EA

Find the charge density giving rise to a spherically symmetric electric potential

$$\varphi(r) = \frac{q}{r} e^{-\gamma r}.$$

Use Gaussian units. What is the total charge?

ANSWER

$$\varphi(r) = \frac{q}{r} + q\left[e^{-\gamma r} - 1\right]/r.$$

The first term gives the usual point charge at $r = 0$. The second term is nonsingular there, so we can use

$$\nabla^2 \left[e^{-\gamma r} - 1\right]/r = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left[e^{-\gamma r} - 1\right] = \gamma^2 e^{-\gamma r}/r.$$

Thus

$$\begin{aligned} \rho(\underset{\sim}{x}) &= -\nabla^2 \varphi(r)/4\pi \\ &= q\,\delta_3(\underset{\sim}{x}) - \frac{q\gamma^2}{4\pi r} e^{-\gamma r}. \end{aligned}$$

The total charge is zero by Gauss's theorem, since $E(r) = -\partial\varphi/\partial r$ decreases faster than r^{-2} as $r \to \infty$.

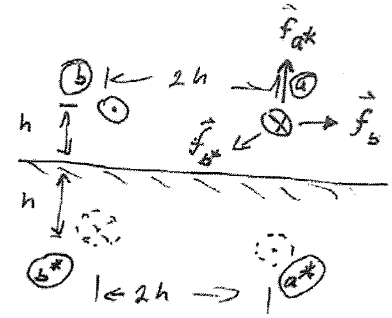
E B

Use method of Images:

Each current has parallel image current in opposite direction

B-field produced by long straight wire:

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$



Force on current: $\vec{F} = (I\, d\vec{l} \times \vec{B})$

with $d\vec{l}$ into page for current (a), we have $\vec{f}_b = f_b\,\hat{x}$;

$\vec{f}_{a*} = f_{a*}\,\hat{y}$; $\vec{f}_{b*} = f_{b*}\left(\frac{-1}{\sqrt{2}}[\hat{x}+\hat{y}]\right)$

now, $f_b = f_{a*} = \frac{\mu_0 I^2}{4\pi h}$; $f_{b*} = \frac{\mu_0 I^2}{4\sqrt{2}\pi h}$

So $\vec{f} = \sum \vec{f}_i = \frac{\mu_0 I^2}{8\pi h}(\hat{x}+\hat{y})$

EB

A point charge q is placed a distance $d > R$ from the center of an EQUALLY CHARGED isolated conducting sphere of radius R.

(a) If $d = 2R$, is the point charge attracted or repelled by the sphere?

(b) Same question for $d = 1.5R$?

Show your work. No points for guessing, but you may find it helpful to check limiting cases.

ANSWER

This is solved in Jackson p. 61. The sphere must be an equipotential, so we need an image charge

$$q_1 = -\frac{R}{d}q \quad \text{at} \quad x_1 = R^2/d \text{ from center.}$$

However the total charge on the sphere is q, so we need a second image charge

$$q_2 = \left(1 + \frac{R}{d}\right)q \quad \text{at} \quad x_2 = 0.$$

The real point charge is at $x = d$. Thus the attractive force is proportional to $\dfrac{R/d}{(d - R^2/d)^2}$ and the repulsive force is proportional to

$$\frac{\left(1 + \frac{R}{d}\right)}{d^2}. \qquad (\text{Both} \times q^2/4\pi\varepsilon_0).$$

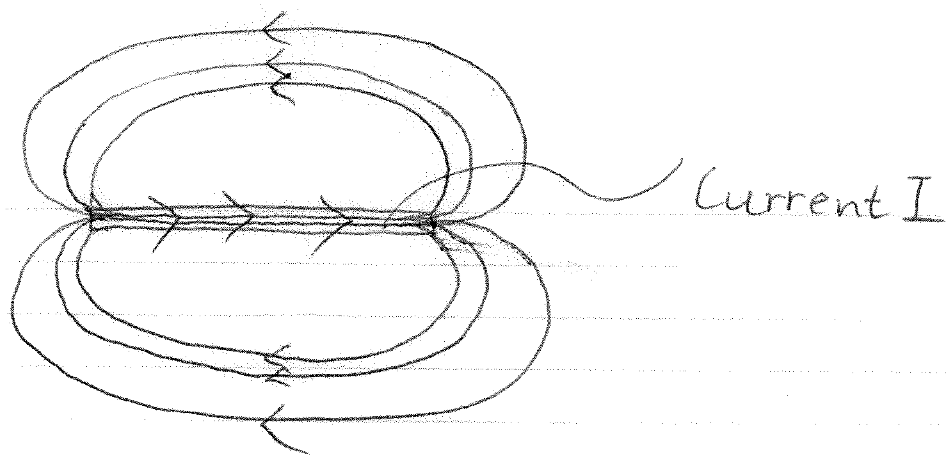
(a) $d = 2R$: attraction $\sim 2/9$, repulsion $\sim 3/8$.

