MA

Mechanics Problem 1. pdf

Mechanics Problem 1: Easiest

An air molecule is roughly spherical with a radius R_{air} of about 2×10^{-10} meter. The number of such molecules per unit volume is about $n=3 \times 10^{25}\ /m^3$.

a) On the average, how far does an air molecule travel between collisions with other such molecules?

b) How does this compare with the average separation between molecules?

Solution:

a) This is equivalent to a single molecule of radius $2R_{air}$ moving through a collection of stationary point particles that represent the centers of the other atoms. In time τ, the molecule has moved a distance $v\tau$, and has swept out a cylindrical volume $V = \pi(2R_{air})^2 v\tau = \sigma v\tau$, where $\sigma=\pi(2R_{air})^2$. Within this volume are $nV=n\sigma v\tau$ point atoms with which the moving atom has collided. Thus, the average distance between collisions is:

$L=v\tau/n\sigma v\tau=1/n\sigma$

$\sigma=4\pi R_{air}^2 = 5 \times 10^{-19}$ meter^2

and $L= 7 \times 10^{-8}$ m.

b) The average separation of each molecule is about $n^{-1/3} = 3 \times 10^{-9}$ m. Thus, the typical molecule goes about 20 times the average molecular separation between collisions.

Mechanics Problem 2 . pdf

MB

Mechanics Problem 2: Easy

A thin uniform rod of mass m and length L, with its bottom end resting on a frictionless table, is released from rest at an angle θ_o to the vertical. Find the force exerted by the table upon the stick at an infinitesimally small time after its release.

Solution:

Since there is no friction, the forces acting on the stick are the normal force N, and gravity mg. Within an infinitesimal time of the release of the stick, the equations of motion are:

$N - mg = m\ddot{y}$

$\frac{1}{2} NL\sin\theta_o = 1/12 mL^2 \ddot{\Theta}$,

where y is the vertical coordinate of the center of mass and $1/12\ mL^2$ is the moment of inertia about a horizontal axis through the center of mass of the rod.

Since $y = \frac{1}{2} L \cos\theta$

$\ddot{Y} = -1/2L(\dot{\Theta}^2 \cos\theta + \ddot{\Theta} \sin\theta) = -\frac{1}{2} L \ddot{\Theta} \sin\theta_o$, since initially $\dot{\Theta} = 0$ and $\theta = \theta_o$.

Hence,

$N = mg + m\ddot{y}$
$= mg - \frac{1}{2} mL\ddot{\Theta} \sin\theta_o$
$= mg - 3N \sin^2 \theta_o$

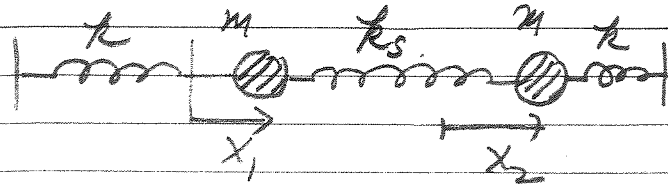
or

$N = mg/(1 + 3\sin^2\theta_o)$

MC1

SOLUTION TO PhD EXAM
CLASSICAL MECHANICS

(A)



$$m\frac{d^2x_1}{dt^2} = -kx_1 - k_s(x_1 - x_2)$$ IF $x_2 > x_1$ FORCE ON 1 IS → + ✓

$$m\frac{d^2x_2}{dt^2} = -kx_2 - k_s(x_2 - x_1)$$ IF $x_2 > x_1$ FORCE ON 2 IS ← − ✓

(B) $x_1 = Ae^{i\omega t}$ $x_2 = Be^{i\omega t}$

$$m(-\omega^2)A = -kA - k_s(A-B)$$

1 $$[m(-\omega^2) + (k+k_s)]A = k_s B$$

$$m(-\omega^2)B = -kB - k_s(B-A)$$

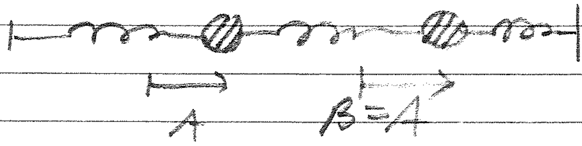
2 $$[m(-\omega^2) + k + k_s]B = k_s A$$ multiply 1 & 2

$$[(k+k_s) - m\omega^2]^2 \cancel{AB} = k_s^2 \cancel{AB}$$

$$(k+k_s) - m\omega^2 = \pm k_s$$

+ solution $k + k_s - m\omega^2 = +k_s$ $\omega = \sqrt{\frac{k}{m}}$

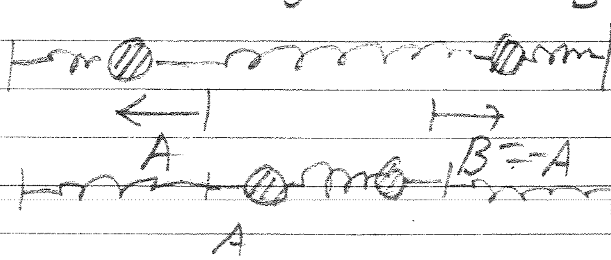
c):



