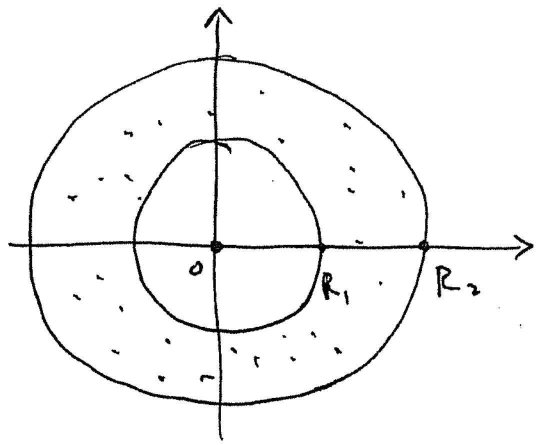
EM-A Solution



(1) According to Gauss theorem, E inside ($r < R_1$) is $\boxed{E=0}$.

(2) Outside of the layer ($r > R_2$), according to Gauss theorem, the flux of $\bar{E}$ is.

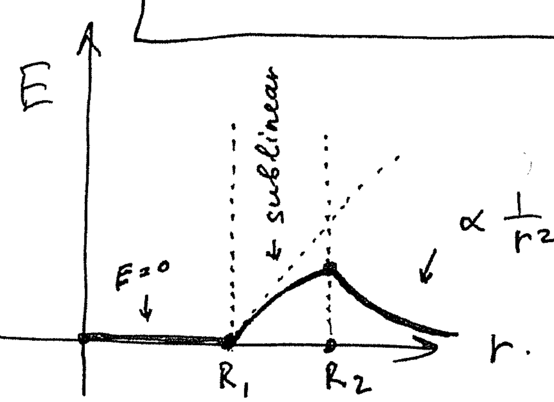
$4\pi r^2 \cdot E = \cancel{Q_{total}}\, 4\pi \cdot Q_{total}$, where Q_{total} is the total charge of the layer: $Q_{total} = \rho \cdot \frac{4}{3}\pi (R_2^3 - R_1^3)$.

$$\Rightarrow \boxed{E_{(r>R_2)} = \frac{4}{3}\pi\rho(R_2^3 - R_1^3)\cdot\frac{1}{r^2}}$$

(3). For $R_1 < r < R_2$, again using Gauss theorem we get:

$$4\pi r^2 \cdot E = 4\pi \cdot \rho \cdot \left(\frac{4}{3}\pi r^3 - \frac{4}{3}\pi R_1^3\right) \Rightarrow$$

$$\boxed{E_{(R_1<r<R_2)} = \frac{4}{3}\pi\rho\left(r - \frac{R_1^3}{r^2}\right)}$$



EM-B SOLUTION

A cylindrical conductor of radius A contains a cylindrical hole of radius B. The axis of the hole is parallel to the axis of the conductor and the distance between them is $D < A-B$. The current density j is uniform in the metal. Find the magnetic field in the hole.

Answer: By Ampère $\oint \underset{\sim}{B} \cdot d\underset{\sim}{s} = \mu_0 I$. Thus $2\pi r B(r) = \mu_0 \pi j r^2$. Putting in directions

$$\underset{\sim}{B} = \frac{1}{2}\mu_0 \, \underset{\sim}{\dot{j}} \times \underset{\sim}{r} \, .$$

Represent the hole by a cylinder with $-\underset{\sim}{\dot{j}}$ centered at D. Thus

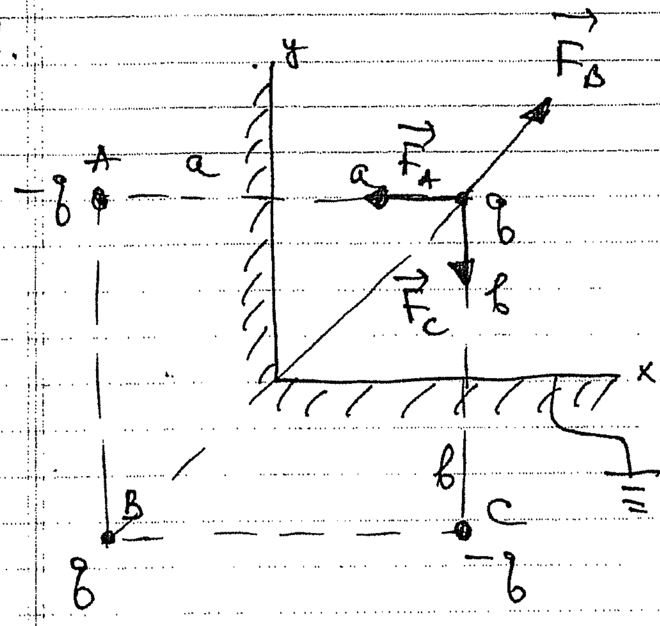
$$\underset{\sim}{B} = \frac{1}{2}\mu_0 \left[\underset{\sim}{\dot{j}} \times \underset{\sim}{r} - \underset{\sim}{j} \times (\underset{\sim}{r} - \underset{\sim}{D}) \right]$$