(b) $d = 1.5R$: attraction $\sim 72/75$, repulsion $\sim 20/27$.

So at $d = 2R$ the point charge is repelled, but at $d = 1.5R$ it is attracted.

EB1

1.



F_{brake} direction ; $-\hat{y}$ (opposite to the motion)

$$v \cdot F_{brake} = P_{ower,\, dissipated} = I^2 R \quad — ⊛$$

$$R \simeq \frac{l_{bx}}{g\, l_{by} l_z}, \qquad I = \frac{\mathcal{E}_{mf}}{R} \simeq \frac{B l_{bx} v}{R}$$

$$⊛ \Rightarrow F_{brake} = \frac{1}{v} \frac{\mathcal{E}_{mf}^2}{R} = \frac{1}{v} \cdot \frac{g\, l_{by} l_z}{l_{bx}} \cdot (B l_{bx} v)^2$$

$$= \underline{g B^2 v\, l_{bx} l_{by} l_z}$$

2. $\vec{F}_{brake} = m \dot{\vec{v}}$

$$\Rightarrow -F_{brake} = m \dot{v} \Rightarrow -A v = m \dot{v}, \quad A \equiv g B^2 l_{bx} l_{by} l_z$$

$$\Rightarrow \frac{dv}{v} = -\frac{A}{m} dt \Rightarrow \ln v/v_0 = -\frac{A}{m} t$$

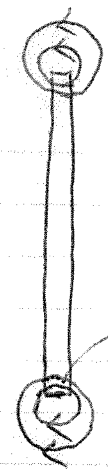
$$\Rightarrow v = v_0 e^{-\frac{A}{m} t}$$

For $v = \frac{v_0}{2}$, $\frac{1}{2} = e^{-\frac{A}{m} t}$

$$\Rightarrow t_{1/2 v_0} = \frac{m}{A} \ln 2 = \underline{\frac{m}{g B^2 l_{bx} l_{by} l_z} \ln 2}$$

EB1

3.



$v \cdot F_{brake} = \text{Power, dissipated} = 2 \cdot I^2 R$

$I = \frac{\mathcal{E}_{mf}}{R}$, $\mathcal{E}_{mf} \approx B\,lb_x v$, $R \approx \frac{4\,lb_x}{g\,lb_x l_z} = \frac{4}{g\,l_z}$

$\therefore F_{brake} = \frac{2}{v} \cdot \frac{\mathcal{E}_{mf}^2}{R} = \frac{2}{v}(B\,lb_x v)^2 \cdot \frac{g\,l_z}{4}$

$= \frac{1}{2}\, g B^2 v\, lb_x^2\, l_z$

Any numerical factors of order "1" are ~~4~~ all equally valid.

4. For 1, $lb_1 \Rightarrow lb_x$, $lb_2 \Rightarrow lb_y$.

So $F_{brake,1} = g B^2 v\, lb_1\, lb_2\, l_z$

For 3, $lb_1 \Rightarrow lb_y$, $lb_2 \Rightarrow lb_x$.

So $F_{brake,3} = \frac{1}{2} g B^2 v\, lb_2^2\, l_z$

Thus $\frac{F_{brake,1}}{F_{brake\,3}} \approx 2\,\frac{lb_1}{lb_2} \gg 1$

$\therefore$ Part I orientation gives much bigger braking force.

