A | Qualifiers-thermo - A - . pdf TA

Problem 1 (blackbody radiation)

(a) The black body radiation fills a cavity of volume ***V***. The radiation energy is

$$U(T,V)=\frac{4\sigma}{c}VT^4 \quad , \text{ the radiation pressure is } \quad P=\frac{4\sigma}{3c}T^4$$

Consider an isentropic (quasi-static and adiabatic) process of the cavity expansion (***TdS=dU+PdV=0***). The radiation pressure performs work during the expansion and the temperature of radiation will drop. Find how ***T*** and ***V*** are related for this process.

(b) (5) Assume that the cosmic microwave background (CMB) radiation was decoupled from the matter when both were at 3000K. Currently, the temperature of CMB radiation is 2.7 K. What was the radius of the universe at the moment of decoupling, compared to now? Consider the process of expansion as isentropic.

(a) The equation that describes the isentropic (quasi-static adiabatic) process for the photon gas:

$$dS=0 \qquad dU=-PdV \qquad P=\frac{4\sigma}{3c}T^4 \qquad U(T,V)=\frac{4\sigma}{c}VT^4$$

$$\frac{4\sigma}{c}V4T^3dT+\frac{4\sigma}{c}T^4dV=-\frac{4\sigma}{3c}T^4dV \qquad 16\frac{\sigma}{c}VT^3dT=-\frac{16}{3}\frac{\sigma}{c}T^4dV$$

$$\frac{dT}{T}=-\frac{1}{3}\frac{dV}{V} \qquad VT^3=const$$

(b) $$\frac{V_i}{V_f}=\left(\frac{R_i}{R_f}\right)^3=\left(\frac{T_f}{T_i}\right)^3$$ or $TR=const$

Thus, at the moment of decoupling, the radius of the universe was ~ 1000 time smaller

$\boxed{\text{TB}}$ TB

(a) $\dfrac{\partial^2 V}{\partial T \partial p} = \dfrac{\partial^2 V}{\partial p \partial T} \Rightarrow -\dfrac{R}{p^2} = -f(p)$

(b) Integrating the first eqn.,

$$V = \frac{RT}{p} - \frac{a}{T} + g(p)$$

$\hookrightarrow$ arbitrary fn.

and, from the second eq., with

$$f(p) = \frac{R}{p^2},$$

$$V = \frac{RT}{p} + h(T)$$

$\hookrightarrow$ arbitrary fn.

These are satisfied by

$$h(T) = -\frac{a}{T} + \text{const}$$

$$g(p) = \text{const}$$

$$V = \frac{RT}{p} - \frac{a}{T} + \text{const}$$

(c) $\dfrac{\partial}{\partial p}\left[C_p\right] = \dfrac{\partial}{\partial p}\left[T\left(\dfrac{\partial S}{\partial T}\right)_p\right]_T$

$$= T\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial p}\right)_T = T\frac{\partial}{\partial T}\frac{\partial(S,T)}{\partial(P,T)} = T\frac{\partial}{\partial T}\frac{\partial(V,P)}{\partial(P,T)}$$

$$= -T\frac{\partial}{\partial T}\left(\frac{\partial V}{\partial T}\right)_p = -T\frac{\partial}{\partial T}\left[\frac{R}{p} + \frac{a}{T^2}\right]$$

$$= \frac{2a}{T^2} \quad \text{so}$$

$$C_p(P,T) = \frac{2aP}{T^2} + C_p(0,T)$$

TC 1: Solution

(a) For a Fermi gas, the density is

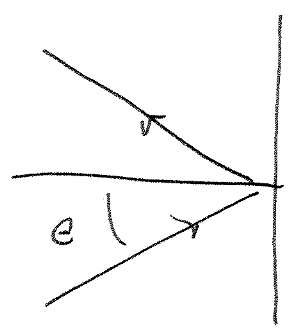
$$n = \frac{1}{h^3}\int \underbrace{(2s+1)}_{\text{spin}} \frac{d^3p}{\exp\left[(E-E_F)/k_BT\right]+1}$$

$$\rightarrow \frac{1}{h^3}(2s+1)\cdot 4\pi \frac{p_F^3}{3} \quad \text{as } T \to 0$$

