TP - A Solution

A)

$$dV = \left.\frac{\partial V}{\partial T}\right|_P dT + \left.\frac{\partial V}{\partial P}\right|_T dP$$ is a perfect differential. Thus $\frac{\partial^2 V}{\partial P \partial T} = \frac{\partial^2 V}{\partial T \partial P}$

gives

$$\left.\frac{\partial \beta}{\partial P}\right|_T = -\left.\frac{\partial \kappa_T}{\partial T}\right|_P$$

B) $$\left.\frac{\partial \kappa_T}{\partial T}\right|_P = \frac{(10 \cdot 10^{-12} - 7 \cdot 10^{-12}) m^2 / N}{1000^0 K} = 3 \cdot 10^{-15} = -\left.\frac{\partial \beta}{\partial P}\right|_T$$

$$\Delta\beta = \left.\frac{\partial \beta}{\partial P}\right|_T \Delta P = -3 \cdot 10^{-15}\, 10^3 \frac{10^5 N / m^2}{1 atm} = -3 \cdot 10^{-7} /^0 K$$

Solution to Problem TP-B

(a) Equation of State:

$$PV = NkT. \tag{1}$$

First Law:

$$dQ = dE + PdV. \tag{2}$$

E depends only on T.

$$NC_V = \left(\frac{\partial Q}{\partial T}\right)_V = \frac{dE}{dT} \tag{3}$$

$$NC_P = \left(\frac{\partial Q}{\partial T}\right)_P = \frac{dE}{dT} + \frac{d}{dT}(PdV) = \frac{dE}{dT} + Nk. \tag{4}$$

Therefore

$$C_P = C_V + k. \tag{5}$$

(b) From (1),

$$PdV + VdP = NkdT. \tag{6}$$

From (3) and (2) with $dQ = 0$,

$$dT = \frac{dE}{NC_V} = -\frac{PdV}{NC_V}. \tag{7}$$

Combining (6) and (7)

$$\gamma PdV + VdP = 0, \tag{8}$$

with

$$\gamma = 1 + \frac{k}{C_V} = \frac{C_P}{C_V}, \tag{9}$$

where the last equality made use of (5). If C_V is taken independent of T (exact for a monatomic ideal gas), then (8) can be integrated to give

$$PV^\gamma = \text{constant}. \tag{10}$$

(c) Helium gas is monatomic, so that the only contribution to E is translational motion. Therefore $E = \frac{3}{2}kT$, $C_V = \frac{3}{2}k$, and $\gamma = \frac{5}{3}$. Equation (10) then implies that the P increases by a factor of $2^5 = 32$.

TP – C1 Solution

(a) For quantum particles confined in a 2D "box: $k_x = \frac{\pi n_x}{L_x}$ $\quad k_y = \frac{\pi n_y}{L_y}$ $\quad k = \sqrt{k_x^{\ 2} + k_y^{\ 2}}$

$$N(k) = \frac{1}{4}\frac{\pi k^2}{\frac{\pi}{L_x}\times\frac{\pi}{L_y}} = \frac{k^2(area)}{4\pi} \quad G(k) = \frac{k^2}{4\pi} \quad G(\varepsilon) = \frac{1}{4\pi}\frac{2m\varepsilon}{\hbar^2} \qquad g^{2D}(\varepsilon) = \frac{(2s+1)m}{2\pi\hbar^2} = \frac{m}{\pi\hbar^2}$$

So $g^{2D}(\varepsilon)$ does not depend on energy.

(b) The Fermi energy is given by: $N = \int_0^{E_F} g(\varepsilon)(area)d\varepsilon = A\frac{m}{\pi\hbar^2}E_F \qquad E_F = \frac{\pi\hbar^2}{m}\left(\frac{N}{A}\right)$

(c) The average energy per electron:

$$U_{tot} = N\langle\varepsilon\rangle = \int_0^{E_F}\varepsilon\times g(\varepsilon)(area)d\varepsilon = \frac{m(area)}{\pi\hbar^2}\int_0^{E_F}\varepsilon\, d\varepsilon = \frac{m(area)}{\pi\hbar^2}\frac{1}{2}E_F^2$$

$$U_{tot} = N\langle\varepsilon\rangle = \int_0^{E_F}\varepsilon\times g(\varepsilon)(area)d\varepsilon = \frac{mA}{\pi\hbar^2}\int_0^{E_F}\varepsilon\, d\varepsilon = \frac{mA}{\pi\hbar^2}\frac{1}{2}E_F^2 \qquad \langle\varepsilon\rangle = \frac{mA}{\pi\hbar^2 N}\frac{1}{2}E_F^2 = \frac{E_F}{2}$$

(d) $$E_F = \frac{\pi\hbar^2}{m}\left(\frac{N}{A}\right) = \frac{\pi\times\left(1.05\cdot10^{-34}\,Js\right)^2}{0.2\times9.1\cdot10^{-31}kg}1\cdot10^{16}m^{-2} = 1.9\cdot10^{-21}J = k_B\times140K$$

So, at 300K the system is not degenerate.