$\boxed{ED\,1}$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{r}\cdot\vec{p}(t_r)}{r^2} + \frac{\hat{r}\cdot\dot{\vec{p}}(t_r)}{cr}\right]$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{\dot{\vec{p}}(t_r)}{r} \qquad \text{where } t_r = t - \frac{r}{c}$$

a) $\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$

now $\frac{\partial\vec{A}}{\partial t} = \frac{\mu_0}{4\pi}\frac{\ddot{\vec{p}}(t_r)}{r}\underbrace{\frac{\partial t_r}{\partial t}}_{=1} = \frac{\mu_0\epsilon_0}{4\pi\epsilon_0}\frac{\ddot{\vec{p}}(t_r)}{r} = \frac{1}{4\pi\epsilon_0}\frac{\ddot{\vec{p}}(t_r)}{c^2 r}$

For $\vec{\nabla}V$, only operating on numerator of third term gives result $\propto \frac{1}{r}$

$$\Rightarrow \vec{\nabla}V = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\hat{r}}{cr}\cdot\frac{\partial\dot{\vec{p}}(t_r)}{\partial r}\hat{r} = \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{\hat{r}}{cr}\cdot\ddot{\vec{p}}(t_r)\right)\frac{\partial t_r}{\partial r}\hat{r}$$

$$= \left(\frac{1}{4\pi\epsilon_0}\right)\frac{-(\hat{r}\cdot\ddot{\vec{p}}(t_r))}{c^2 r}\hat{r}$$

So $\boxed{\vec{E} = \left(\frac{1}{4\pi\epsilon_0 c^2}\right)\frac{[(\hat{r}\cdot\ddot{\vec{p}}(t_r))\hat{r} - \ddot{\vec{p}}(t_r)]}{r}}$

b) $\vec{B} = \vec{\nabla}\times\vec{A} = \left(\frac{\mu_0}{4\pi}\right)\left(\vec{\nabla}\times\frac{\dot{\vec{p}}(t_r)}{r}\right)$

① use $(\nabla\times f\vec{v}) = f(\vec{\nabla}\times\vec{v}) - (\vec{v}\times\vec{\nabla}f)$ where $\vec{v} = \dot{\vec{p}}(t_r)$ and $f = \frac{1}{r}$

$$\Rightarrow \vec{B} = \left(\frac{\mu_0}{4\pi}\right)\left[\left(\frac{1}{r}\right)\left(\vec{\nabla}\times\dot{\vec{p}}(t_r)\right) - \left(\dot{\vec{p}}(t_r)\times -\frac{1}{r^2}\hat{r}\right)\right]$$

neglect second term because $\propto r^{-2}$.

$\vec{B} \sim \left(\frac{\mu_0}{4\pi}\right)\left(\frac{1}{r}\right)\left(\vec{\nabla}\times\dot{p}(t_r)\hat{p}\right)$ use ① again with $f = \dot{p}(t_r)$, $\vec{v} = \hat{p}$

$$\Rightarrow \vec{B} \sim \left(\frac{\mu_0}{4\pi}\right)\left(\frac{1}{r}\right)\left[\dot{p}\underbrace{(\vec{\nabla}\times\hat{p})}_{=0} - \left(\hat{p}\times\underbrace{\vec{\nabla}\dot{p}(t_r)}_{=\ddot{p}(t_r)\vec{\nabla}t_r = \ddot{p}(t_r)\left(-\frac{1}{c}\right)\hat{r}}\right)\right]$$

So $\vec{B} \sim \left(\frac{\mu_0}{4\pi rc}\right)\left(\hat{p}\times\ddot{p}(t_r)\hat{r}\right) = \boxed{-\left(\frac{\mu_0}{4\pi c r}\right)\left(\hat{r}\times\ddot{\vec{p}}(t_r)\right) = \vec{B}}$

(b) With $\vec{p}(t) = p_0(t)\,\hat{z} \Rightarrow \ddot{\vec{p}}(t_r) = \ddot{p}_0(t_r)\,\hat{z}$

so $\vec{E} = \left(\frac{1}{4\pi\epsilon_0 c^2}\right)\left[\frac{\ddot{p}_0(t_r)(\hat{r}\cdot\hat{z})\hat{r} - \ddot{p}_0(t_r)\hat{z}}{r}\right]$