So

$$n \propto (2s+1)\, p_F^3$$

The corresponding pressure is



$$P = \frac{1}{h^3}\int_0^{p_F} \underbrace{(2s+1)\, d^3p\, v\cos\theta}_{\text{flux}} \cdot \underbrace{2p\cos\theta}_{\text{momentum transfer}} \quad d^3p$$

$$\propto (2s+1)\, p_F^5$$

so

$$\frac{P_1}{P_2} = \left(\frac{2s_1+1}{2s_2+1}\right)\left(\frac{p_1}{p_2}\right)^5 = \frac{2s_1+1}{2s_2+1}\left(\frac{n_1}{n_2}\right)^{5/3}\left(\frac{2s_2+1}{2s_1+1}\right)^{5/3}$$

At equilibrium, $P_1 = P_2$,

so

$$\left(\frac{n_1}{n_2}\right) = \left(\frac{2s_1+1}{2s_2+1}\right)^{2/5} \cong 7/58$$

(b) For $T \to \infty$, both components are classical ideal gases, with

$$P = nkT$$

$$P_1 = P_2 \Rightarrow n_1 = n_2$$

$\boxed{TC2}$

3B. $E = -WL$ TC2

$E = -Wa(n_\downarrow - n_\uparrow)$ $\beta = 1/kT$

$(n_\downarrow + n_\leftrightarrow + n_\uparrow) = N$

$Z = \left(e^{+\beta Wa} + 1 + e^{-\beta Wa}\right)^N$

$$\boxed{\frac{\bar{E}}{W} = \bar{L} = -\frac{1}{W}\frac{\partial \ln Z}{\partial \beta} = \frac{Na\left(e^{\beta Wa} - e^{-\beta Wa}\right)}{\left(e^{-\beta Wa} + 1 + e^{+\beta Wa}\right)}}$$

Problem 2 (partition function)

The ground level of the neutral lithium atom is doubly degenerate (that is, d_0=2). The first excited level is 6-fold degenerate (d_1=6). and is at an energy 1.2 eV above the ground level.

(a) In the outer atmosphere of the sun, which is at a temperature of about 6000 K, what fraction of the neutral lithium is in the excited level? Since all the other levels of *Li* are at a much higher energy, it is safe to assume that they are not significantly occupied.

(b) Find the average energy of Li atom at temperature ***T*** (again, consider only the ground state and the first excited level). Find the contribution of these levels to the specific heat per mole, $\boldsymbol{C_V}$, and sketch $\boldsymbol{C_V}$ as a function of ***T***.

If the ground level energy is defined as zero and ***E*** is the energy of excited level:

$$Z = \sum_i d_i \exp(-\beta\varepsilon_i) = 2 + 6\exp(-\beta E)$$

TQ1

The probability that the atom is in its excited level:

TQ1

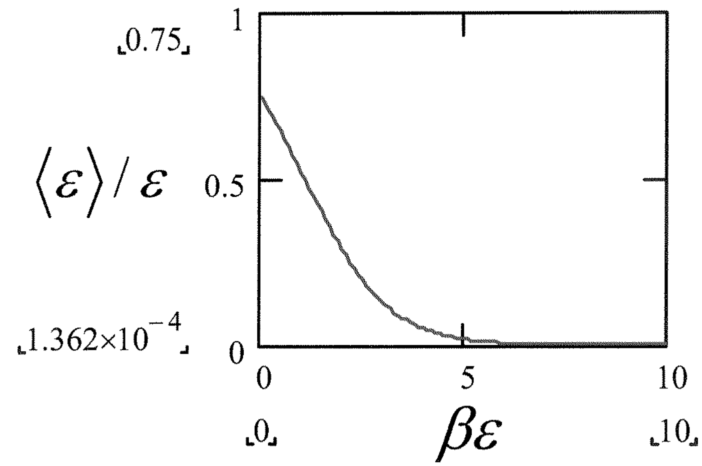
$$P(E) = \frac{6\exp(-\beta E)}{Z} = \frac{3\exp(-\beta E)}{1+3\exp(-\beta E)} = \frac{3}{3+\exp(\beta E)}$$

