Hydrodynamics and the Spectral Form Factor

Brian Swingle (Brandeis)

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It from Qubit
Simons Collaboration on
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Quantum chaos?

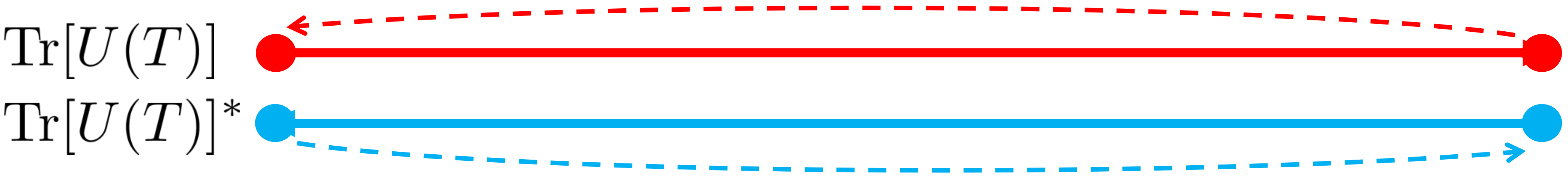
- Classical chaos: “deterministic randomness”, butterfly effect, …
- Quantum chaology [Berry]: quantum signatures of classical chaos, semi-classical limit, focused on “single-particle” systems
  - Quantum chaotic with regular classical limit [Rozenbaum-Bunimovich-Galitski]
- Quantum chaos: “deterministic quantum randomness”, …
  - Sensitivity: echoes, quantum butterfly effect
  - Randomness: effective randomness in energy eigenstates, [Srednicki, Deutsch] random matrix-like energy level statistics [Wigner, Bohigas-Giannoni-Schmit]
  - Complexity: growth of circuit complexity, complexity of eigenstates
  - Thermalization: approach to equilibrium, transport and hydrodynamics

Now is a good time to look at quantum chaos

- [Experimental] It is increasingly possible to probe long-time dynamics of isolated quantum many-body systems; far from equilibrium experiments are often natural and directly probe chaos
- [Quantum information] Many new insights and tools from quantum information shed new light on the physics of chaos
- [Quantum gravity] Quantum chaos matters for the black hole information problem, e.g. in the context of AdS/CFT
- [Strong correlation physics] Study of quantum chaos gives us new insights into transport in strongly correlated systems, possibly new principled computational tools based on chaotic dynamics

This talk – chaos in the spectrum

- *Hydrodynamic theory of the connected spectral form factor*, 2012.01436, w/ Mike Winer
- Spectral form factor: $\mathrm{SFF}(T) = \langle |\mathrm{Tr}[U(T)]|^2 \rangle_{\mathrm{disorder}}.$



Chaos → random matrix behavior at “intermediate” time: $\mathrm{SFF}(T) \propto T$

Question: under what conditions is this random matrix behavior realized?

Prior work

- Significant older literature, e.g. Altshuler-Shklovskii ’86, Argaman-Imry-Smilansky ‘93, reviews: D'Alessio-Kafri-Polkovnikov-Rigol, …
- Analytic results for many-body models: Bertini-Kos-Prosen, Dubertrand-Muller, Chan-De Luca-Chalker, Saad-Shenker-Stanford, Garcia-Garcia-Verbaarschot, Altland-Sonner, …
- RMT Onset: Schiulaz-Torres-Herrera-Santos, Gharibyan-Hanada-Shenker-Tezuka, Friedman-Chan-De Luca-Chalker, Altland-Bagrets, …
- Fluctuating hydrodynamics: Dubovsky-Hui-Nicolis-Son, Grozdanov-Polonyi, Haehl-Loganayagam-Rangamani, Crossley-Glorioso-Liu, Jensen-Pinkani-Fokeeva-Yarom, Chen-Lin-Delacretaz-Hartnoll, …

