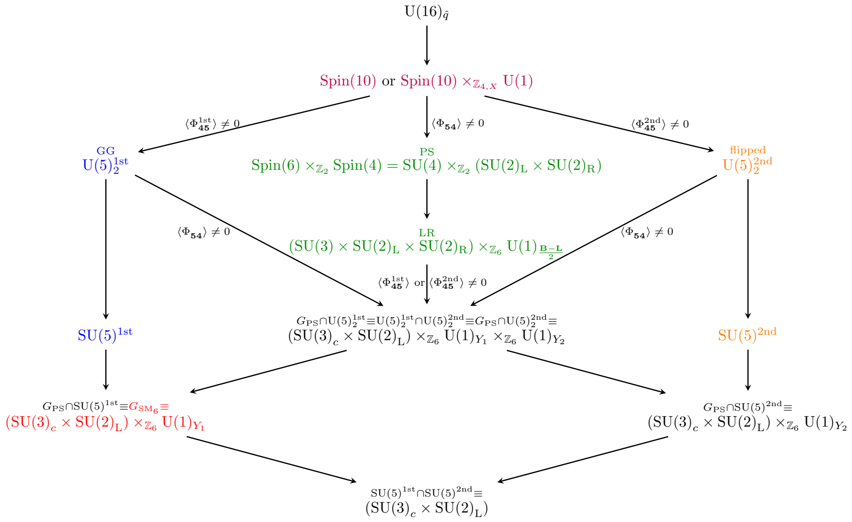
**Standard Model:**

| | $\mathfrak{su}(3)$ | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)^{Y_1}$ | $\mathfrak{u}(1)^{X_1}$ | | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)^{Y_1}$ | $\mathfrak{u}(1)^{X_1}$ |
|---|---|---|---|---|---|---|---|---|
| $u_L$, $d_L$ | $\mathbf{3}$ | $\mathbf{2}$ | 1 | 1 | $\nu_{eL}$, $e_L$ | $\mathbf{2}$ | −3 | −3 |
| $\bar{u}_R$ | | $\mathbf{1}$ | −4 | 1 | $\bar{\nu}_{eR}$ | $\mathbf{1}$ | 0 | 5 |
| $\bar{d}_R$ | | $\mathbf{1}$ | 2 | −3 | $\bar{e}_R$ | $\mathbf{1}$ | 6 | 1 |

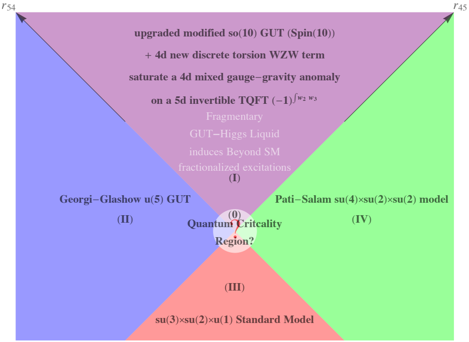
**New Parton:**

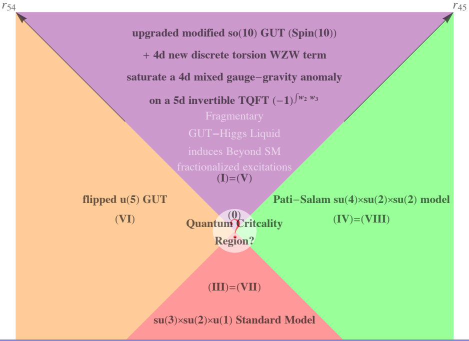
| | $\mathfrak{u}(1)^{\text{dark}}_{\text{gauge}}$ | $\mathfrak{su}(3)$ | $\mathfrak{su}(2)$ | $\mathfrak{u}(1)_{Y_1}$ | $\mathfrak{u}(1)_{X_1}$ | $\mathfrak{su}(5)$ | $\mathfrak{so}(10)$ |
|---|---|---|---|---|---|---|---|
| $c$ | 1 | $\mathbf{3}$ | $\mathbf{1}$ | −2 | −2 | $\mathbf{5}$ | $\mathbf{10}$ |
| $f$ | 1 | $\mathbf{1}$ | $\mathbf{2}$ | 3 | −2 | | |
| $c'$ | 1 | $\bar{\mathbf{3}}$ | $\mathbf{1}$ | 2 | 2 | $\bar{\mathbf{5}}$ | |
| $f'$ | 1 | $\mathbf{1}$ | $\mathbf{2}$ | −3 | 2 | | |

