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On Topological Boundaries in 2+1d

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Introduction

- Topological Quantum Field Theory (TQFT) in 2+1d has been an active research topic in high energy physics, condensed matter physics, and mathematics in the past few decades.

- It includes the familiar Chern-Simons theory (CS) and also the finite group gauge theory.

- It has applications in conformal field theory, topological order, knot theory…

Boundary

- Both in high energy physics and in condensed matter systems, it is common to study a 2+1d TQFT on a manifold with a 1+1d boundary.
- CS/WZW, Quantum Hall effects…
- Generally, the boundary can host many different kinds of edge modes.
- What about a trivially gapped boundary? Namely, a topological boundary condition?

Gapped boundary

- Given a 2+1d TQFT, does it admit a gapped boundary?

- We will restrict ourselves to bosonic TQFTs in this talk.

- Generally, a TQFT may not admit a gapped boundary. In this case, its boundary is forced to have gapless edge modes.

Gapped boundary

- Example: $U(1)_{2N}$ CS theory does **NOT** admit a gapped boundary.

$$\mathcal{L} = \frac{2N}{4\pi} ada$$

Its boundary can be the chiral boson CFT.

- Example: $\mathbb{Z}_N$ gauge theory **DOES** admit gapped boundary conditions.

$$\mathcal{L} = \frac{N}{2\pi} adb$$

Gapped boundary

- Question: What are the sufficient and necessary conditions for a 2+1d bosonic TQFT to have a gapped boundary?

- It's an old question with many interesting results in the literature.

[Kapustin-Saulina 2010, Davydov-Mueger-Nikshych-Ostrik 2010, Kitaev-Kong 2011, Fuchs-Schweigert-Valentino 2012, …, Levin 2013, Barkeshli-Jian-Qi 2013 … , Freed-Teleman 2020, …]

- We will build on these old results to derive new ones.

Main results

- We find new obstructions to a 2+1d bosonic TQFT admitting a gapped boundary. For abelian TQFTs, these new obstructions arise from the phases of the partition functions on closed three-manifolds.
- **Theorem** [KKOSS]

An abelian bosonic TQFT (G, θ) with one-form symmetry G has a gapped boundary **iff**

$$\xi_n \equiv \frac{\sum_a \theta_a^n}{|\sum_a \theta_a^n|} = +1$$

$$\text{for all } n \text{ such that } \gcd\left(n, \frac{2|G|}{\gcd(2|G|, n)}\right) = 1$$

Here θ_a is the spin of the anyon a.

Outline

- Lightening review of 2+1d TQFT
- Abelian TQFT & Lens spaces
- Non-abelian TQFT

Lightening Review
of 2+1d TQFT

Review of 2+1d TQFT

- The only local operator is the identity operator.

- Finitely many topological lines in spacetime. For example, they can be the Wilson lines in Chern-Simons theory.

- These lines are the worldlines of the microscopic anyon excitations.

- We will review some, but not all, basic properties of a general 2+1d TQFT.

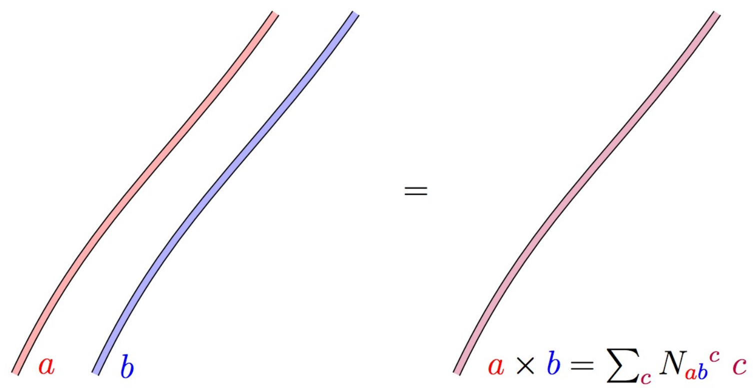
Fusion

- Two anyons can be fused together with a fusion rule

$$a \times b = \sum_c N_{ab}^c \, c \quad , \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}$$

- An anyon a is called abelian if its fusion with an arbitrary anyon b has a single anyon on the RHS, i.e. $a \times b = c$.
- A TQFT with only abelian anyons is called an abelian TQFT.