Chaotic Dynamics

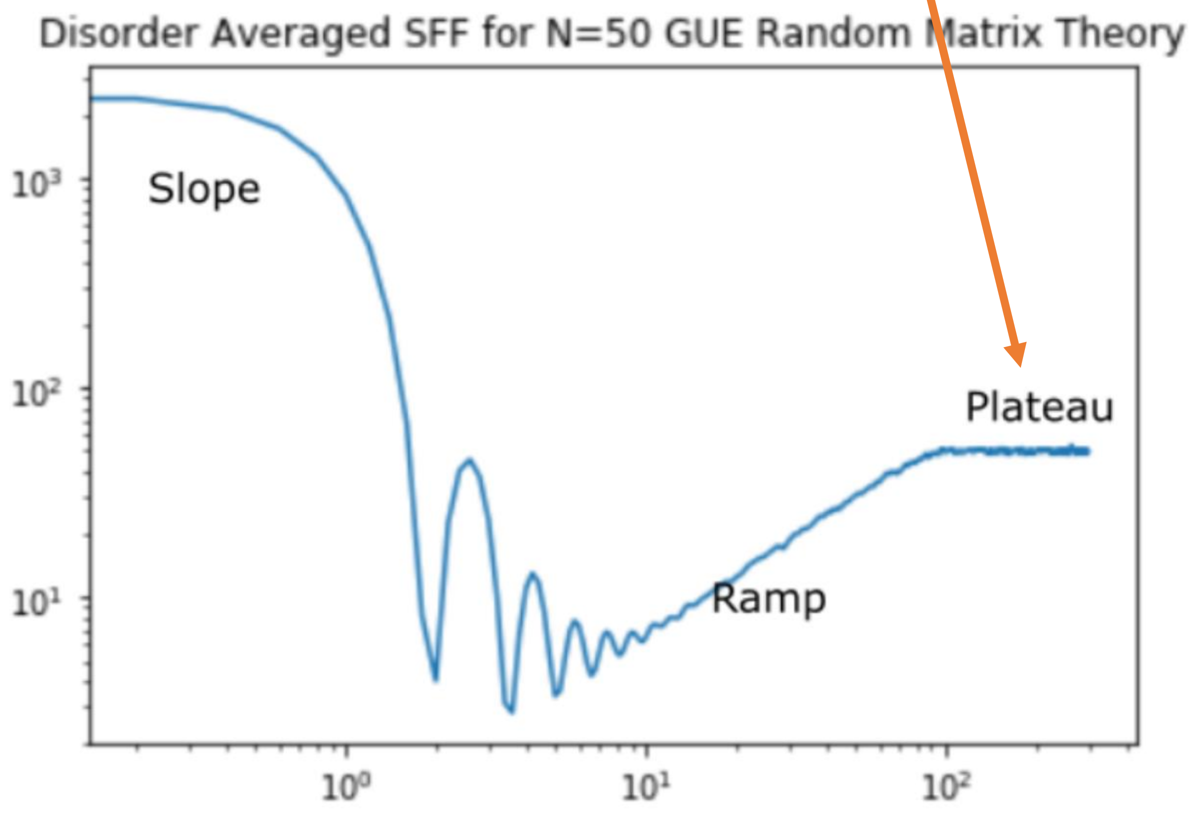
Random matrix theory (RMT)

Expect plateau after exponentially long time

$$dP \propto \prod_{ij} dH_{ij} \exp\left(-\mathrm{Tr}[V(H)]\right)$$

$$dP = \frac{1}{\mathcal{Z}} \prod_{i<j} |E_i - E_j|^{\beta} \prod_i e^{-V(E_i)}$$

Data: Dyson index and potential



$$\mathrm{SFF}(T,f) = \overline{|\mathrm{Tr}[f(H)e^{iHT}]|^2} = \overline{\sum_{i,j} f(E_i)f(E_j)e^{i(E_i-E_j)T}}$$

f = filter function

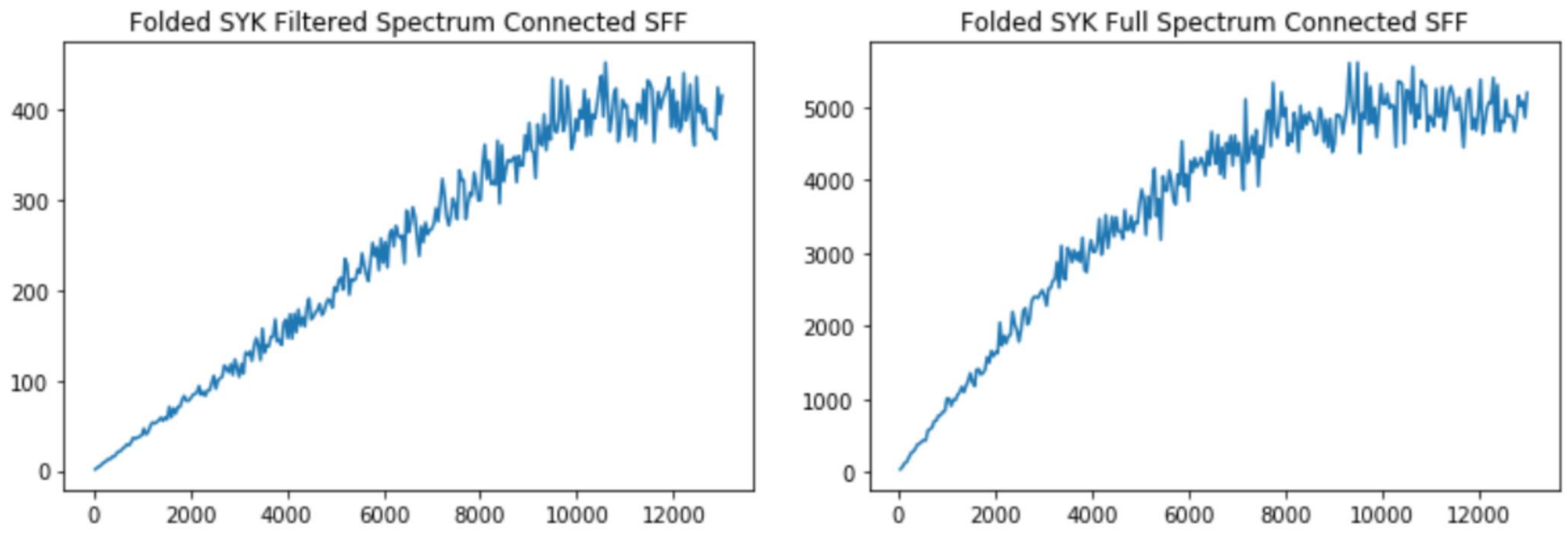
Filtering

$$\mathrm{SFF}(T,f)_{\mathrm{ramp}} = \int dE f^2(E)\frac{T}{\pi\boldsymbol{\beta}}.$$