# Lie Group Embedding and Symmetry Breaking

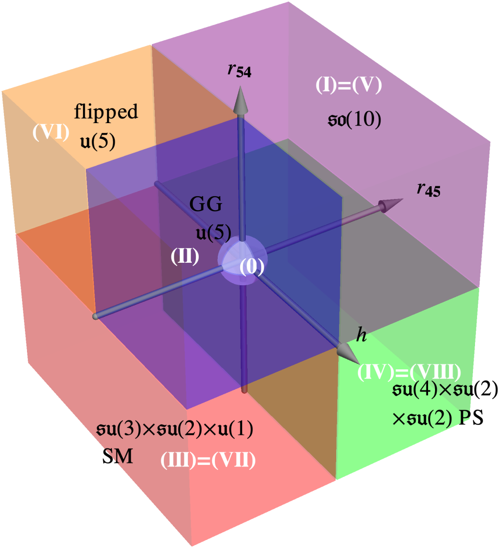



$$\mathrm{U}(\boldsymbol{\Phi}_{\mathbf{R}}) = \Big( r_{45} (\Phi_{45})^2 + \lambda_{45} (\Phi_{45})^4 \Big) + \Big( r_{54} (\Phi_{54})^2 + \lambda_{54} (\Phi_{54})^4 \Big) + h\ \Phi_{45} \cdot (\langle \Phi_{45}^{\mathrm{1st}} \rangle - \langle \Phi_{45}^{\mathrm{2nd}} \rangle) + \dots .$$


$r_{54}$
$r_{45}$
upgraded modified so(10) GUT (Spin(10))
+ 4d new discrete torsion WZW term
saturate a 4d mixed gauge−gravity anomaly
on a 5d invertible TQFT $(-1)^{\int w_2 w_3}$
Fragmentary
GUT−Higgs Liquid
induces Beyond SM
fractionalized excitations
(I)
Georgi−Glashow u(5) GUT
(II)
(0)
Quantum Critcality
Region?
Pati−Salam su(4)×su(2)×su(2) model
(IV)
(III)
su(3)×su(2)×u(1) Standard Model


$r_{54}$
$r_{45}$
upgraded modified so(10) GUT (Spin(10))
+ 4d new discrete torsion WZW term
saturate a 4d mixed gauge−gravity anomaly
on a 5d invertible TQFT $(-1)^{\int w_2 w_3}$
Fragmentary
GUT−Higgs Liquid
induces Beyond SM
fractionalized excitations
(I)=(V)
flipped u(5) GUT
(VI)
(0)
Quantum Critcality
Region?
Pati−Salam su(4)×su(2)×su(2) model
(IV)=(VIII)
(III)=(VII)
su(3)×su(2)×u(1) Standard Model

# Quantum Phase Diagram (Moduli space or Landscape)




$$\mathrm{U}(\Phi_{\mathbf{R}}) = \Big(r_{45}(\Phi_{45})^2 + \lambda_{45}(\Phi_{45})^4\Big) + \Big(r_{54}(\Phi_{54})^2 + \lambda_{54}(\Phi_{54})^4\Big) + h\ \Phi_{45} \cdot (\langle \Phi_{45}^{\text{1st}} \rangle - \langle \Phi_{45}^{\text{2nd}} \rangle) + \dots.$$

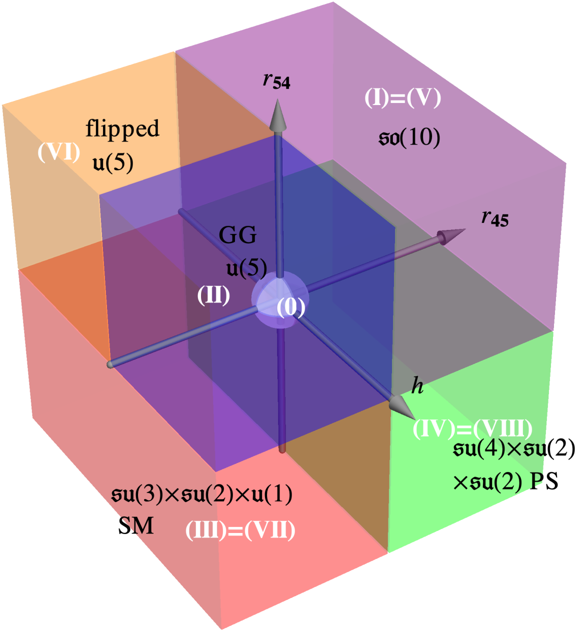
# Outline

1. **Ultra Unification** : Quantum Field Theory /Matter.
- Anomalies, Cobordisms.
- Logic: Three (two+one) Assumptions.
- Consequences: Gapped Topological/Gapless CFT sectors, Extra dim.
- (Gauge-) symmetry-breaking, -preserving, -extension mass or energy gap.
- Dirac/Majorana mass vs Interaction induced vs Topological mass/energy gap.
- $\Omega^5_{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4} = \Omega^4_{\mathrm{Pin}^+} = \mathbb{Z}_{16}$ - Path Integral.

2. **Quantum Criticality Beyond the Standard Model**
- $\Omega^5_{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathrm{Spin}(10)} = \mathbb{Z}_2$ and $w_2 w_3$ anomaly. - 4d boundary or 5d bulk criticality.
- GUT-Higgs fragmentary fermionic parton + Dark Gauge sector.

3. **Combined Synthesis - Conclusion**
- 15n vs 16n Weyl fermion scenarios (SM or GG vs other GUTs).
- Topological criticality or phase transition (topological order).
- Non-invertible categorical Higher symmetries retraction.