Figure taken from [Delmastro-Gomis]



Topological spin

- An anyon in spacetime should be thought of as a ribbon, rather than a line with zero width. This can be thought of as a point-splitting regularization for the Wilson line.
- We will also write $\theta_a = \exp(2\pi i\, h_a)$, where $h_a \in \mathbb{R}/\mathbb{Z}$ is the spin of the microscopic anyon excitation (defined modulo 1).

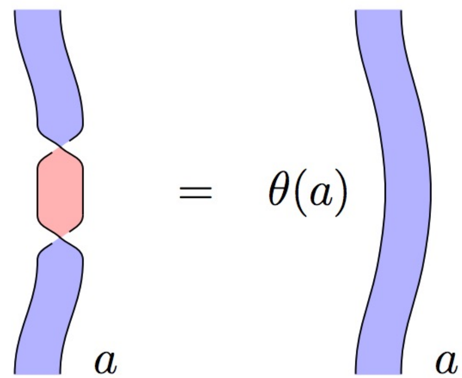
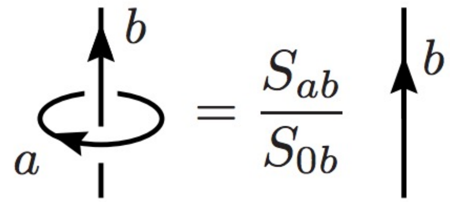


Figure taken from
[Delmastro-Gomis]

S and T matrices

- A representation of the modular group. The S matrix determines the braiding between two anyons:



- The T matrix is defined in terms of the spins of the anyons,

$$T_{ab} = \theta_a \delta_{ab}$$

Figure taken from [Barkeshli-Bonderson-Meng-Wang]

Chiral central charge

- The chiral central charge c_- captures the perturbative gravitational anomaly of the boundary edge modes.

- Therefore, if $c_- \neq 0$, then the TQFT does **NOT** admit a gapped boundary. This is the first obstruction to a gapped boundary.

- If $c_- = 0 \bmod 8$, we can stack an appropriate power of some invertible QFTs (such as the $(E_8)_1$ Chern-Simons theory) to cancel the gravitational anomaly.

Chiral central charge

- While $c_- = 0 \bmod 8$ is a necessary condition for gapped boundaries, it is not sufficient.
- Many $c_- = 0 \bmod 8$ TQFTs, such as $U(1)_2 \times U(1)_{-4}$, do **NOT** admit a gapped boundary.
- These gapless edges are <u>not</u> protected by ordinary global symmetries or anomalies.
- We will henceforth assume $c_- = 0 \bmod 8$. In this case, there is a scheme in which the TQFT partition function is topological and independent the choice of the framing of the three-manifold.

Abelian TQFT
& Lens Spaces

Abelian TQFT

- For simplicity, let us start with abelian TQFTs. Every abelian TQFT can be described by an abelian Chern-Simons theory. [Belov-Moore 2005] (see also [Stirling 2008])

- The fusion of the anyons form an abelian finite group G. It is the one-form global symmetry of the abelian TQFT. The symmetry generators are the anyons.

- Gauging a one-form symmetry in a 2+1d bosonic TQFT [Moore-Seiberg 1989] is known as condensing the corresponding anyons in condensed matter theory.

’t Hooft anomalies of one-form symmetry

- ’t Hooft anomaly means that there is an obstruction to gauging the one-form symmetry.
- This obstruction is captured by the spins θ_a of the anyons. [Gaiotto-Kapustin-Seiberg-Willet 2014, Gomis-Komargodski-Seiberg 2016, Hsin-Lam-Seiberg 2018]
- The one-form symmetry generated by the boson lines (i.e., $\theta_a = 1$) is free of the ‘t Hooft anomalies.
- In an abelian TQFT, however, the set of all boson lines might NOT be closed under fusion.