MOVE TOGETHER
INNER SPRING DOESN'T MATTER

− solution $k + k_s - m\omega^2 = -k_s$ $\omega = \sqrt{\frac{k+2k_s}{m}}$

c):



MOVE IN OPPOSITE DIRECTIONS

NOTE INNER SPRING GETS STRETCHED TWICE
EXPLAINS WHY $2k_s$

SOLUTIONS TO PhD EXAM
CLASSICAL MECHANICS.

$x_1 = A_1 \cos\omega_1 t + A_2 \cos\omega_2 t$

$x_2 = B_1 \cos\omega_1 t + B_2 \cos\omega_2 t$

$\underbrace{A_1 = B_1 \ \& \ A_2 = -B_2}$
CONDITION ON NORMAL MODES.

$x_2 = 0$ AT $t = 0$ $\quad B_1 = -B_2$

$B_2 = -A_2 \quad B_1 = +A_2 \Rightarrow A_1 = A_2.$

D.

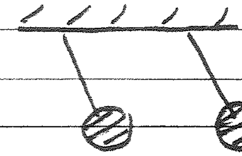
$$x_1 = 2A_1(\cos\omega_1 t + \cos\omega_2 t)$$
$$x_2 = A_1(\cos\omega_1 t - \cos\omega_2 t)$$

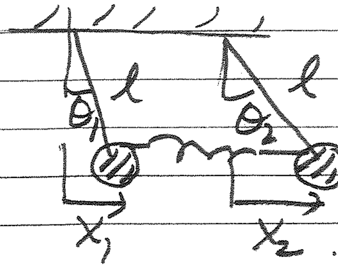
$\cos\omega_1 t + \cos\omega_2 t = 2\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_2-\omega_1}{2}t\right)$

$\cos\omega_1 t - \cos\omega_2 t = 2\sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_2-\omega_1}{2}t\right)$

MC2

SOLUTION TO PhD EXAM.
CLASSICAL MECHANICS





$x_1 \cong l\theta_1$

$$\tau_1 = I\alpha = ml^2\ddot{\theta}_1 \approx -mg\sin\theta_1 - kl(\theta_1-\theta_2)$$

($\sin\theta_1 = \theta_1$)

IF $\theta_2 > \theta_1$ SPRING FORCE IS $\rightarrow$ ON 1

$$\tau_2 = I\alpha_2 = ml^2\ddot{\theta}_2 \cong -mg\,\theta_2 - kl(\theta_2-\theta_1)$$

IF $\theta_2 > \theta_1$ SPRING FORCE IS $\leftarrow$ ON 2.

$$\theta_1 = \theta_A e^{i\omega t} \qquad \theta_2 = \theta_B e^{i\omega t}$$

$$ml^2(-\omega^2)\theta_A = -mg\,\theta_A - kl(\theta_B-\theta_A)$$

$$[ml^2(-\omega^2) + mg + kl]\theta_A = kl\,\theta_B$$

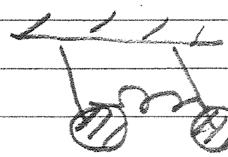
$$[ml^2(-\omega^2) + mg + kl]\,\theta_B = kl\,\theta_A$$

MULTIPLY EACH EQUATION BY OTHER

$$(ml^2(-\omega^2) + mg + kl)^2\theta_A\theta_B = (kl)^2\theta_A\theta_B$$

$$\Rightarrow \quad ml^2(-\omega^2) + mg + kl = \pm kl$$

+ SOLUTION $\quad ml^2(-\omega^2) + mg = 0 \quad \omega = \sqrt{\frac{g}{l}}$



OSCILLATE WITH $\theta_A = \theta_B$
SPRING DOES NOT MATTER.