general Dyson index

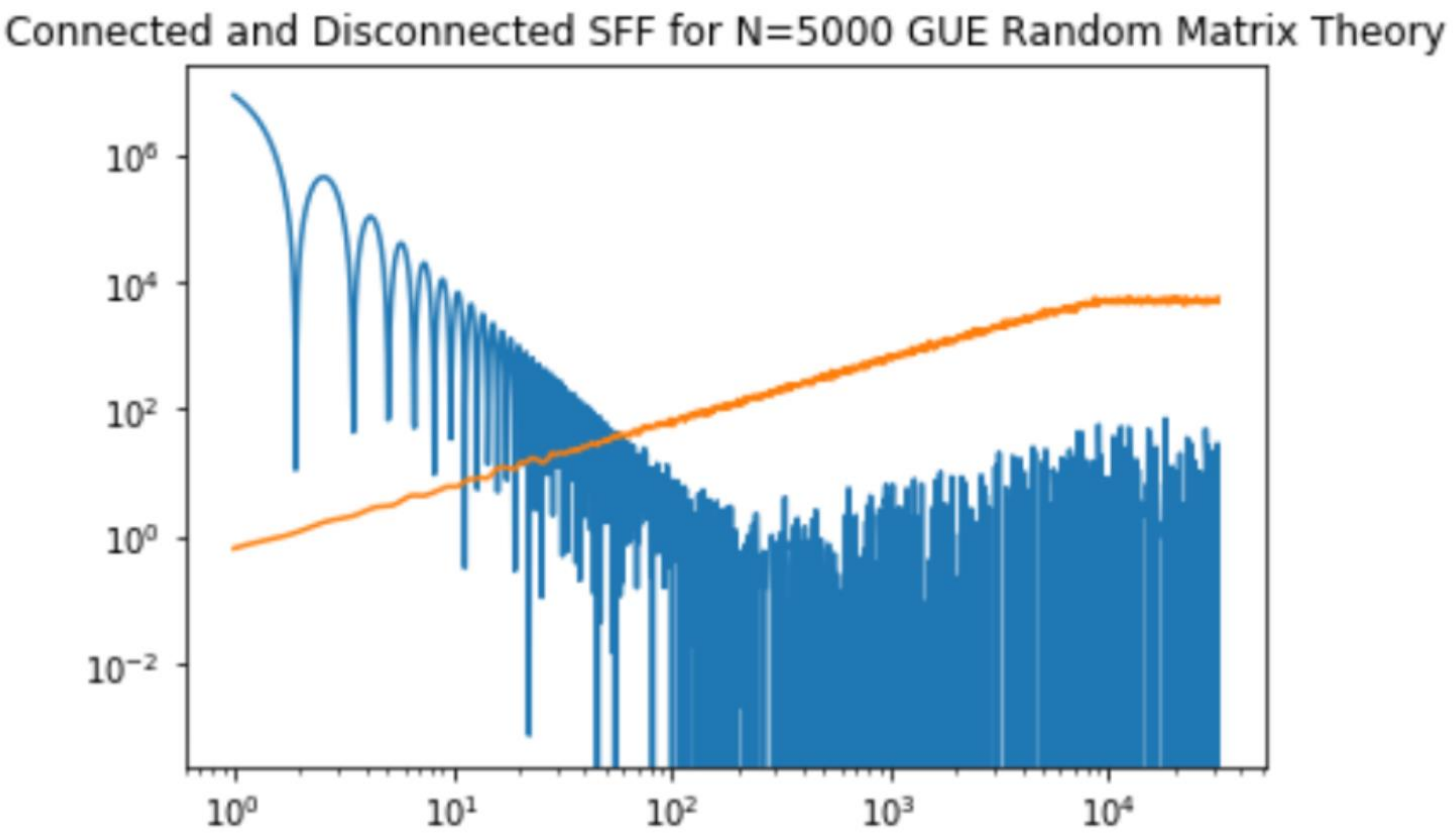
- "Thermal" SFF: $\int dE \exp(-2\beta E)\frac{T}{\pi\boldsymbol{\beta}} = \frac{T}{2\pi\beta\boldsymbol{\beta}}(e^{-2\beta E_{\mathrm{min}}} - e^{-2\beta E_{\mathrm{max}}})$
- Gaussian filter: $\int dE \frac{T}{\pi\boldsymbol{\beta}}\exp\left(\frac{(E-\bar{E})^2}{2\sigma^2}\right) = \frac{\sqrt{2\pi}\sigma T}{\pi\boldsymbol{\beta}}$

[also discussed in Gharibyan et al.]