# Quantum Phase Diagram (Moduli space or Landscape)



# Deformation class of QFT: cobordism class

For 4d SM/BSM physics, we propose such a 5d cobordism invariant:

$$\mathbf{Z}_{\text{5d-iTQFT}}^{(\mathrm{p},\nu)} \equiv \boxed{\exp(\frac{2\pi \mathrm{i}}{16} \cdot \nu \cdot \eta(\mathrm{PD}(\mathcal{A}_{\mathbb{Z}_4} \mod 2))\Big|_{M^5})} \cdot \boxed{\exp(\mathrm{i}\pi \cdot \mathrm{p} \cdot \int_{M^5} w_2 w_3)},$$

with $\nu \in \mathbb{Z}_{16}$, $\mathrm{p} \in \mathbb{Z}_2$, .

- The $\mathcal{A}_{\mathbb{Z}_4} \in \mathrm{H}^1(M, \mathbb{Z}_{4,X})$ is a cohomology class discrete gauge field of the $\mathbb{Z}_{4,X}$-symmetry.
- A 4d Atiyah-Patodi-Singer (APS) $\eta$ invariant $\equiv \eta_{\mathrm{Pin}^+} \in \mathbb{Z}_{16}$.
A 4d (3+1d) topological superconductor, protected by time-reversal $\mathrm{T}^2 = (-1)^F$.
- Stiefel-Whitney (SW) characteristic class: $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(10)})$.

# Redefine Lie group $\mathrm{U}(5)_{\hat{q}}$ and SM/GUT Higher Symmetries

- Redefine Lie group of $u(5)$ or $su(5) \times u(1)$ Lie algebra:

$$\mathrm{U}(5)_{\hat{q}} \equiv \frac{\mathrm{SU}(5) \times_{\hat{q}} \mathrm{U}(1)}{\mathbb{Z}_5} \equiv \{(g, \mathrm{e}^{\mathrm{i}\,\theta}) \in \mathrm{SU}(5) \times \mathrm{U}(1) \big| (\mathrm{e}^{\mathrm{i}\,\frac{2\pi n}{5}}\mathbb{I}, 1) \sim (\mathbb{I}, \mathrm{e}^{\mathrm{i}\,\frac{2\pi n \hat{q}}{5}}), n \in \mathbb{Z}_5\}.$$

(1) $\mathrm{U}(5)_1 \cong \mathrm{U}(5)_4 \cong \mathrm{U}(5)_{5m+1} \cong \mathrm{U}(5)_{5m-1}$.

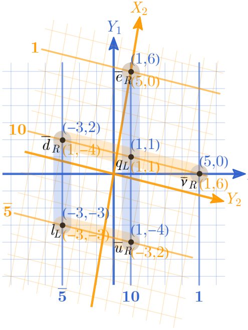
(2) $\mathrm{U}(5)_2 \cong \mathrm{U}(5)_3 \cong \mathrm{U}(5)_{5m+2} \cong \mathrm{U}(5)_{5m-2}$. GG or Baar's flipped

(3) $\mathrm{U}(5)_0 \cong \mathrm{U}(5)_{5m} \cong \mathrm{SU}(5) \times \mathrm{U}(1)$.

- Higher symmetry:

| Higher symmetries of 4d SMs or GUTs with SM matters | | | | |
|---|---|---|---|---|
| QFT | $Z(G_g)$ | $\pi_1(G_g)$. $\pi_1(G_g)^\vee$ | 1-form $e$ sym $G^e_{[1]}$ | 1-form $m$ sym $G^m_{[1]}$ |
| $G_{\mathrm{SM}_q} \equiv \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_{\tilde{Y}}}{\mathbb{Z}_q}$ | $\mathbb{Z}_{6/q} \times \mathrm{U}(1)$ | $\mathbb{Z}$. $\mathrm{U}(1)$ | $\mathbb{Z}^e_{6/q,[1]}$ | $\mathrm{U}(1)^m_{[1]}$ |
| $G_{\mathrm{SM}_6} \equiv \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_{\tilde{Y}}}{\mathbb{Z}_6}$ | $\mathrm{U}(1)$ | $\mathbb{Z}$. $\mathrm{U}(1)$ | 0 | $\mathrm{U}(1)^m_{[1]}$ |
| SU(5) (GG or flipped) | $\mathbb{Z}_5$ | 0. 0 | 0 | 0 |
| $\mathrm{U}(5)_{\hat{q}}$ (GG or flipped) | $\mathrm{U}(1)$ | $\mathbb{Z}$. $\mathrm{U}(1)$ | 0 | $\mathrm{U}(1)^m_{[1]}$ |
| $G_{\mathrm{PS}_{q'}} \equiv \frac{\mathrm{SU}(4)\times\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}}{\mathbb{Z}_{q'}}$ | $\mathbb{Z}_4 \times_{\mathbb{Z}_{q'}} (\mathbb{Z}_2 \times \mathbb{Z}_2)$ | $\mathbb{Z}_{q'}$. $\mathbb{Z}_{q'}$ | $\mathbb{Z}^e_{2/q',[1]}$ | $\mathbb{Z}^m_{q',[1]}$ |
| $G_{\mathrm{PS}_2} \equiv \frac{\mathrm{SU}(4)\times\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}}{\mathbb{Z}_2}$ | $\mathbb{Z}_4 \times_{\mathbb{Z}_2} (\mathbb{Z}_2 \times \mathbb{Z}_2)$ | $\mathbb{Z}_2$. $\mathbb{Z}_2$ | 0 | $\mathbb{Z}^m_{2,[1]}$ |
| Spin(10) | $\mathbb{Z}_4$ | 0. 0 | 0 | 0 |