Gauging the one-form symmetry

- In this HEP language, the result of [Kapustin-Saulina 2010, Fuchs-Schweigert-Valentino 2012, Levin 2013, Barkeshli-Jian-Qi 2013] can be phrased as follows:

An abelian TQFT has a gapped boundary if and only if there is a non-anomalous one-form symmetry subgroup $L \subset G$ with $|L|^2 = |G|$.

- If we gauge L, the TQFT becomes trivial.
- In CMT, this is known as condensing a Lagrangian subgroup L of anyons.

Example: $\mathbb{Z}_2$ gauge theory

- The $\mathbb{Z}_2$ gauge theory can be described using a pair of one-form $U(1)$ gauge fields a, b [Maldacena-Moore-Seiberg 2001, Banks-Seiberg 2010, Kapustin-Seiberg 2014]:

$$\frac{2}{2\pi} adb$$

- This is the low energy theory of the toric code [Kitaev 1997].
-

Anyon	**1**	e $\exp\left(i\oint a\right)$	m $\exp(i\oint b)$	f $\exp(i\oint a + i\oint b)$
Spin θ_a	+1	+1	+1	-1

- Fusion: $e \times e = 1, \;\; m \times m = 1, \;\; f \times f = 1, \;\; e \times m = m \times e = f$

Example: $\mathbb{Z}_2$ gauge theory

Anyon	**1**	e $\exp\left(i\oint a\right)$	m $\exp(i\oint b)$	f $\exp(i\oint a + i\oint b)$
Spin θ_a	+1	+1	+1	-1

- One-form symmetry group $G = \mathbb{Z}_2^{(e)} \times \mathbb{Z}_2^{(m)}$. Mixed anomaly between $\mathbb{Z}_2^{(e)}$ and $\mathbb{Z}_2^{(m)}$.
- Two Lagrangian subgroups: $L = \mathbb{Z}_2^{(e)}$ and $L = \mathbb{Z}_2^{(m)}$.
- Gauging either one of them (but not both) gives the trivial theory.

Example: $\mathbb{Z}_2$ gauge theory

Two gapped boundary conditions:

Boundary condition	$\boldsymbol{a}\| = \mathbf{0}$	$\boldsymbol{b}\| = \mathbf{0}$
Lagrangian subgroup	$L = \mathbb{Z}_2^{(e)}$	$L = \mathbb{Z}_2^{(m)}$
Subgroup broken by the b.c.	$\mathbb{Z}_2^{(m)}$ is broken	$\mathbb{Z}_2^{(e)}$ is broken

Obstructions labeled by 3-Manifolds

- Let $Z(M)$ be the partition function of an abelian TQFT (G, θ) on a closed three-manifold M.
- We will show that (assuming $c_- = 0 \bmod 8$)

Theorem [KKOSS]: An abelian TQFT (G, θ) has a gapped boundary **only if**

$$Z(M) > 0$$

for any three-manifold M with $\gcd(|G|, |H_1(M)|) = 1$.

Obstructions labeled by 3-Manifolds

Theorem: An abelian TQFT (G, θ) has a gapped boundary **only if** $Z(M) > 0$ for any manifold M with $\gcd(|G|, |H_1(M)|) = 1$.

Proof:

- An abelian TQFT has a gapped boundary iff one can gauge a non-anomalous one-form symmetry subgroup L with $|L|^2 = |G|$ to obtain the trivial theory.

$$1 = \frac{|H^0(M,L)|}{|H^1(M,L)|} \sum_{A \in H^2(M,L)} Z(M, A)$$

Partition function of trivial theory	Positive normalization factor from the volume of the gauge group	Sum over 2-form L gauge field A	Partition function of the abelian TQFT G coupled to L gauge field

Obstructions labeled by 3-Manifolds

- It is easy to show that $\gcd(|G|, |H_1(M)|) = 1$ if and only if
$$H^2(M, G) = 0$$
- Hence, there is no nontrivial 2-form gauge field for the one-form symmetry G or its Lagrangian subgroup L, i.e., $H^2(M, L) = 0$.
- Since gauging the Lagrangian subgroup L is now trivial:
$$Z(M) = \frac{1}{|L|}$$
- In particular, $Z(M) > 0$. Q.E.D.