Thouless time

$$Z(T,f) = \mathrm{Tr}[f(H)e^{-iHT}]$$
$$\mathrm{SFF}_{\mathrm{disc}} = |\overline{Z(T,f)}|^2$$
$$\mathrm{SFF}_{\mathrm{conn}} = \mathrm{SFF} - \mathrm{SFF}_{\mathrm{disc}}$$

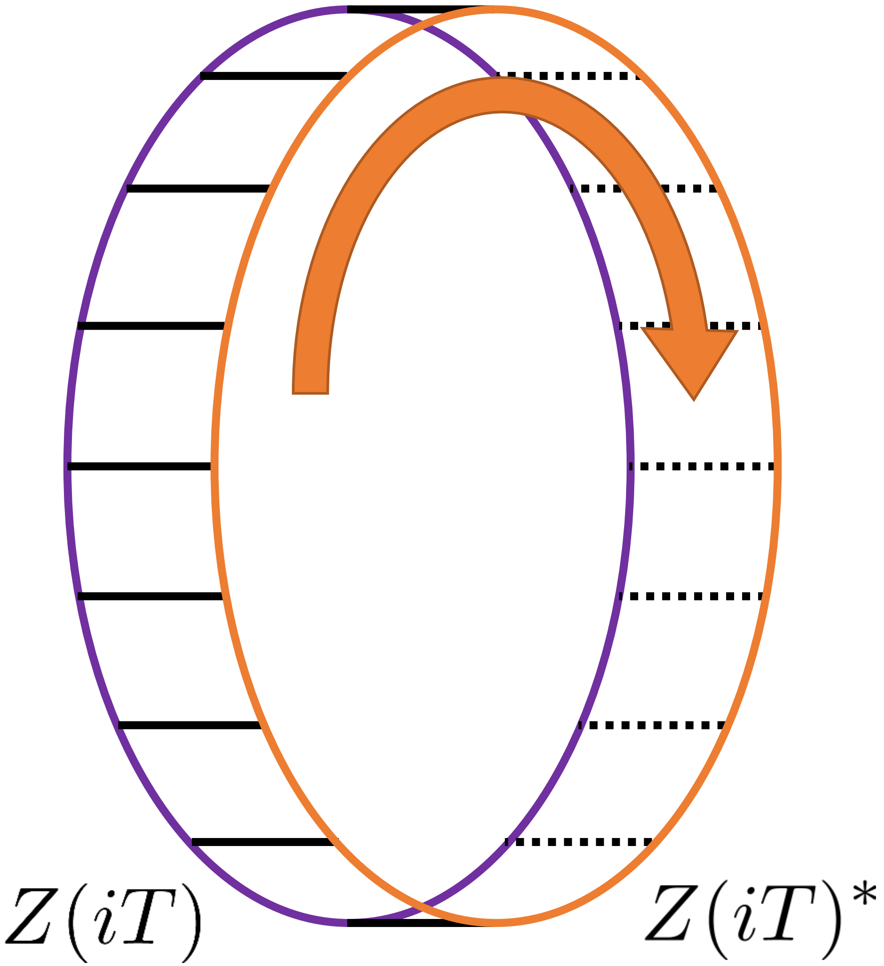


- In a particular chaotic local Hamiltonian system, random matrix theory will not be accurate at "early" time
- Thouless time = time required to be "close" to the pure random matrix result in the connected SFF, typically at least log(system size)

[connected to diffusion, many-body context: Gharibyan et al. '18, Friedman et al. '19]

Goals

- Regimes of time:
  - Very early time (non-universal)
  - Hydrodynamic regime
  - RMT ramp regime
  - RMT plateau regime
- Can we explain the observed RMT universality and relate it to other notions associated with quantum chaos?
- Can we compute the Thouless time and the whole crossover function in the "hydrodynamic" regime?



Correlated contours, free relative time shift

Nearly-conserved sectors

Energy diffusion

- Imagine breaking all other symmetries: all that remains is energy diffusion → minimal slow dynamics in a local Hamiltonian system
- At time T, there are an extensive number of almost conserved modes:

$$k_T \sim \frac{1}{\sqrt{DT}} \qquad N_T \sim \sum_k \theta(k_T - |k|) \sim V \int \frac{d^d k}{(2\pi)^d} = \frac{V S_d}{(2\pi)^d} \frac{k_T^d}{d}$$

- If each sector is random matrix like, then the SFF should correspond to a sum of many almost-independent ramps → sectors are labelled by amplitudes of nearly-conserved energy fluctuations

Nearly-block Hamiltonians

- Decoupled sectors ($\alpha = 1, \cdots, \Omega_0$) + transitions: $H = H_0 + V$
- Decoupled limit, no level repulsion:

$$\mathrm{SFF} \sim T \int \frac{dE}{2\pi} f^2(E) \times (\text{\# of sectors at energy } E)$$

- To compute full SFF, we need to sum over return amplitudes

$$|\psi_{(\alpha,i)}(T)\rangle = \sum_{\beta=1}^{\Omega_0} \sqrt{p_{\alpha\to\beta}(T)} |\phi_{\beta,(\alpha,i)}(T)\rangle,$$

$$\langle\psi_{(\alpha,i)}(0)|\psi_{(\alpha,i)}(T)\rangle = \sqrt{p_{\alpha\to\alpha}(T)} \langle\psi_{(\alpha,i)}(0)|\phi_{\alpha,(\alpha,i)}(T)\rangle$$

Averaging

- SFF is assembled by summing these amplitudes, taking the squared magnitude, and then averaging (denoted by overline)
- Key assumption: the average decouple different sectors

$$\sum_{i,j} \overline{\langle \psi_{(\alpha,i)}(0) | \phi_{\alpha,(\alpha,i)}(T) \rangle \langle \psi_{(\beta,j)}(0) | \phi_{\beta,(\beta,j)}(T) \rangle^*} = \delta_{\alpha,\beta} \mathrm{SFF}_\alpha(T)$$