TP – C2 Solutions

To find the entropy $S(V,T)$ use

$$TdS = dU + pdV$$

$$dS = \left.\frac{\partial S}{\partial T}\right|_V dT + \left.\frac{\partial S}{\partial V}\right|_T dV \qquad dU = \left.\frac{\partial U}{\partial T}\right|_V dT + \left.\frac{\partial U}{\partial V}\right|_T dV$$

Then

$$\left.\frac{\partial S}{\partial T}\right|_V = \frac{1}{T}\left.\frac{\partial U}{\partial T}\right|_V = \frac{f}{2}\frac{Nk_B}{T} \quad \text{or} \quad S \sim \frac{f}{2} Nk_B \ln T$$

$$\left.\frac{\partial S}{\partial V}\right|_T = \frac{1}{T}\left(\left.\frac{\partial U}{\partial V}\right|_T + P\right) = \frac{Nk_B}{V} \quad \text{or} \quad S \sim Nk_B \ln V$$

Thus $S \sim Nk_B \ln\{T^{f/2}V\}$

B) For an adiabatic process $dS = 0 \Rightarrow T^{f/2}V = \text{constant}$

C) $dQ = dU + pdV = \left.\frac{\partial U}{\partial T}\right|_V dT + (\left.\frac{\partial U}{\partial V}\right|_T + p)dV$

$$\left.\frac{dQ}{dT}\right)_V = \left.\frac{\partial U}{\partial T}\right|_V = \frac{f}{2}Nk_B \qquad \text{and} \qquad C_V = \frac{f}{2}k_B$$

$$\left.\frac{dQ}{dT}\right)_p = \left.\frac{\partial U}{\partial T}\right|_V + (\left.\frac{\partial U}{\partial V}\right|_T + p)\left.\frac{\partial V}{\partial T}\right|_P = \left.\frac{\partial U}{\partial T}\right|_V + (\left.\frac{\partial U}{\partial V}\right|_T + p) - 1\left(\frac{\left.\frac{\partial p}{\partial T}\right)_V}{\left.\frac{\partial p}{\partial V}\right)_T}\right)$$

Substituting

$$\left.\frac{\partial p}{\partial T}\right|_V = \frac{Nk_B}{V} \quad \& \quad \left.\frac{\partial p}{\partial V}\right|_T = -\frac{Nk_BT}{V^2} + 2b_2\frac{N^2}{V^3}$$

gives $$C_P = C_V + \frac{Nk_B^2TV}{Nk_BTV - 2b_2N^2} = \frac{f}{2}k_B + \frac{k_B}{1 - 2b_2\frac{N/V}{k_BT}}$$

TP – D1 Solution

The magnetic moment μ is given by

$$\mu = \frac{\sum_{lm} m\tilde{\mu} e^{-\beta\epsilon_{lm}}}{\sum_{lm} e^{-\beta\epsilon_{lm}}} \equiv \frac{1}{\beta Z}\frac{\partial Z}{\partial B}, \tag{1}$$

where

$$Z = \sum_{lm} e^{-\beta\epsilon_{lm}} \equiv \sum_{l=0}^{\infty} e^{-\beta\tilde{\epsilon}l(l+1)} \sum_{m=-l}^{l} e^{\beta\tilde{\mu}Bm}, \tag{2}$$

and $\beta = (kT)^{-1}$. In both parts of the problem, B is small, so we may expand to the lowest nontrivial order:

$$Z \approx \sum_{l=0}^{\infty} e^{-\beta\tilde{\epsilon}l(l+1)} \sum_{m=-l}^{l} \left[1 + \frac{1}{2}(\beta\tilde{\mu}B)^2 m^2\right], \tag{3}$$

where the linear term in B vanishes because of the symmetry of the sum over m.

case 1 Here $\beta\tilde{\epsilon} \gg 1$ so that the sum over l is restricted to small l. The lowest nontrivial order is $l = 1$, so that we can truncate the series there. We therefore obtain

$$Z \approx 1 + 3e^{-2\beta\tilde{\epsilon}} + (\beta\tilde{\mu}B)^2 e^{-2\beta\tilde{\epsilon}}. \tag{4}$$