Lens space

Theorem: An abelian TQFT (G,θ) has a gapped boundary **only if** $Z(M) > 0$ for any manifold M with $\gcd(|G|, |H_1(M)|) = 1$.

- Let us look for the simplest three-manifold M with finite $|H_1(M)|$.
- Lens space

$$L(n,1) = S^3/\mathbb{Z}_n$$

has $H_1(L(n,1),\mathbb{Z}) = \mathbb{Z}_n$.

- For an abelian TQFT (G,θ), choose n such that $\gcd(|G|, n) = 1$.

Higher central charges

- For an abelian TQFT (G, θ), the lens space partition function is

$$Z\big(L(n,1)\big) = (ST^{-n}S)_{00} = |G|^{-1} \sum_a \theta_a^{-n}.$$

- The phase of the partition function (changing $n \to -n$)

$$\xi_n \equiv \frac{\sum_a \theta_a^n}{|\sum_a \theta_a^n|} \in U(1)$$

is known as the higher central charge [Ng-Schopieray-Wang 2018, Ng-Rowell-Wang-Zhang 2020].

- It reduces to the ordinary chiral central charge when $n = 1$:

$$\exp(2\pi i \frac{c_-}{8}) = \frac{\sum_a \theta_a}{|\sum_a \theta_a|}$$

Higher central charges

Theorem [Ng-Rowell-Wang-Zhang 2020]

An abelian bosonic TQFT (G, θ) has a gapped boundary **only if**

$$\xi_n \equiv \frac{\sum_a \theta_a^n}{|\sum_a \theta_a^n|} = +1$$

for all n such that $\gcd(n, |G|) = 1$

- Remark: Their theorem applies to nonabelian TQFTs as well, but the formula for ξ_n will be slightly generalized.

Sufficient and necessary conditions

Theorem [KKOSS]

An abelian bosonic TQFT (G,θ) has a gapped boundary **iff**

$$\xi_n \equiv \frac{\sum_a \theta_a^n}{|\sum_a \theta_a^n|} = +1$$

for all n such that $\gcd\left(n, \frac{2|G|}{\gcd(2|G|,n)}\right) = 1$

- Remark: We use the prime factorization property of abelian TQFTs.

Example: $U(1)_{2N_1} \times U(1)_{-2N_2}$

- $U(1)_{2N_1} \times U(1)_{-2N_2}$ has vanishing chiral central charge $c_- = 0$.

- One-form symmetry group $G = \mathbb{Z}_{2N_1} \times \mathbb{Z}_{2N_2}$.

- What are the conditions on N_1, N_2 such that $U(1)_{2N_1} \times U(1)_{-2N_2}$ admits a gapped boundary?

- Lagrangian subgroup L (which obeys $|L|^2 = |G|$) exists iff $N_1 N_2$ is a perfect square.

Example: $U(1)_{2N_1} \times U(1)_{-2N_2}$

Recall that $\left(\frac{a}{p}\right) =$

1 if $a = x^2 \bmod p$ and $a \neq 0$
-1 otherwise
0 if $a = 0 \bmod p$

- The higher central charges are the Jacobi symbols:

$$\xi_n \equiv \frac{\sum_a \theta_a^n}{|\sum_a \theta_a^n|} = \left(\frac{N_1 N_2}{n}\right) \quad , \ \gcd(n, 2N_1N_2) = 1$$

- **Math Fact**: A positive integer N is a perfect square iff

$\left(\frac{N}{n}\right) = +1$ for all primes n such that $\gcd(n, 2N) = 1$

- We have thus demonstrated that the $U(1)_{2N_1} \times U(1)_{-2N_2}$ has a gapped boundary **iff** all the higher central charges vanish.