- Final formula: SFFs of each sector, weighted by a return probability

$$\mathrm{SFF}(T, f) = \sum_\alpha f(E_\alpha)^2 p_{\alpha \to \alpha}(T) \mathrm{SFF}_\alpha(T)$$

[Winer-S]

Linear diffusion

$$p(\epsilon_{k,\text{final}}, T) = \frac{\exp\left(-\frac{(\epsilon_{k,\text{final}} - e^{-\gamma_k T}\epsilon_k)^2}{2\sigma^2(T)}\right)}{\sqrt{2\pi\sigma^2(T)}}$$

$$\int d\epsilon_k p(\epsilon_{k,\text{final}} = \epsilon_k, T) = \frac{1}{1 - e^{-\gamma_k T}}$$

$$\sum_\alpha p_{\alpha\to\alpha}(T) = \prod_k \frac{1}{1 - e^{-Dk^2 T}} = \exp\left(V\left(\frac{1}{4\pi DT}\right)^{d/2}\zeta(1 + d/2)\right)$$

exclude zero mode, quasi-continuous wavevector regime

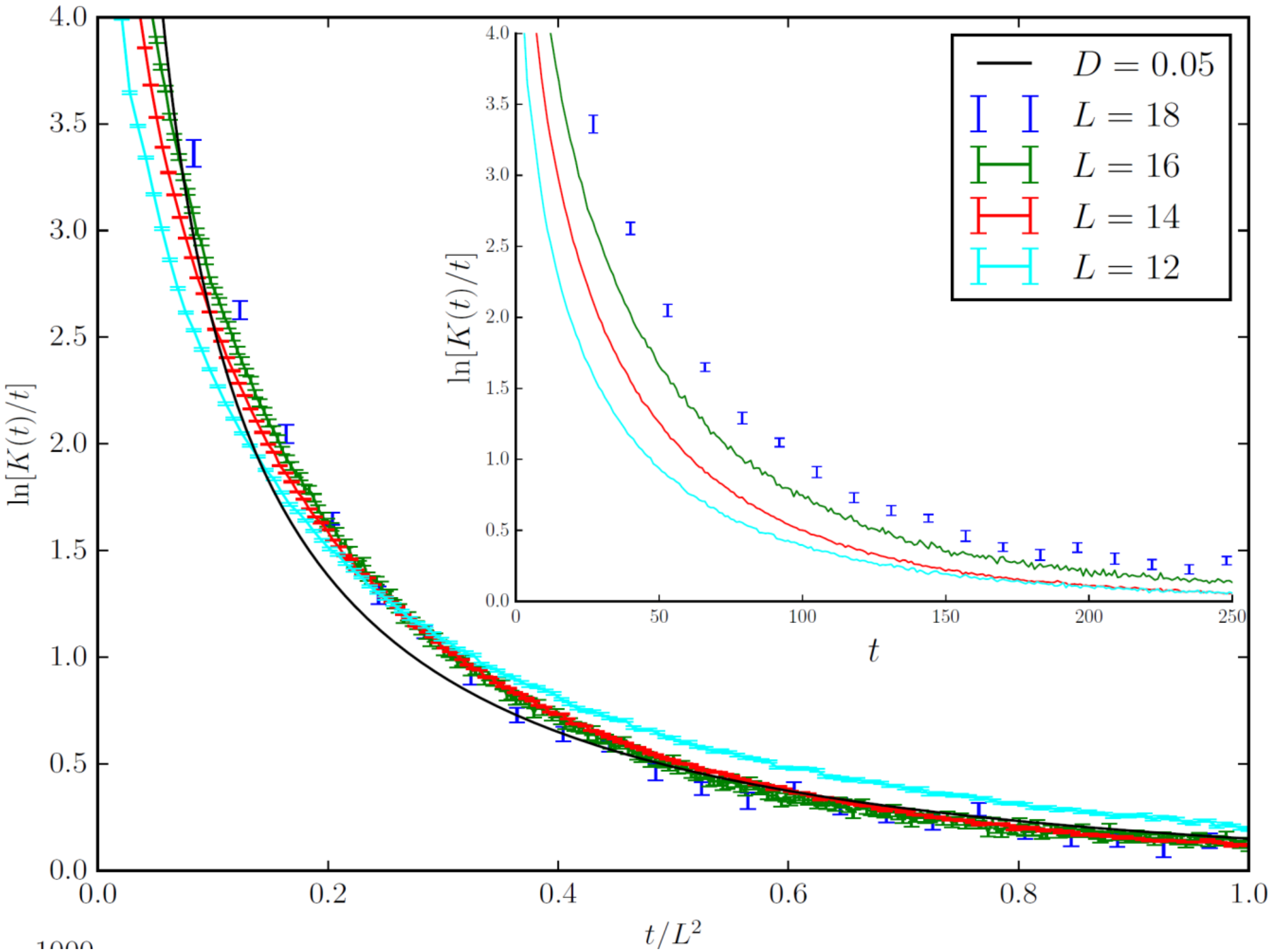
[Winer-S, special case previously obtained for a d=1 Floquet model with large onsite dimension Friedman et al. ‘19]

$$T = t_{\text{Th}} = \frac{L^2 \log\frac{1}{\epsilon}}{(2\pi)^2 D} \longrightarrow \sum_\alpha p_{\alpha\to\alpha}(T) = 1 + 2d\epsilon + O(\epsilon^2)$$

periodic box

Comparison with numerical data

- Consistent with numerical data from [Friedman et al.], which derives the previous formula (in the context of U(1) conservation) in d=1 with large onsite dimension
- We show that it arises generally from linearized diffusion; and we can compute corrections