$$= \frac{\mu_0}{2} \underset{\sim}{\dot{j}} \times \underset{\sim}{D} \, .$$

The magnetic field in the hole is uniform. Its direction is perpendicular to both axes and to the line joining them.

EM-C1 Solutions

1.



Use the method of images

$$\vec{F}_A = \frac{q^2}{(2a)^2}(-\hat{x}) \qquad \vec{F}_c = \frac{q^2}{(2b)^2}(-\hat{y})$$

$$\vec{F}_B = \frac{q^2}{(2a)^2+(2b)^2}\left(\hat{x}\cos\theta + \hat{y}\sin\theta\right)$$

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \qquad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

$$\vec{F}_{net} = \vec{F}_A + \vec{F}_B + \vec{F}_c = \frac{q^2}{4}\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2}\right)\hat{x}$$

$$+ \frac{q^2}{4}\left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right)\hat{y}$$

To convert to SI, replace $q^2 \rightarrow \dfrac{q^2}{4\pi\varepsilon_0}$

EM-C2 SOLUTION

a) The charge that accumulates on plates is $Q = \int_0^t I\,dt$

So $Q(t) = \frac{1}{2}\alpha t^2 = \sigma \pi a^2 \Rightarrow \boxed{\sigma = \frac{\alpha t^2}{2\pi a^2}}$

(b) $\vec{E} = \frac{\sigma}{\epsilon_0} \Rightarrow \boxed{\vec{E}(t) = \frac{\alpha t^2}{2\pi \epsilon_0 a^2}\hat{z}}$

c) Ampere's Law + Maxwell correction $\Rightarrow$

$$\oint_{P} \vec{B}\cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d}{dt}\int \vec{E}\cdot d\vec{a}$$

Let P be circle of radius s about axis

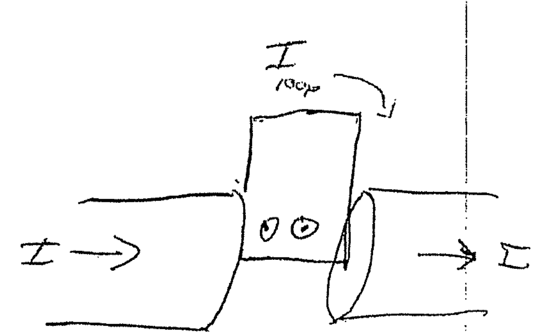
$$\Rightarrow B\, 2\pi \cancel{s} = \mu_0 \cancel{\epsilon_0} \frac{\cancel{2}\alpha t \cancel{\pi} s^{\cancel{2}}}{\cancel{2\pi\epsilon_0} a^2} \Rightarrow B = \frac{\mu_0 \alpha s t}{2\pi a^2}\hat{\phi}$$

d) Flux $\Phi_B = \int B\cdot da = \frac{\mu_0 \alpha t W}{2\pi a^2}\int_0^a s\,ds = \frac{\mu_0 \alpha t W \cancel{a^2}}{4\pi \cancel{a^2}}$

So $\boxed{\Phi_B = \frac{\mu_0 \alpha W t}{4\pi}}$ (note, interestingly, independent of a!)

e) $\Phi = \Phi(t) \Rightarrow \mathcal{E}_{loop} = -\frac{d\Phi}{dt} = \frac{-\mu_0 \alpha W}{4\pi} = I_{loop} R$