Gauging back and forth

- Gauging a finite symmetry is invertible. In the context of abelian orbifolds in 1+1d, this is related to the "quantum symmetry" [Vafa 1986]:

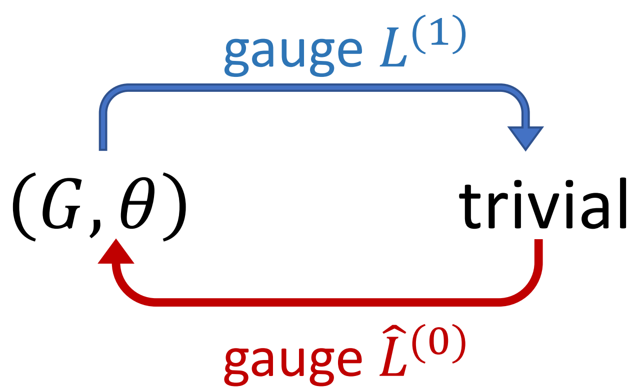
$$\text{1+1d:} \quad \mathbb{Z}_N^{(0)} \underset{\text{gauging}}{\longleftrightarrow} \mathbb{Z}_N^{(0)}$$

- In 2+1d, gauging a one-form symmetry is the inverse of gauging a zero-form symmetry [Gaiotto-Kapustin-Seiberg-Willett 2014, Tachikawa 2017]:

$$\text{2+1d:} \quad \mathbb{Z}_N^{(1)} \underset{\text{gauging}}{\longleftrightarrow} \mathbb{Z}_N^{(0)}$$

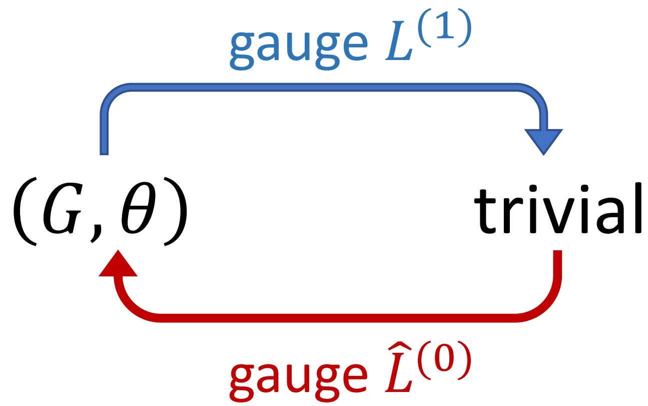
Gauging back and forth

- Recall: An abelian TQFT (G, θ) has a gapped boundary iff there is a non-anomalous one-form symmetry subgroup $L \subset G$ such that when we gauge L, it becomes trivial.
- Conversely, such an abelian TQFT (G, θ) can be obtained by coupling the trivial theory to a discrete $\hat{L}$ gauge field.
- In other words, it is an abelian finite group gauge theory, possibly with a Dijkgraaf-Witten twist.



Topological boundaries of abelian TQFT

- On the other hand, any finite group gauge theory admits a topological boundary (e.g. the Dirichlet boundary).
- **Fact:** An abelian TQFT has a gapped boundary iff it is an abelian Dijkgraaf-Witten gauge theory.



Topological boundaries of abelian TQFT

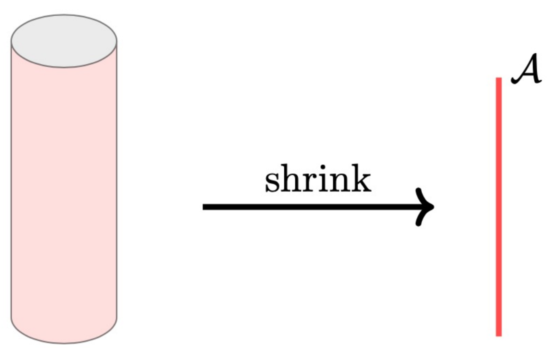
The following conditions for an abelian bosonic TQFT (G, θ) are equivalent to each other:

- ∃ Topological boundary
- ∃ Lagrangian subgroup
- Abelian Dijkgraaf-Witten gauge theory
- $\sum_a \theta_a^n > 0$ for all n such that $\gcd\left(n, \frac{2|G|}{\gcd(n,2|G|)}\right) = 1$ [KKOSS].