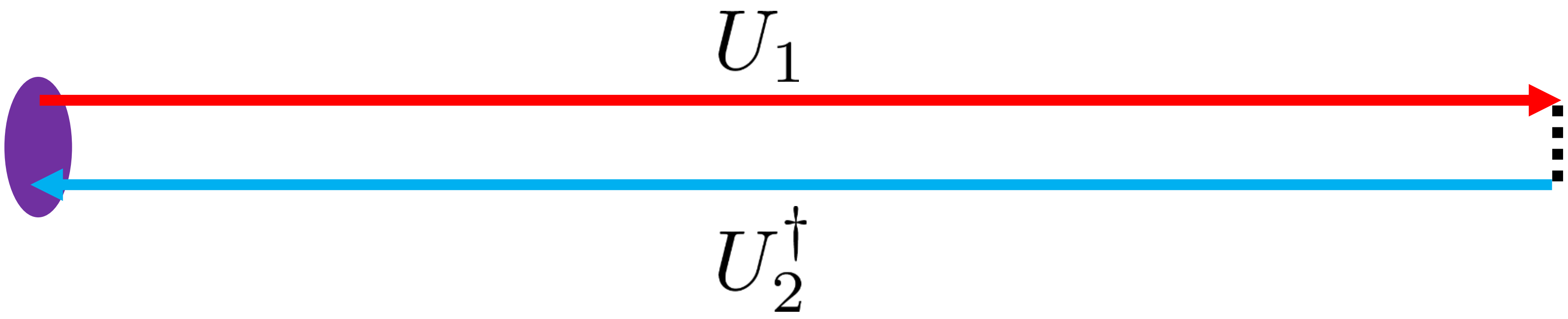


[data from Friedman et al. 1906.07736]

Fluctuating hydrodynamics

Closed time path (CTP) formalism

$$\mathrm{Tr}[U_1(T)\rho U_2(T)^\dagger]$$

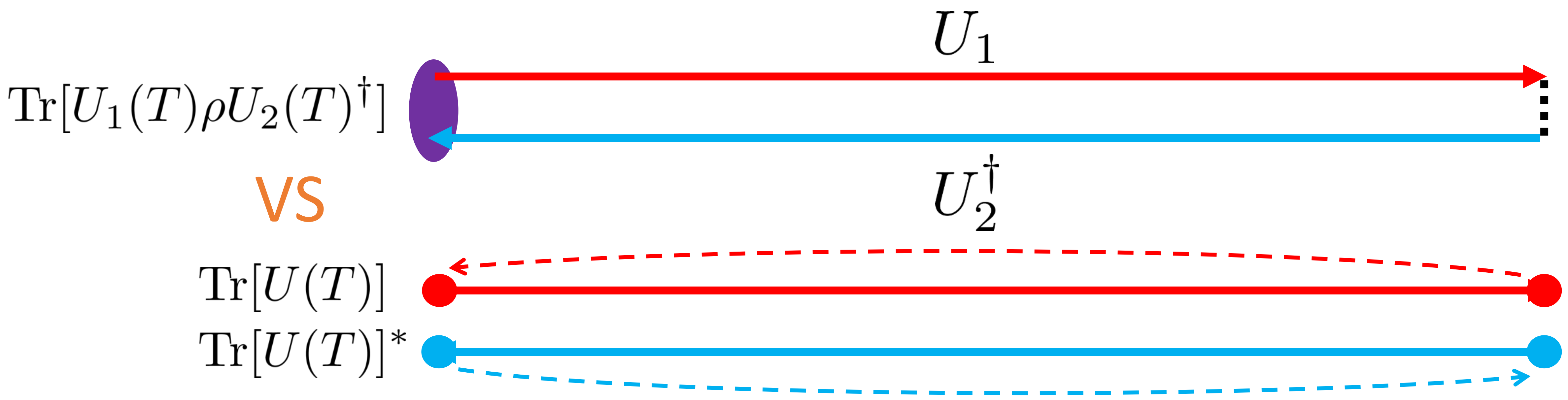


- Symmetric (classical, r-type) and antisymmetric (quantum, a-type) variables, powerful set of rules that govern allowed effective actions
- Let’s focus on energy diffusion as a simple example $\epsilon = c\beta^{-1}\partial_t \phi_r$

$$L = -\phi_a(\partial_t - D\Delta)\epsilon + i\beta^{-2}\kappa(\nabla\phi_a)^2$$ [Glorioso-Liu, Chen-Lin et al.]