# Categorical Symmetry and Its Retraction




| SM fermion spinor field | $\mathrm{U}(1)_{Y_1}$ | $\mathrm{U}(1)_{X_1}$ | $\mathbb{Z}_{4,X}$ | $\mathbb{Z}_2^F$ | $\mathrm{U}(1)_{X_2}$ | $\mathrm{U}(1)_{Y_2}$ | $\mathrm{SU}(5)^{\text{1st}}$ | $\mathrm{SU}(5)^{\text{2nd}}$ |
|---|---|---|---|---|---|---|---|---|
| $u_L$ | 1 | 1 | 1 | 1 | 1 | 1 | $(\mathbf{3},\mathbf{2})$ in $\mathbf{10}$ | |
| $d_L$ | 1 | 1 | 1 | 1 | 1 | 1 | | |
| $\nu_L$ | −3 | −3 | 1 | 1 | −3 | −3 | $(\mathbf{1},\mathbf{2})$ in $\bar{\mathbf{5}}$ | |
| $e_L$ | −3 | −3 | 1 | 1 | −3 | −3 | | |
| $\bar{u}_R$ | −4 | 1 | 1 | 1 | −3 | 2 | in $\mathbf{10}$ | in $\bar{\mathbf{5}}$ |
| $\bar{d}_R$ | 2 | −3 | 1 | 1 | 1 | −4 | in $\bar{\mathbf{5}}$ | in $\mathbf{10}$ |
| $\bar{\nu}_R = \nu_L$ | 0 | 5 | 1 | 1 | 1 | 6 | in $\mathbf{1}$ | in $\mathbf{10}$ |
| $\bar{e}_R = e_L^+$ | 6 | 1 | 1 | 1 | 5 | 0 | in $\mathbf{10}$ | in $\mathbf{1}$ |

- $[(\mathrm{U}(1)_{X_1} \times_{\mathbb{Z}_{4,X}} \mathrm{U}(1)_{X_2}) \rtimes \mathbb{Z}_2^{\text{flip}}]$ gauge theory.

1-form magnetic symmetries from $\mathrm{U}(1)_{X_1}$ and $\mathrm{U}(1)_{X_2}$ (not broken by electric gauge charged objects). Symmetry generator = topological defect = charge operator:

$U^{\text{flip}}_{\theta_m} \equiv U^{X_1}_{\theta_m} + U^{X_2}_{\theta_m} \equiv \exp(\mathrm{i}\theta_m \oiint_{\Sigma^2} \star \mathrm{d}\, V_{X_1}) + \exp(\mathrm{i}\theta_m \oiint_{\Sigma^2} \star \mathrm{d}\, V_{X_2})$.

$U^{\text{flip}}_{\theta_m,\theta'_m} \equiv U^{\text{flip}}_{\theta'_m,\theta_m} \equiv (U^{X_1}_{\theta_m} \times U^{X_2}_{\theta'_m} + U^{X_1}_{\theta'_m} \times U^{X_2}_{\theta_m})$.

$U^{\text{flip}}_{\theta_m,\theta'_m} \times U^{\text{flip}}_{\vartheta_m,\vartheta'_m} = U^{\text{flip}}_{\theta_m+\vartheta_m,\theta'_m+\vartheta'_m} + U^{\text{flip}}_{\theta_m+\vartheta'_m,\theta'_m+\vartheta_m}$.

Fusion rule for gauge-invariant topological magnetic 2-surface operator is beyond group laws:

- Part of gauge group $[(\mathrm{U}(5)^{\text{1st}}_{\hat{q}=2} \cup \mathrm{U}(5)^{\text{2nd}}_{\hat{q}=2}) \rtimes \mathbb{Z}_2^{\text{flip}}] = [\mathrm{Spin}(10)]$.

# Summary

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- GUT-Higgs fragmentary fermionic parton + Dark Gauge sector.

3. **Combined Synthesis - Conclusion**
- 15n vs 16n Weyl fermion scenarios (SM or GG vs other GUTs).
- Topological criticality or phase transition (topological order).
- Non-invertible categorical Higher symmetries retraction.

# Back Up Slides:

# Summary

1. Discrete internal symmetry of baryon minus lepton number $\mathbf{B}-\mathbf{L}$, the electroweak hypercharge $Y$ (Wilzek-Zee '79): $\mathbb{Z}_{4,X\equiv 5(\mathbf{B}-\mathbf{L})-4Y}$. (Garcia-Etxebarria-Montero 1808.00009, Wan-JW 1910.14668, JW 2006.16996, 2008.06499, 2012.15860)

$\Omega^5_{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4} = \Omega^4_{\mathrm{Pin}^+} = \mathbb{Z}_{16}$. $\quad \Omega^5_{\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathbb{Z}_{4,X}\times G_{\mathrm{SM}}} = \mathbb{Z}_{16}\times\mathbb{Z}_4^{\cdots}\times\mathbb{Z}_2^{\cdots}\times\mathbb{Z}^5$.

Ultra Unification: 15 Weyl fermions 3 generations, $-3$ mod 16 anomaly, cancel by something new.