$= \frac{\ddot{p}_0(t_r)}{4\pi\epsilon_0 c^2 r}\left[\cos\theta\,\hat{r} - \hat{z}\right]$; but $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

so $\boxed{\vec{E} = \left(\frac{\ddot{p}_0(t_r)}{4\pi\epsilon_0}\right)\left(\frac{\sin\theta}{c^2 r}\right)\hat{\theta}} = E(r,t)\,\hat{\theta}$

similarly, $\vec{B} \sim \frac{-\mu_0}{4\pi c r}\left(\ddot{p}_0(t_r)\right)\underbrace{(\hat{r}\times\hat{z})}_{= \hat{r}\times(-\sin\theta)\hat{\theta} = -\sin\theta\,\hat{\phi}}$

so $\boxed{\vec{B} = \left(\frac{\mu_0 \ddot{p}_0(t_r)}{4\pi c}\right)\left(\frac{\sin\theta}{r}\right)\hat{\phi}} = B(r,t)\,\hat{\phi}$

$E(r,t)$ and $B(r,t)$ can be read off from here.

(c) Poynting Vector: $\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$

$\vec{S} = \frac{[\ddot{p}_0(t_r)]^2}{16\pi\epsilon_0 c^3}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$

$\hat{r} = (\hat{\theta}\times\hat{\phi})$

Total radiated power $\Rightarrow$ $P = \oint \vec{S}\cdot d\vec{a}$

$P = \frac{\ddot{p}_0^{\,2}}{16\pi\epsilon_0 c^3}\int \frac{\sin^2\theta}{\cancel{r^2}}\cdot \cancel{r^2}\,dr\,\sin\theta\,d\theta\,d\phi$

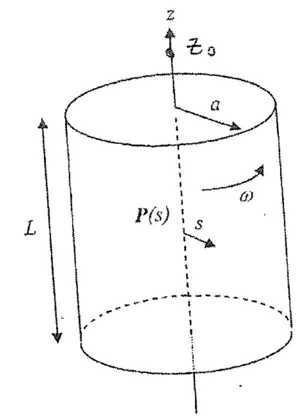
$= \left(\frac{\ddot{p}_0^{\,2}}{16\pi\epsilon_0 c^3}\right) 2\pi\cdot\frac{4}{3} = \frac{\ddot{p}_0^{\,2}}{6\pi\epsilon_0 c^3}$

ED2

$\vec{P} = \frac{P_0}{a} s \hat{s}$

bound volume charge density $\rho_b = -\vec{\nabla}\cdot\vec{P}$

$\Rightarrow \rho_b = -\frac{1}{s}\frac{\partial}{\partial s}\left(\frac{P_0 s^2}{a}\right) = -2\frac{P_0}{a}$ = constant.



bound surface charge density? $\sigma_b = \vec{P}\cdot\hat{n}$

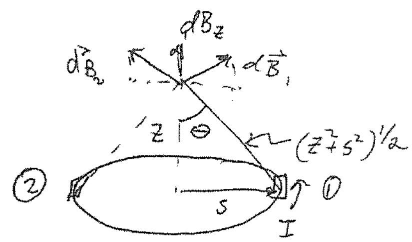
$\Rightarrow \sigma_b = \frac{P_0 a}{a}(\hat{s}\cdot\hat{s}) = P_0$

An element of bound charge moves with velocity $v = \omega s$

$\Rightarrow J = \rho v = \rho_b \omega s$

◦ Now, divide cyl. into rings ⇒ disks ⇒ cylinder. Consider contribution to B from ring of rotating charge $I = \rho_b \omega s\, ds\, dz$

Contribution from current element ①



$$d\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{(I d\vec{l} \times \hat{r})}{r^2} = \frac{\mu_0 I dl}{4\pi r^2}(\hat{\phi}\times\hat{r})$$

but, component parallel to plane of loop canceled by $d\vec{B}_2 \Rightarrow$ only z-component remains:

$$d\vec{B}_z = \frac{\mu_0}{4\pi}\frac{I dl}{r^2}\sin\theta\,\hat{z} = \frac{\mu_0}{4\pi} I dl \cdot \frac{s}{r^3}\hat{z} \quad \text{where } r = (z^2+s^2)^{1/2}, \quad dl = s\,d\phi$$

$$\text{so } d\vec{B}_z = \frac{\mu_0}{4\pi}\frac{I s\, dl}{(z^2+s^2)^{3/2}}\hat{z} \Rightarrow \vec{B}_z = \frac{\mu_0}{2}\frac{I s^2}{(z^2+s^2)^{3/2}}\hat{z} = \frac{\mu_0}{2}\frac{\rho_b \omega s^3 ds\, dz}{(z^2+s^2)^{3/2}}$$