E = 1.2 eV, ***T*** = 6000K (~0.5 eV), β***E*** = 2.32, exp(β***E***) ≈ 10:

$$P(E) = \frac{3}{3+10} = 0.23$$

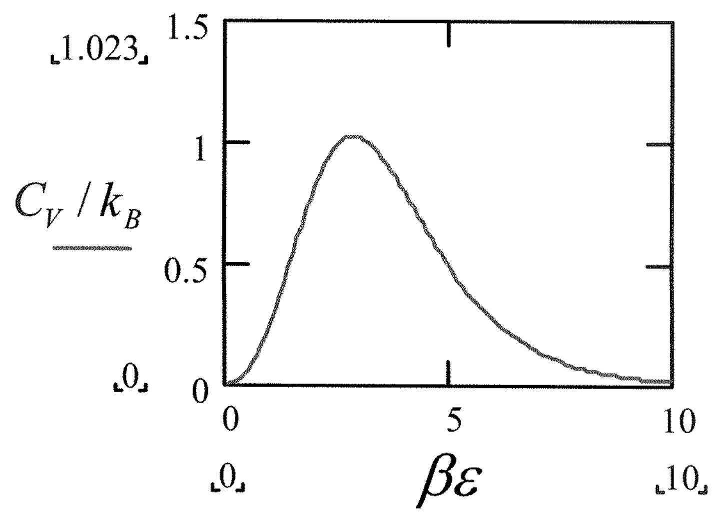
The average energy per atom:

$$\langle \varepsilon \rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{\varepsilon \times 6\exp(-\beta\varepsilon)}{2+6\exp(-\beta\varepsilon)} = \frac{3\varepsilon}{\exp(\beta\varepsilon)+3}$$



The specific heat:

$$C_V = \left(\frac{\partial \langle \varepsilon \rangle}{\partial T}\right)_V = \frac{-3\varepsilon \times \varepsilon \exp(\beta\varepsilon)\frac{\partial \beta}{\partial T}}{[\exp(\beta\varepsilon)+3]^2} = k_B \frac{3(\beta\varepsilon)^2 \exp(\beta\varepsilon)}{[\exp(\beta\varepsilon)+3]^2}$$



Problem TD2 Solution

(a) It is less error prone to do the problem in the reference frame of the satellite. According to kinetic theory, the net force on the satellite F $[\vec{F} = F_z \hat{z}]$ is given by (assume satellite moves in $+z$ direction)

$$\frac{F_z}{L^2} = \left(\int_{p_z > 0} - \int_{p_z < 0} \right) d^3p \, \frac{2p_z^2}{m} f(\vec{p}) \qquad (1)$$

[Note: $2p_z$ = momentum xfer/collision; $\left|\frac{p_z}{m}\right| f(\vec{p})\, d^3p$ = # collisions/time/area]

where the distribution function $f(\vec{p})$ has the normalization $\int d^3p \, f(\vec{p}) = n$. The first integral is the Force/area on the trailing face, while the second integral is for the leading edge.

In the satellite frame, there is a wind of speed V in the negative z direction, so that

$$f(\vec{p}) = f_0(\vec{p} + m\vec{V}), \text{ where } f_0(p) \propto e^{-p^2/2mkT}$$

and $\vec{V} = V\hat{z}$. We write Eq. (1) as

$$F_z = L^2 \int_{p_z > 0}^{\infty} d^3p \, \frac{2p_z^2}{m} \left[f_0(\vec{p} + m\vec{V}) - f_0(\vec{p} - m\vec{V}) \right] \qquad (2)$$

and expand to first order in V!

$$F_z \simeq L^2 \int_{p_z>0} d^3p \; 4p_z^2 \; V \frac{\partial f_0(p)}{\partial p_z}$$, now integrate by parts

$$= -8VL^2 \int_{p_z>0} d^3p \; p_z f_0(p) = -4VL^2 n \langle |p_z| \rangle_0$$

where $$\langle |p_z| \rangle_0 = \int_{-\infty}^{\infty} dp_z |p_z| e^{-p_z^2/2mkT} \Big/ \int_{-\infty}^{\infty} dp_z e^{-p_z^2/2mkT}$$

$$= \sqrt{2mkT/\pi}$$

$$\therefore \quad F_z = -\underbrace{\left(4L^2 n \sqrt{2mkT/\pi}\right)}_{\text{let} \equiv K} V + O(V^3)$$

(b) $$M \frac{dV}{dt} = -KV$$

$$V = V_0 e^{-Kt/M} = V_0/2$$

or $Kt/M = \ln 2$

giving $t \sim 8 \times 10^7 \,\text{s} \sim 2.5 \,\text{yr}$