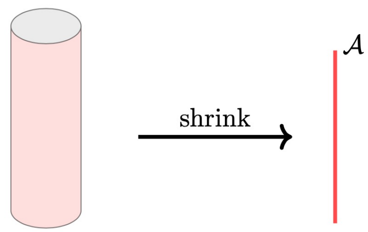
Non-abelian TQFT

Empty tube

- Let us assume a (non-abelian) TQFT admits a topological boundary condition.
- Then we can cut a small cylindrical tube and impose the topological boundary condition on the surface of the cylinder.
- Since everything is topological, we can shrink the radius of the cylinder at will.



Empty tube

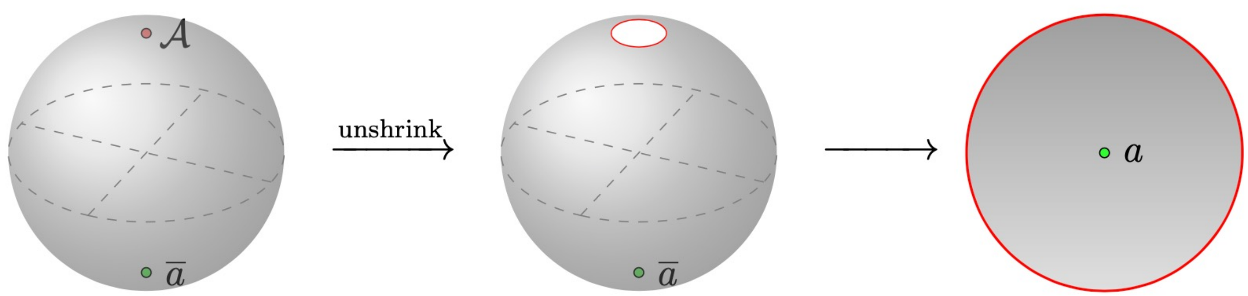


- This shrinking defines a line defect, which is a direct sum of simple anyons a :

$$\mathcal{A} = \bigoplus_a Z_{0a} a \quad , \quad Z_{0a} \in \mathbb{Z}_{\geq 0}$$

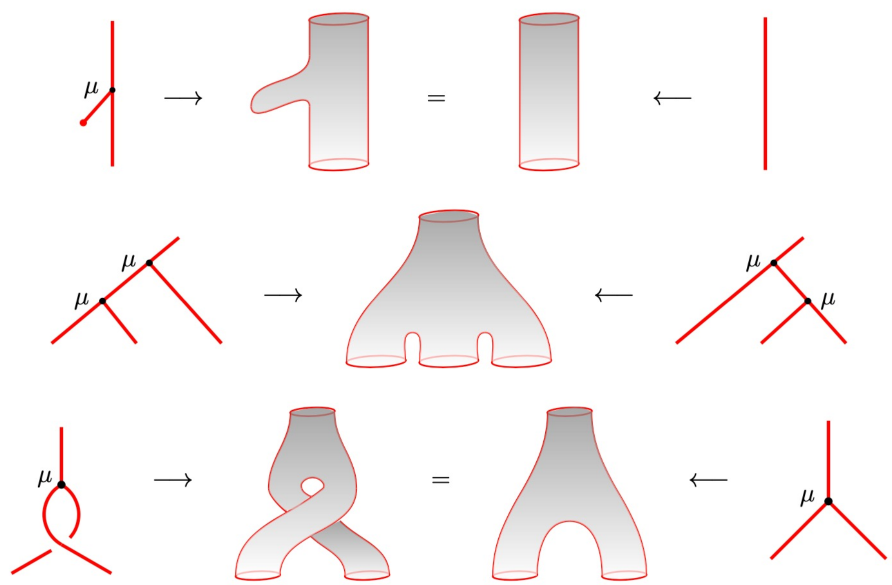
- Consider wrapping $\mathcal{A}$ and the anyon $\bar{a}$ on the S^1 of $S^2 \times S^1$:

$$Z_{0a} = \dim \mathcal{H}(D^2; a)$$



Lagrangian algebra

- The fact that $\mathcal{A} = \bigoplus_a Z_{0a} a$ arises from an empty tube implies several nontrivial properties. It defines what's called a Lagrangian algebra.