Finally, using the last equality of Eq. (1) and keeping only the leading order in B and $e^{-2\beta\tilde{\epsilon}}$, one finds

$$\mu \approx \frac{2\tilde{\mu}^2 B}{kT} \exp\left(-\frac{2\tilde{\epsilon}}{kT}\right). \tag{5}$$

case 2 Here $\beta\tilde{\epsilon}$ is small, so there will be a large number of terms in the sum, which will be dominated by terms where $l \gg 1$. Therefore in this limit we may replace the sums by integrals and $l(l+1)$ by l^2, giving

$$\begin{aligned} Z &\approx \int_0^\infty dl\, e^{-\beta\tilde{\epsilon}\, l^2} \int_{-l}^{l} dm \left[1 + \frac{1}{2}(\beta\tilde{\mu}B)^2 m^2\right] \\ &= \int_0^\infty dl\, e^{-\beta\tilde{\epsilon}\, l^2} \left[2l + \frac{1}{3}(\beta\tilde{\mu}B)^2 l^3\right]. \end{aligned} \tag{6}$$

The integrals are elementary, and in the same manner as for case 1, one obtains

$$Z \approx \frac{1}{\beta\tilde{\epsilon}} + \frac{1}{6}\left(\frac{\tilde{\mu}B}{\tilde{\epsilon}}\right)^2, \tag{7}$$

and

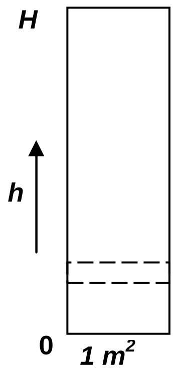
$$\mu \approx \frac{\tilde{\mu}^2 B}{3\tilde{\epsilon}}. \tag{8}$$

To find a weaker criterion for (8), look back at Eq. (2). The largest values of l that are important are given by $l(l+1)\beta\tilde{\epsilon} \sim 1$ or $l \sim (\beta\tilde{\epsilon})^{-\frac{1}{2}}$. Therefore that largest m in the second summation will also be of this order, and the largest exponent in this term will be $\sim \beta\tilde{\mu}B(\beta\tilde{\epsilon})^{-\frac{1}{2}}$ giving the condition that this quantity should be small, or $\tilde{\mu}B \ll \sqrt{\tilde{\epsilon}kT}$. Since $kT \gg \tilde{\epsilon}$, this is weaker than the criterion given for case 2 above.

TP – D2 Solution

(a) The number of molecules within dh is: $dN(h \leftrightarrow h+dh) = n_0 \exp\left(-\frac{mgh}{k_B T}\right)(area)dh$

Where m is the mass of the molecule, M_{pl} and R_{pl} are the mass of the planet and its radius, respectively, n_o is the density of molecules at the planet's surface.



$$M_{atm} = 4\pi R_{pl}^2 \int_0^\infty m n_0 \exp\left(-\frac{mgh}{k_B T}\right) dh = 4\pi R_{pl}^2 m n_0 \frac{k_B T}{mg} \int_0^\infty \exp(-x) dx = \frac{4\pi R_{pl}^2 k_B T n_0}{g}$$

$$M_{atm} = \frac{4\pi R_{pl}^2 k_B T n_0}{g} = \frac{4\pi G M_{pl} P_0}{g^2}$$

$$\frac{M_{atm}}{M_{pl}} = \frac{4\pi G P_0}{g^2} = \frac{4\pi \times 6.67 \cdot 10^{-11} Nm^2/kg^2 \times 10^5 Pa}{(10m/s^2)^2} = 8.4 \times 10^{-7}$$

This is the same results you would get by considering the equilibrium condition:

Where F is the total force exerted on the atmosphere by the plant's surface: $F = M_{atm} g$

$$F = P_0 4\pi R_{pl}{}^2 = M_{atm} g \qquad P_0 4\pi \frac{GM_{pl}}{g} = M_{atm} g \qquad \frac{M_{atm}}{M_{pl}} = P_0 4\pi \frac{G}{g^2}$$

(b) The average potential energy of the molecules in the atmosphere is found by:

$$\langle U \rangle = \frac{\int_0^\infty mgh \times n_0 \exp\left(-\frac{mgh}{k_B T}\right)(area)dh}{\int_0^\infty n_0 \exp\left(-\frac{mgh}{k_B T}\right)(area)dh} = \left[\frac{mgh}{k_B T} = y\right] = \frac{\int_0^\infty (k_B T) y \times \exp(-y) \times \frac{k_B T}{mg} dy}{\int_0^\infty \exp(-y) \times \frac{k_B T}{mg} dy} = k_B T$$

(c) Heat capacity: $E(T) = K(T) + U(T) = \frac{5}{2} k_B T + k_B T \qquad C = \frac{dE}{dT} = \frac{7}{2} k_B = C_P$