2. 3+1d "deconfined quantum criticality-like" phenomena between Georgi-Glashow $su(5)$ and Pati-Salam $su(4)\times su(2)\times su(2)$ models

$\Omega^5_{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathrm{Spin}(10)} = \mathbb{Z}_2$.

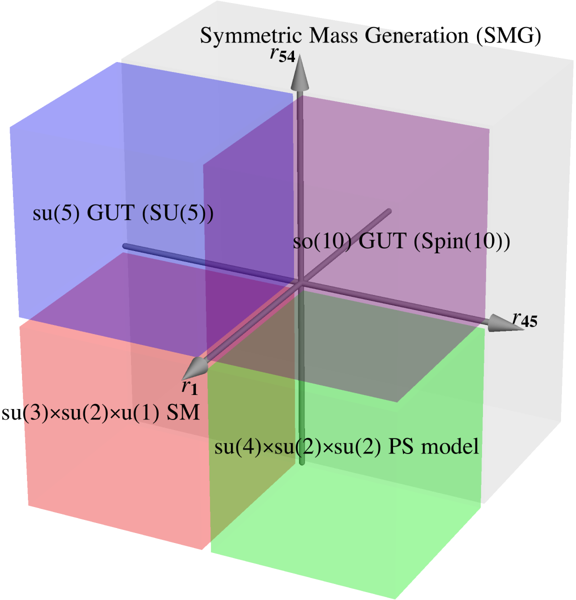
3 generation of SM Weyl fermions + GUT-Higgs WZW requires 1 mod 2 anomaly.

A modified $so(10)$ GUT (with a Spin(10) gauge group) plus a new 4d discrete torsion class of Wess-Zumino-Witten-like term that saturates the 4d anomaly from 5d $w_2 w_3$. Naturally require a double-spin structure $\mathrm{DSpin}\equiv(\mathbb{Z}_2^F\times\mathbb{Z}_2^{F'})\rtimes\mathrm{SO}$.

$G^{\text{modified}}_{so(10)\text{-GUT}} \equiv (\mathrm{DSpin}\times_{\mathbb{Z}_2^F}\mathrm{Spin}(10))\times_{\mathbb{Z}_2^{F'}}\mathrm{U}(1)'$.

| SM fermion spinor field | SU(3) | $\mathrm{SU}(2)_{\mathrm{L}}$ | $\mathrm{SU}(2)_{\mathrm{R}}$ | $\mathrm{U}(1)_{\frac{\mathbf{B}-\mathbf{L}}{2}}$ | $\mathrm{U}(1)_{Y_1}$ | $\mathrm{U}(1)_{\tilde{Y}_R}$ | $\mathrm{U}(1)_{\mathrm{EM}}$ | $\mathrm{U}(1)_{X_1}$ | $\mathbb{Z}_{4,X}$ | $\mathbb{Z}_2^F$ | $\mathrm{U}(1)_{X_2}$ | $\mathrm{U}(1)_{Y_2}$ | $\mathrm{SU}(5)^{\mathrm{1st}}$ | $\mathrm{SU}(5)^{\mathrm{2nd}}$ | $G_{\mathrm{PS}}$ | Spin(10) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $u_L$ | **3** | $q_L$ : **2** | **1** | 1/6 | 1 | 4 | 2/3 | 1 | 1 | 1 | 1 | 1 | (**3**,**2**) in **10** | | **4**, **2**, 1 | **16** |
| $d_L$ | **3** | | **1** | 1/6 | 1 | −2 | −1/3 | 1 | 1 | 1 | 1 | 1 | | | | |
| $\nu_L$ | **1** | $l_L$ : **2** | **1** | −1/2 | −3 | 0 | 0 | −3 | 1 | 1 | −3 | −3 | (**1**,**2**) in $\bar{\mathbf{5}}$ | | | |
| $e_L$ | **1** | | **1** | −1/2 | −3 | −6 | −1 | −3 | 1 | 1 | −3 | −3 | | | | |
| $\bar{u}_R$ | $\bar{\mathbf{3}}$ | **1** | $q_R$ : **2** | −1/6 | −4 | −1 | −2/3 | 1 | 1 | 1 | −3 | 2 | in **10** | in $\bar{\mathbf{5}}$ | $\bar{\mathbf{4}}$, 1, **2** | |
| $\bar{d}_R$ | $\bar{\mathbf{3}}$ | **1** | | −1/6 | 2 | −1 | 1/3 | −3 | 1 | 1 | 1 | −4 | in $\bar{\mathbf{5}}$ | in **10** | | |
| $\bar{\nu}_R = \nu_L$ | **1** | **1** | $l_R$ : **2** | 1/2 | 0 | 3 | 0 | 5 | 1 | 1 | 1 | 6 | in **1** | in **10** | | |
| $\bar{e}_R = e_L^+$ | **1** | **1** | | 1/2 | 6 | 3 | 1 | 1 | 1 | 1 | 5 | 0 | in **10** | in **1** | | |

## Quantum Phase Diagram (Moduli space or Landscape)



## 5d invertible TQFT/SPTs and 4d Anomalies via 5d Cobordism

Kapustin'14, Freed-Hopkins'16 (systematic)
Unitarity of Lorentz $\sim$ Reflection positivity of Euclidean.
$d$d invertible TQFT with reflect.pos in Euclidean signature
$\Rightarrow$ anomaly of $(d-1)$d reflect.pos Euclidean QFT.
$\Rightarrow$ anomaly of $(d-2)+1$d unitary Lorentz QFT. Take $d=5$.