So $\boxed{I_{loop} = \frac{\mu_0 \alpha W}{4\pi R}}$

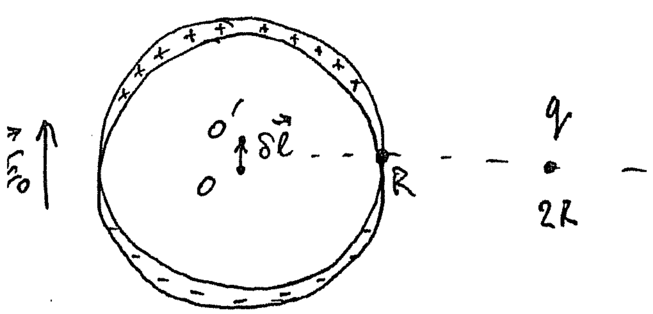


f) As in diagram Φ_B is out of page and increases w/t, so I_{loop} must oppose this $\Rightarrow$ CW as shown

EM-D1 Solution.

In a uniform external electric field, the ball is polarized and can be represented by two positively and negatively charged balls, shifted with respect to each other along $\vec{y}$ by a vector $\delta\vec{\ell}$. The uniform charge density of each ball is ρ, so each ball has total charge of $Q = \pm\frac{4}{3}\pi R^3\rho$.

The electric field outside of the ball is equivalent to the field produced by a small (point) dipole $\vec{p} = Q\cdot\delta\vec{\ell} = V\cdot\vec{\mathbb{P}}$, where V - volume of the ball, and $\vec{\mathbb{P}}$ is polarization of the ball ($\vec{\mathbb{P}} \equiv \frac{p}{V}$, i.e. dip. moment per unit volume).



The field of a point dipole is at a distance r:

$$\vec{E} = \frac{3}{r^5}\cdot(\vec{p}\cdot\vec{r})\cdot\vec{r} - \frac{\vec{p}}{r^3}, \text{ or}$$

$$\vec{E} = \frac{3V}{r^5}(\bar{\mathbb{P}}\cdot\bar{r})\bar{r} - \frac{V\vec{\mathbb{P}}}{r^3}.$$

At $\bar{r}\perp\vec{\mathbb{P}}$, the scalar product $(\bar{p}\cdot\bar{r})$ is zero, and

$$\boxed{\vec{E}_q = -V\frac{\bar{\mathbb{P}}}{r^3}}$$ - at position of charge q, $(r=2R)$

The total el. field at $(x,y)=(2R,0)$ is

$$E_{total} = E_0 - V\cdot\frac{\mathbb{P}}{r^3} = E_0 - \frac{4}{3}\pi R^3\cdot\frac{\mathbb{P}}{(2R)^3} =$$

$$= E_0 - \frac{\pi}{6}\cdot\mathbb{P}$$

Hence, the force acting on charge q is

$$\boxed{F = q\cdot E_{total} = q\left(E_0 - \frac{\pi}{6}\cdot\mathbb{P}\right).} \qquad (*)$$

Now we need to find polarization of the ball as a function of E_0, $\mathbb{P}(E_0)$.

The el. field inside the ball is $\vec{E} = \vec{E}_0 - \frac{4\pi}{3}\vec{\mathbb{P}}$. $\vec{\mathbb{P}} = \alpha\cdot\vec{E}$, so $\left(1+\frac{4\pi}{3}\alpha\right)\vec{E} = \vec{E}_0$, or

$$\vec{E} = \vec{E}_0\cdot\frac{3}{2+\varepsilon}, \quad \text{where } \boxed{\varepsilon \equiv 1+4\pi\alpha}.$$

Polarization of the ball is $\vec{\mathbb{P}} = \alpha\vec{E} = \frac{3\alpha}{2+\varepsilon}\cdot\vec{E}_0 =$

$= \frac{3}{4\pi}\cdot\frac{\varepsilon-1}{\varepsilon+2}\cdot\vec{E}_0$; Hence, in a uniform el. field $\vec{E}_0$, the ball has a dipole moment:

$$\vec{p} = V\cdot\vec{\mathbb{P}} = R^3\cdot\frac{\varepsilon-1}{\varepsilon+2}\cdot\vec{E}_0. \quad \text{Hence, } (*)-$$

answer.

$$\boxed{F_{\text{on } q} = q\left(E_0 - \frac{\pi}{6}\cdot\frac{3}{4\pi}\cdot\frac{\varepsilon-1}{\varepsilon+2}\cdot E_0\right) = qE_0\left(1-\frac{1}{8}\cdot\frac{\varepsilon-1}{\varepsilon+2}\right)}$$