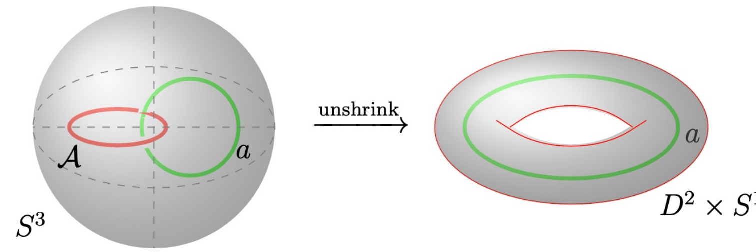


Lagrangian algebra

- The Lagrangian algebra $\mathcal{A} = \bigoplus_a Z_{0a} a$ is the generalization of a Lagrangian subgroup for abelian TQFTs.
- In the case of an abelian TQFT with a topological boundary, all the $Z_{0a} = 0$ or 1. The fusion of the set of anyons with $Z_{0a} = 1$ forms a subgroup of the one-form symmetry.
- For a Lagrangian algebra, the set of anyons with $Z_{0a} \neq 0$ are generally <u>not</u> closed under fusion.
- Example: Let $\mathcal{T}$ be a bosonic TQFT with anyons labeled by a_i. Let $\bar{\mathcal{T}}$ be its orientation reversal with anyons labeled by $\tilde{a}_i$. Then the tensor product TQFT $\mathcal{T} \times \bar{\mathcal{T}}$ has a Lagrangian algebra:

$$\mathcal{A} = \bigoplus_i (a_i \otimes \tilde{a}_i)$$

RCFT



- We can put such a TQFT on a Riemann surface times an interval. We impose the topological boundary condition on one end.
- On the other end, we impose the boundary condition supporting the chiral RCFT.
- Compactifying the interval gives a holomorphic CFT.
- Therefore, the existence of a topological boundary implies that the chiral algebra of the boundary RCFT can be extended to a single module. (See [Moore-Seiberg 1989].)
- In particular, one can show that $S_{ab}Z_{0b} = Z_{0a}$ and $T_{ab}Z_{0b} = Z_{0a}$.

Topological boundaries of TQFT

The conditions in each column are equivalent to each other: [Kapustin-Saulina 2010, Davydov-Mueger-Nikshych-Ostrik 2010, Kitaev-Kong 2011, Fuchs-Schweigert-Valentino 2012, …, Levin 2013, Barkeshli-Jian-Qi 2013… , Freed-Teleman 2020, KKOSS]

Abelian TQFT	Non-abelian TQFT
∃ Topological boundary	∃ Topological boundary
∃ Lagrangian subgroup	∃ Lagrangian algebra
Abelian Dijkgraaf-Witten gauge theory	Turaev-Viro theory / Drinfeld center
$\sum_a \theta_a^n > 0$ for all n such that $\gcd\left(n, \frac{2\|G\|}{\gcd(n,2\|G\|)}\right) = 1$ [KKOSS]	?

Summary

- **Theorem**: An abelian TQFT (G, θ) has a gapped boundary **only if** $Z(M) > 0$ for any manifold M with $\gcd(|G|, |H_1(M)|) = 1$.
- **Theorem:** An abelian bosonic TQFT with one-form symmetry G has a gapped boundary **iff**

$$\xi_n \equiv \frac{\sum_a \theta_a^n}{|\sum_a \theta_a^n|} = +1$$

for all n such that $\gcd\left(n, \frac{2|G|}{\gcd(2|G|, n)}\right) = 1$

- It would be interesting to find an analogous computable sufficient and necessary condition for non-abelian TQFTs.

Thank you!