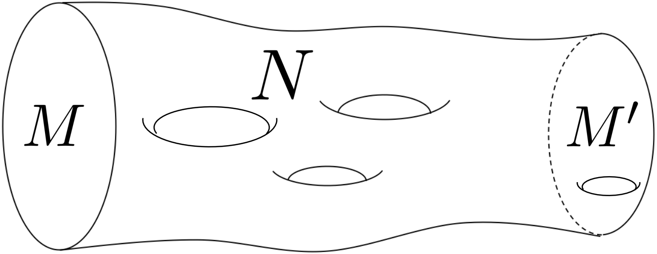
Here we only concern a cobordism group $\Omega_G^d \equiv \mathrm{TP}_d(G)$,
Also a bordism group $\Omega_d^G = \pi_d(MTG) \equiv \mathrm{colim}_{k\to\infty}\pi_{d+k}(MTG)_k$. Note $(\mathrm{TP}_d(G))_{\text{tors}} = (\Omega_d^G)_{\text{tors}}$.
tor: a torsion group (only a finite group part).

Tools: Pontryiagin-Thom construction, Thom-Madsen-Tillmann spectra, Adams spectral sequence, and Freed-Hopkins's theorem

Given a $G$ structure, we will later show co/bordism group (abelian group classification) and $d$d topological terms (invertible TQFT or Symmetry Protected Topological states [SPTs]) and the anomaly of a $(d-2,1)$d unitary Lorentz QFT.

# Classify iTQFT/SPTs and Anomalies via Cobordism

Bordism group (abelian): $\Omega_d^G \equiv \mathrm{TP}_d(G)$
- $+$: the disjoint union.
- Closure: Disjoint union of manifolds is a manifold.
- Identity: 0 is the empty manifold.
- Inverse: $[M] + [\bar{M}] = 0$ since $\partial(M \times [0,1]) = M \sqcup \bar{M}$.
- Associativity and commutativity: true for disjoint union.



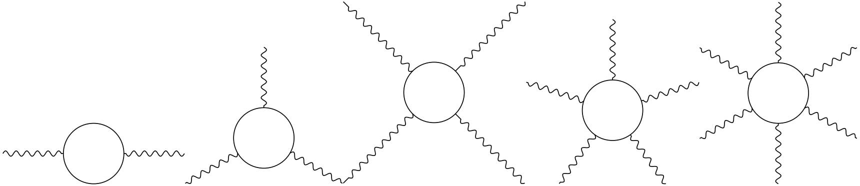

Spin cobordism:Kapustin-Thorngren-Turzillo-Wang'14 (proposed), Freed-Hopkins'16 (systematic), Wan-JW'18 arXiv:1812.11967: Encode higher-symmetry/classifying space.

Application to SM: Garcia-Etxebarria-Montero'18, JW-Wen'18, Davighi-Gripaios-Lohitsiri'19, Wan-JW'19 arXiv:1910.14668

# Examples $\Omega^d_G \equiv \mathrm{TP}_d(G)$: $d$d-iTQFT/SPTs and $(d-1)$d anomaly

| $d$ | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|
| $\mathrm{TP}_d(\mathrm{Spin} \times \mathrm{U}(1))$ | $\mathbb{Z}_2$ | $\mathbb{Z}^2$ | 0 | $\mathbb{Z}^2$ | 0 | ... |
| $\mathrm{TP}_d(\mathrm{Spin}^c)$ | 0 | $\mathbb{Z}^2$ | 0 | $\mathbb{Z}^2$ | 0 | ... |

bulk $d$d: 3d, 5d, 7d, 9d, 11d, etc, with $\mathbb{Z}$ class
boundary $(d-1)$d: 2d(=1+1d), 4d(=3+1d), 6d, 8d, 10d w/ $\mathbb{Z}$ class



2d-3d: $\mathrm{U}(1)_A^2$ and grav$^2$. iTQFT as $\mathrm{CS}_3^{(\mathrm{U}(1))}$ and $\mathrm{CS}_3^{(\mathrm{TM})}$.
4d-5d: $\mathrm{U}(1)_A^3$ and $\mathrm{U}(1)_A$-grav$^2$. iTQFT as $\mathrm{CS}_5^{(\mathrm{U}(1))}$ and $c_1^{(\mathrm{U}(1))}\mathrm{CS}_3^{(\mathrm{TM})}$.
$\mathbb{Z}$ class: free part, or **perturbative local anomaly**.

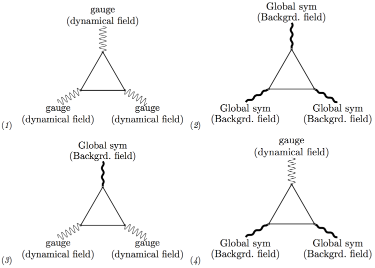
Wan-JW'18 arXiv:1812.11967: Encode higher-symmetry/classifying space.