For a disk
$$\vec{B}_z = \frac{\mu_0 \rho \omega dz}{2}\int_{s=0}^{a}\frac{s^3 ds}{(z^2+s^2)^{3/2}}\hat{z} = \frac{\mu_0 \rho\omega dz}{2}\left(\frac{s^2+2z^2}{(s^2+z^2)^{1/2}}\right)\Bigg|_0^a \hat{z}$$

$$= \frac{\mu_0 \rho\omega dz}{2}\left[\frac{a^2+2z^2}{(a^2+z^2)^{1/2}} - \frac{2z^2}{z}\right] = \frac{\mu_0\rho\omega dz}{2}\left[(a^2+z^2)^{1/2} + \frac{z^2}{(a^2+z^2)^{1/2}} - 2z\right]\hat{z}$$

Finally, integrate over disks to find total contribution from cylinder

$$\vec{B}_{tot} = \frac{\mu_0 \rho \omega}{2} \int_{z_0}^{z_0+L} \left[(a^2+z^2)^{1/2} + \frac{z^2}{(a^2+z^2)^{1/2}} + 2z \right] dz \, \hat{z}$$

$$= \frac{\mu_0 \rho \omega}{2} \left[\frac{1}{2}\left(a^2 \ln\left(z + \sqrt{a^2+z^2}\right) + z\sqrt{a^2+z^2} \right) + \frac{1}{2}\left(z\sqrt{a^2+z^2} - a^2 \ln\left(z+\sqrt{a^2+z^2}\right)\right) + z^2 \right]_{z_0}^{z_0+L} \hat{z} = \frac{\mu_0 \rho \omega}{2} \left[z\sqrt{a^2+z^2} + z^2 \right]_{z_0}^{z_0+L} \hat{z}$$

$$= \frac{\mu_0 \rho \omega}{2} \left\{ (z_0+L)\sqrt{a^2+(z_0+L)^2} + (z_0+L)^2 - z_0\sqrt{a^2+z_0^2} - z_0^2 \right\} \hat{z}$$

$$\boxed{\vec{B}^{bulk}_{TOT} = -\left(\frac{\mu_0 P_0 \omega}{a}\right)\left\{ (z_0+L)\sqrt{a^2+(z_0+L)^2} + (z_0+L)^2 - z_0\sqrt{a^2+z_0^2} - z_0^2 \right\} \hat{z}}$$

Now surface current density gives

$$I^{surf} = \sigma \omega a \, dz, \quad d\vec{B}_z = \frac{\mu_0}{4\pi} \frac{I \, dl}{r^2} \sin\theta \, \hat{z}$$

where $dl = a\,d\phi$; $\sin\theta = \frac{a}{(a^2+z^2)^{1/2}}$, $r^2 = (a^2+z^2)$

So, integrating over ϕ:

$$\vec{B}_z = \left(\frac{\mu_0}{4\pi}\right)(2\pi a)\frac{(\sigma \omega a)}{(a^2+z^2)} \frac{a}{(a^2+z^2)^{1/2}} dz \, \hat{z} = \left(\frac{\mu_0}{2}\right)(\sigma \omega a^3)(a^2+z^2)^{-\frac{3}{2}} dz \, \hat{z}$$

$$\text{So } \vec{B}^{surf}_{tot} = \left(\frac{\mu_0 \sigma \omega a^3}{2}\right) \int_{z_0}^{z_0+L} (a^2+z^2)^{-\frac{3}{2}} dz \, \hat{z} = \left(\frac{\mu_0 \sigma \omega a^3}{2}\right) \left(\frac{z}{a^2\sqrt{a^2+z^2}}\right)\Bigg|_{z_0}^{z_0+L} \hat{z}$$

$$\boxed{\vec{B}^{surf}_{tot} = \left(\frac{\mu_0 P_0 \omega a}{2}\right) \left[\frac{z_0+L}{\sqrt{a^2+(z_0+L)^2}} - \frac{z_0}{\sqrt{a^2+z_0^2}} \right] \hat{z}}$$

$$\vec{B}_{tot} = \vec{B}^{bulk}_{tot} + \vec{B}^{surf}_{tot}.$$