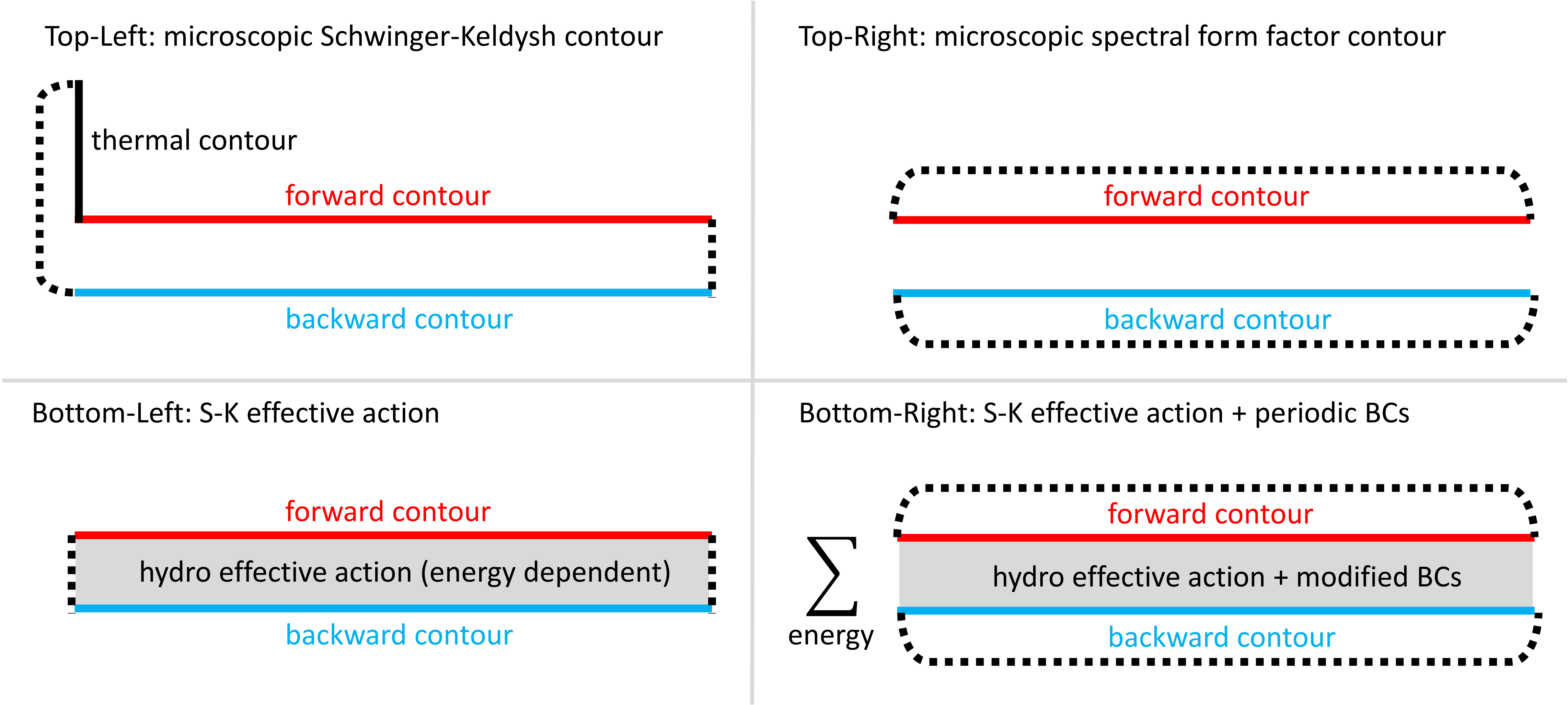
- Uncompleting the square in a-type variable leads to representation of path integral in terms of fluctuating energy diffusion

Ramp from modified CTP on the SFF contours



Assumptions:

- At cutoff scale, same hydro action with modified boundary conditions
- Modified hydro action gives dominant saddle point for SFF
- There is some averaging, e.g. disorder, that connects the contours



Top-Left: microscopic Schwinger-Keldysh contour
thermal contour
forward contour
backward contour
Top-Right: microscopic spectral form factor contour
forward contour
backward contour
Bottom-Left: S-K effective action
forward contour
hydro effective action (energy dependent)
backward contour
Bottom-Right: S-K effective action + periodic BCs
forward contour
hydro effective action + modified BCs
backward contour
energy

Spatial zero mode

- Times long compared to the longest lifetime: $L = -\phi_a \partial_t \epsilon$
- Up to the usual ambiguities of regulating the measure, the path integral reduces to an integral over the zero-frequency components

$$\mathrm{SFF} \propto \int d\phi_a(k=0,\omega=0)d\epsilon(k=0,\omega=0) \propto T\int dE$$

- For higher powers, many different ways to connect contours, reproduces expected result from RMT (to leading order)

e.g. GUE symmety: $\overline{Z^k(Z^*)^k} = k!\mathrm{SFF}^k$

[Winer-S, previously discussed for SYK in Saad-Shenker-Stanford]

Full path integral

$$\mathrm{SFF} = \int \mathcal{D}\epsilon \mathcal{D}\phi_a \exp(iS_{\mathrm{hydro}})$$

$$\mathcal{D}\epsilon\mathcal{D}\phi_a = \prod_x \prod_{\ell=0}^{T/\Delta t-1} \frac{d\epsilon(x, t=\ell\Delta t) d\phi_a(x, t=\ell\Delta t)}{2\pi}$$

$$S_{\mathrm{hydro}} = \int dV\, dt \Big(-\phi_a(\partial_t - D\Delta)\epsilon - icT^2\kappa\phi_a\Delta\phi_a \Big)$$

eigenvalues of $dt\partial_t$: $T/\Delta t$ complex numbers $i\omega$ obeying $(i\omega+1)^{T/\Delta t} = 1$

$$\mathrm{SFF} = \prod_k \prod_\omega \frac{1}{i\omega - \lambda_k \Delta t} = \prod_k \frac{1}{1-e^{\lambda_k T}} \qquad \lambda_k = -Dk^2$$

exactly reproduces prior calculation

[Winer-S]

Interactions?