# Interpretation of ABJ (axial or chiral) anomaly

$$\mathbf{Z}_{\text{Dirac}\Psi}[A] \text{ or } \mathbf{Z}_{\psi_L,\psi_R}[A] \xrightarrow{\psi_{L/R}\to \mathrm{e}^{\pm\,\mathrm{i}\,\alpha}\psi_{L/R}}$$

$$\int[D\bar{\Psi}][D\Psi]\exp\left(\mathrm{i}\int_{M^d}\mathrm{d}^d x(\bar{\Psi}(\mathrm{i}\not{D}_A)\Psi + \alpha_{\mathrm{A}}\left(\partial_\mu J^{\mu,\text{Chiral}} + \frac{2(q\ g)^{d/2}\epsilon^{\mu_1\dots\mu_d}}{(d/2)!(4\pi)^{d/2}}F_{\mu_1\mu_2}\dots F_{\dots\mu_d}\right)\right)\Big),$$

- 't Hooft anomaly of background (Backgrd.) fields.
- Original ABJ: Mixed anomaly between $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$. In 4d, polynomial $\mathrm{U}(1)_A$-$\mathrm{U}(1)_V^2$
- Dynamical gauge anomaly.
- Continuous $\mathrm{U}(1)_A$ may be anomalous, but its discrete $\mathbb{Z}_{N,A}$ can be anomaly-free with $\mathrm{U}(1)_V^2$.




The charge $q$ is quantized, thus $\mathbb{Z}$ class **perturbative local anomaly**.

# Examples $\Omega^d_G \equiv \mathrm{TP}_d(G)$: $d$d-iTQFT/SPTs and $(d-1)$d anomaly

| $d$ | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|
| $\mathrm{TP}_d(\mathrm{Spin} \times \mathrm{SU}(2))$ | $\mathbb{Z}_2$ | $\mathbb{Z}^2$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | ... |
| $\mathrm{TP}_d(\mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{SU}(2))$ | 0 | $\mathbb{Z}^2$ | 0 | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | ... |
| $\mathrm{TP}_d(\mathrm{Spin} \times_{\mathbb{Z}_2} \mathrm{Spin}(10))$ | 0 | $\mathbb{Z}^2$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | ... |

bulk $d$d and boundary $(d-1)$d.
$d = 5$ and $d = 6$:

1st $\mathbb{Z}_2$: Witten anomaly. iTQFT as $c_2(\mathrm{SU}(2))\tilde{\eta} \equiv \tilde{\eta}\mathrm{PD}(c_2(\mathrm{SU}(2)))$.
2nd $\mathbb{Z}_2$ detected on non-spin $M^5$: $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(3)})$.

JW-Wen-Witten'18 1810.00844

| SU(2) isospin | 0 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | 2 | $\frac{5}{2}$ | 3 | $\frac{7}{2}$ | mod 4 | $2r+\frac{1}{2}$ | $4r+\frac{3}{2}$ | mod 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| SU(2) Rep **R** (dim) | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | mod 8 | $\mathbf{4}r+\mathbf{2}$ | $\mathbf{8}r+\mathbf{4}$ | mod 8 |
| Witten SU(2) anomaly | | ✓ | | | | ✓ | | | | ✓ | | |
| New SU(2) anomaly | | | | ✓ | | | | | | | ✓ | |

$\mathbb{Z}_n$ class: torsion part, or **nonperturbative global anomaly**.
JW-Wen arXiv:1809.11171, Wan-JW arXiv:1812.11967: Encode higher-symmetry/classifying space.

## Math Physics Equations: **Ultra Unification** Path (Functional) Integral example

$$\mathbf{Z}_{\mathrm{UU}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \mathbf{Z}_{\substack{\text{5d-iTQFT}/\\ \text{4d-SM+TQFT}}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \mathbf{Z}^{(-\nu_{5\mathrm{d}})}_{\text{5d-iTQFT}}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}^{(\nu_{4\mathrm{d}})}_{\text{4d-TQFT}}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}^{(n_{\nu_{e,R}}, n_{\nu_{\mu,R}}, n_{\nu_{\tau,R}})}_{\mathrm{SM}}[\mathcal{A}_{\mathbb{Z}_4}].$$

$$\mathbf{Z}_{\mathrm{SM}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}][\mathcal{D}A][\mathcal{D}\phi]\ldots \exp(\mathrm{i}\ S_{\mathrm{SM}}[\psi,\bar{\psi},A,\phi,\ldots,\mathcal{A}_{\mathbb{Z}_4}]|_{M^4})$$

$$S_{\mathrm{SM}} = \int_{M^4} \Big(\mathrm{Tr}(F_I \wedge \star F_I) - \frac{\theta_I}{8\pi^2} g_I^2 \mathrm{Tr}(F_I \wedge F_I)\Big) + \int_{M^4} \Big(\bar{\psi}(\mathrm{i}\,\not{D}_{A,\mathcal{A}_{\mathbb{Z}_4}})\psi$$