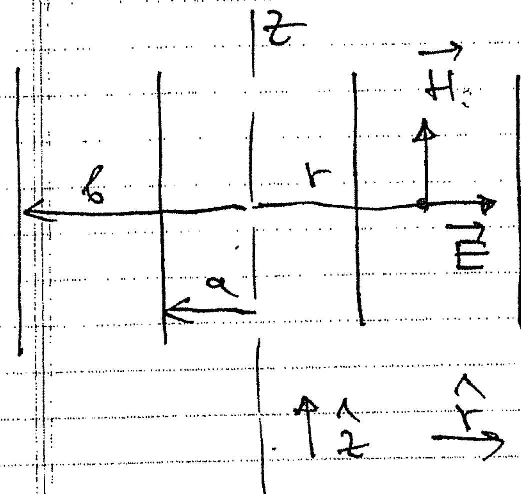
directed vertically (along $\vec{y}$).

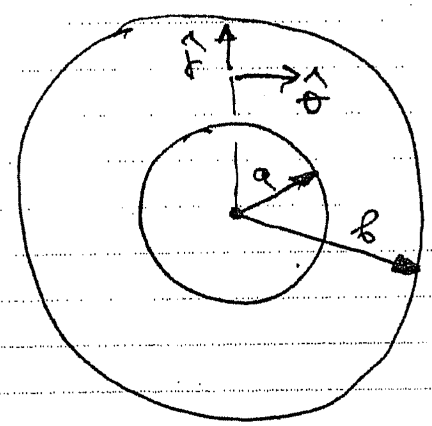
EM-D2 SOLUTION

The Lagrangian is

$$L = -mc^2\sqrt{1-v^2/c^2} + \frac{q}{c}\vec{A}\cdot\vec{v} - q\varphi,$$

where $\vec{A}$ and φ are the vector and electrostatic potentials, respectively





Use cylindrical coordinates r, z, θ

$$\vec{v} = \dot{r}\,\hat{r} + \dot{z}\,\hat{z} + r\dot{\theta}\,\hat{\theta}$$

Eq. of motion for θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

$$\Downarrow$$

$$(1) \quad \frac{d}{dt}\left(\frac{m r^2\dot{\theta}}{\sqrt{1-v^2/c^2}}\right) = q\left[E_\theta + \frac{1}{c}\left(H_r\dot{z} - H_z\dot{r}\right)\right]r$$

At $H=0$ trajectories of electrons are straight lines. As H increases they bend more and more in the plane $\perp \hat{z}$. Electrons ~~no longer~~ stop reaching the anode when at $r=b$ their velocity becomes parallel to the surface of the anode (outer shell), i.e. $\dot{r}\big|_{r=b}=0$. Then, $\dot{\theta}\big|_{r=b}=\frac{v_{max}}{b}$

In this problem $\vec{E}=E_r\hat{r}$ $\vec{H}=H(r)\hat{z}$ and Eq. (1) becomes

$$\frac{d}{dt}\left(\frac{mr^2\dot{\theta}}{\sqrt{1-v^2/c^2}}\right)=-\frac{q}{c}H(r)\,r\frac{dr}{dt}$$

Let us integrate along the trajectory

$$\frac{mr^2\dot{\theta}}{\sqrt{1-v^2/c^2}}\Bigg|_{r=a}^{r=b}=-\frac{q}{2\pi c}\int_a^b 2\pi H r\,dr=\frac{-q\Phi}{2\pi c}$$

We obtain

$$\Phi_c = \frac{2\pi c b}{|e|} p_{max}\,,$$ where $|e|$ is electron's charge (absolute value)

$$E^2 = p^2 c^2 + (mc^2)^2$$

at $r = b$ $\quad p = p_{max} \quad E = |e|V + mc^2$

We have

$$\left(|e|V + mc^2\right)^2 = p_{max}^2 c^2 + (mc^2)^2$$

$$\Downarrow$$

$$\Phi_c = 2\pi c b \sqrt{\frac{2mV}{|e|}\left(1 + \frac{|e|V}{2mc^2}\right)}$$

To convert to SI, replace $2\pi c \to 2\pi$

In nonrelativistic case $|e|V \ll mc^2$

and $$\Phi_c \simeq 2\pi c b \sqrt{\frac{2mV}{|e|}}$$