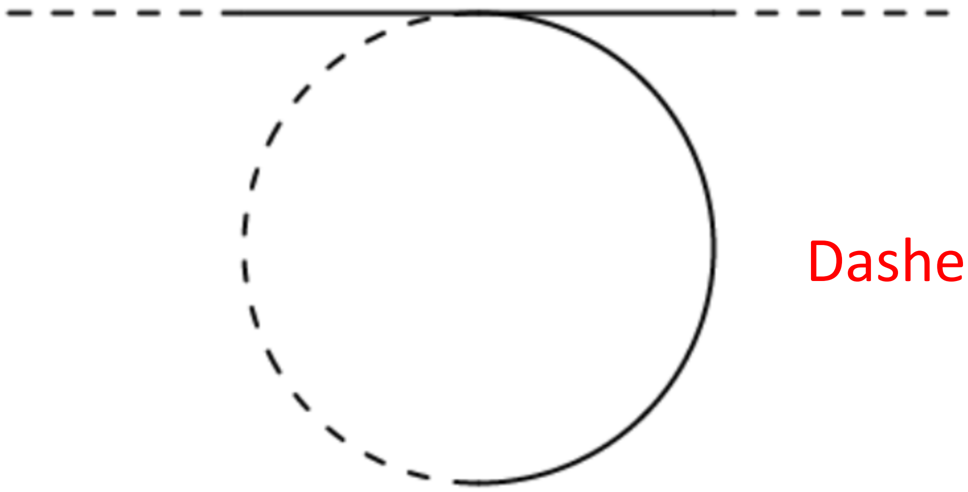
$$\Delta L = \frac{\lambda}{2}\Delta\phi_a\epsilon^2 + \frac{\lambda'}{3}\Delta\phi_a\epsilon^3 + ic\beta^{-2}(\nabla\phi_a)^2(\tilde{\lambda}\epsilon + \tilde{\lambda}'\epsilon^2)$$

[Chen-Lin et al.]

- Modified propagators, e.g. $G_{ar}^{\mathrm{circle}}(t,k) = \sum_n G_{ar}^{\mathrm{CTP}}(t+nT,k)$

$$G_{ar}^{\mathrm{CTP}}(t,k) = i\theta_+(t)e^{-Dk^2t} \longrightarrow G_{ar}^{\mathrm{circle}}(t,k) = i\frac{e^{-Dk^2t}}{1-e^{-Dk^2T}}, t \in (0,T]$$

- Novel diagrams, e.g.



Dashed: a-type, solid: r-type

[Winer-S]

Example: self energy

$$G_{ar}(\omega, k) = \frac{i\omega + Dk^2}{\Sigma D\kappa k^2 + (D^2k^4 + \omega^2)}$$

- Let's look for a self-consistent solution with a constant self energy

$$\Sigma = \lambda' \sum_n \int d^dk k^2 \frac{Dk^2}{\sqrt{D^2k^4 + \Sigma D\kappa k^2}} \exp(-n\sqrt{D^2k^4 + \Sigma D\kappa k^2}T)$$

$$\Sigma \to \lambda' \sum_n \int d^dk k^2 \exp(-nDk^2T) = \lambda' \frac{d}{2DT} \sqrt{\frac{\pi}{DT}}^d \zeta(1 + d/2)$$

Consistent to set Sigma=0 on RHS

- Ignoring all other effects and redoing previous calculation, significant modification for d=1 (in the continuous wavevector regime):

$$\log \mathrm{coeff}(T) = \frac{L}{2\pi} \frac{2}{T\sqrt{\Sigma D\kappa}} \frac{\pi^2}{6} + \cdots$$

caveats: should include all diagrams, effect for a given L may not be dramatic

[Winer-S]

Outlook

Spectral quantum chaos is generic and robust

- Key lesson: hydro → * chaos in the spectral sense: the spatial zero mode gives the usual ramp with correct coefficient (after regulating) and the non-zero modes compute the return probabilities
- After Thouless time, hydro predicts universality: $H = H_0 + g\delta H$

$$\delta|Z|^2 = \mathrm{Tr}\left(i\delta H e^{iHT}\right)\mathrm{Tr}e^{-iHT} - \mathrm{Tr}e^{iHT}\mathrm{Tr}\left(i\delta H e^{-iHT}\right)$$

a-type expectation

- Another perspective from eigenstate thermalization:

$$\langle n|\delta H|m\rangle = \langle \delta H\rangle(E_n)\delta_{nm} + R_{mn} \longrightarrow H = f_{\mathrm{stretch}}(H) + \mathrm{random}$$

- Path integral point of view: spontaneous time translation symmetry breaking, corresponding symmetry cannot be explicitly broken

* terms and conditions apply

Some comments and directions

- Growing number of connections between different manifestations of quantum chaos, from hydrodynamics to eigenstate thermalization to random matrix energy levels (e.g. D'Alessio review); synthesis?
- We provided tools to compute SFFs in systems with slow modes, applications to weakly Floquet systems (hydro paper), symmetry breaking (2106.07674) and glasses (WIP)
- Many directions: Hydro theory of the plateau? Higher moments? RG theory of interactions? …

THANK YOU!