*(Gauge) Symmetry breaking*

$$+|D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}\phi|^2 - \mathrm{U}(\phi) - (\psi_L^\dagger \phi(\mathrm{i}\sigma^2 \psi_L'^{\,*}) + \mathrm{h.c.})\Big)\, \mathrm{d}^4 x$$

$$(-(N_{\mathrm{gen}} = 3) + n_{\nu_{e,R}} + n_{\nu_{\mu,R}} + n_{\nu_{\tau,R}} + \nu_{4\mathrm{d}} - \nu_{5\mathrm{d}}) = 0 \mod 16.$$

$$\mathbf{Z}^{(-\nu_{5\mathrm{d}}=-2)}_{\text{5d-iTQFT}}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}^{(\nu_{4\mathrm{d}}=2)}_{\text{4d-TQFT}}[\mathcal{A}_{\mathbb{Z}_4}] = \sum_{c \in \partial'^{-1}(\partial[\mathrm{PD}(\mathcal{A}^3)])} \mathrm{e}^{\frac{2\pi\,\mathrm{i}}{8}\mathrm{ABK}(c \cup \mathrm{PD}(\mathcal{A}^3))}$$

$$\cdot \frac{1}{2^{|\pi_0(M^4)|}} \sum_{\substack{a \in C^1(M^4,\mathbb{Z}_2),\\ b \in C^2(M^4,\mathbb{Z}_2)}} (-1)^{\int_{M^4} a(\delta b + \mathcal{A}^3)} \cdot \mathrm{e}^{\frac{2\pi\,\mathrm{i}}{8}\mathrm{ABK}(c \cup \mathrm{PD}'(b))}.$$

$$1 \to [\mathbb{Z}_2] \to \mathrm{Spin} \times \mathbb{Z}_{4,X} \times G_{\mathrm{SM/GUT}} \to \mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\mathrm{SM/GUT}} \to 1.$$

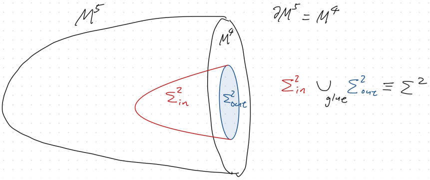
*Symmetry extension trivialize anomaly* (JW-Wen-Witten'17 1705.06728). Fermionic *non-abelian* TQFT.

• Arf-Brown-Kervaire (ABK) for $\Omega_2^{\mathrm{Pin}^-} = \mathbb{Z}_8$. Fidkowski-Kitaev 1+1d fermionic chain with boundary Majorana zero modes.

• $\sum_{\substack{a \in C^1(M^4, \mathbb{Z}_2), \\ b \in C^2(M^4, \mathbb{Z}_2)}} (-1)^{\int_{M^4} a(\delta b + \mathcal{A}^3)}$

braiding statistics of four of strings (2-worldsheets) is non-abelian in nature: quadruple link invariant in non-abelian TQFT. Four loop braiding statistics. There could also be three loop braiding statistics.

• Bulk-boundary: $\mathrm{e}^{\frac{2\pi\, \mathrm{i}}{8} \mathrm{ABK}(c \cup \mathrm{PD}(\mathcal{A}^3))}$. Similarly, boundary-boundary: $\mathrm{e}^{\frac{2\pi\, \mathrm{i}}{8} \mathrm{ABK}(c \cup \mathrm{PD}'(b))}$.




• Hilbert space and ground state degeneracy (GSD), due to the odd class of ABK, show the non-abelian TQFT nature.

What are physical observables for us (from SM) and for 4d TQFT or 5d SPT?

Homotopy group

| | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ |
|---|---|---|---|---|---|---|
| GG $\frac{\mathrm{O}(10)}{\mathrm{U}(5)}$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| PS $\frac{\mathrm{O}(10)}{\mathrm{O}(6)\times\mathrm{O}(4)}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}^2$ | $\mathbb{Z}_2^2$ |

| | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ |
|---|---|---|---|---|---|---|
| Néel $S^2 = \frac{\mathrm{O}(3)\times\mathrm{O}(2)}{\mathrm{O}(2)\times\mathrm{O}(2)} = \frac{\mathrm{O}(3)}{\mathrm{O}(2)} = \frac{\mathrm{SO}(3)\times\mathrm{SO}(2)}{\mathrm{SO}(2)\times\mathrm{SO}(2)} = \frac{\mathrm{SO}(3)}{\mathrm{SO}(2)}$ | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| VBS $S^1 = \frac{\mathrm{O}(3)\times\mathrm{O}(2)}{\mathrm{O}(3)\times\mathrm{O}(1)} = \frac{\mathrm{O}(2)}{\mathrm{O}(1)} = \frac{\mathrm{SO}(3)\times\mathrm{SO}(2)}{\mathrm{SO}(3)\times\mathrm{SO}(1)} = \frac{\mathrm{SO}(2)}{\mathrm{SO}(1)}$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | 0 |

.

$$\begin{aligned}
\langle \Phi_{\mathbf{54}} \rangle &\propto \begin{pmatrix} -3 & & & & \\ & -3 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 2 \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \\
\langle \Phi_{\mathbf{45}}^{\text{1st}} \rangle &\propto \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\langle \Phi_{\mathbf{45}}^{\text{2nd}} \rangle &\propto \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned}$$

Each gauge group $G$ for its Yang-Mills gauge theory. What are the (Weyl spinor) fermion matter contents in which rep of $G$?