− SOLUTION $\quad ml^2(-\omega^2) + mg + kl = -kl$

$$ml^2\omega^2 = 2kl + mg \qquad \omega = \sqrt{\frac{g}{l} + \frac{2k}{m}}$$

$\overline{Tm}$ $\quad \theta_A = -\theta_B$

SOLUTION TO PhD EXAM

$$\theta_A(t) = \theta_{A1} \cos \omega_1 t + \theta_{A2} \cos \omega_2 t$$

$$\theta_B = \theta_{B1} \cos \omega_1 t + \theta_{B2} \cos \omega_2 t \qquad \theta_{B2} = -\theta_{A2}$$

$$\theta_{B1} = \theta_{A1}$$

$$\theta_A = \theta_A = \theta_{A1} + \theta_{A2}$$

$$\theta_B = 0 = \theta_{B1} + \theta_{B2} \qquad \theta_{B2} = -\theta_{B1} = -\theta_{A2} \Rightarrow$$

$$\theta_{A1} = \theta_{A2}$$

$$\theta_A(t) = \frac{\theta_A}{2}(\cos \omega_1 t + \cos \omega_2 t)$$

$$\theta_B(t) = \frac{\theta_A}{2}(\cos \omega_1 t - \cos \omega_2 t)$$

MD 1 Mechanics 4A - Solution

Solution 1.2. Suppose the rocket is moving in the positive x-direction, and the dust cloud starts at $z = 0$. Because the collisions between the rocket and dust particles are inelastic, energy is not conserved. However, we must conserve momentum at all times. If $m(x)$ and $v(x)$ are the mass and velocity of the rocket at point x, then for all x

$$m(x)v(x) = m_0 v_0. \tag{10.6}$$

In particular, for a small displacement δx,

$$m(x)v(x) = m(x+\delta x)v(x+\delta x). \tag{10.7}$$

As the rocket travels from the edge of the cloud to a point x, it sweeps the dust out of a region of volume Ax, so its mass at position x is

$$m(x) = m_0 + A\rho x. \tag{10.8}$$

Expanding equation (10.7) gives us

$$m(x)v(x) = (m(x) + A\rho\delta x)\left(v(x) + \frac{dv}{dx}\delta x + \mathcal{O}(\delta x^2)\right). \tag{10.9}$$

Neglecting terms of second and higher order in δx,

$$\left[A\rho v(x) + m(x)\frac{dv}{dx}\right] = 0, \tag{10.10}$$

or, using (10.8),

$$\frac{A\rho}{m_0 + A\rho x}dx + \frac{dv}{v} = 0. \tag{10.11}$$

Integrating this equation and using the initial condition $v(x=0) = v_0$, we find that

$$v = \frac{dx}{dt} = \frac{m_0 v_0}{m_0 + A\rho x}. \tag{10.12}$$

Integrating again and using the condition that $x(t=0) = 0$, it is easy to show that, for $t > 0$,

$$x(t) = -\frac{m_0}{A\rho} + \sqrt{\frac{2m_0 v_0 t}{A\rho} + \frac{m_0^2}{A^2\rho^2}}. \tag{10.13}$$

Mechanics 4B- Solution

Solution 1.10. In plane-polar coordinates, the Lagrangian for a particle moving in a central potential $V(r)$ is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r), \tag{10.89}$$

where m is the mass of the particle. The potential is given in the question as

$$V(r) = -\frac{k}{r} + \frac{1}{2}br^2. \tag{10.90}$$

The θ-component of Lagrange's equation is

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant} \equiv l. \tag{10.91}$$

The hamiltonian of our system is then

$$H = \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} + V(r) = \frac{p_r^2}{2m} + V_{eff}(r), \tag{10.92}$$

with $p_r = m\dot{r}$ and

$$V_{eff}(r) = \frac{l^2}{2mr^2} + V(r). \tag{10.93}$$

The term $l^2/2mr^2$ is referred to as an "angular momentum barrier." Solving the equations of motion for this hamiltonian is equivalent to solving Lagrange's equations for the Lagrangian:

$$L = \frac{1}{2}m\dot{r}^2 - V_{eff}(r). \tag{10.94}$$

This is a completely general result for the motion of a particle in a central potential and could easily have been our starting point in this problem (e.g., Goldstein, Chapter 3).

It may seem unnecessarily long-winded to go through this procedure, but note that the sign of the angular momentum barrier in (10.94) is *opposite* to what we would have gotten if we had naively replaced $\dot{\theta}$ with l/mr^2 in the Lagrangian (10.89). This is due to the fact that the Lagrangian is a function of the time derivative of the position, and not of the canonical momentum.

The equation of motion from (10.94) is

$$m\ddot{r} = -\frac{d}{dr}V_{eff}(r). \tag{10.95}$$

If the particle is in a circular orbit at $r = r_0$ we require that the force on it at that radius should vanish,

$$\left.\frac{dV_{eff}}{dr}\right|_{r=r_0} = 0. \tag{10.96}$$

Using our expression for V_{eff} (10.93), we derive an expression relating the angular momentum l to the radius of the orbit r_0:

$$\frac{l^2}{mr_0^3} - \frac{k}{r_0^2} - br_0 = 0. \tag{10.97}$$

We are interested in perturbations about this circular orbit. Provided the perturbation remains small, we can expand $V_{eff}(r)$ about r_0,

$$V_{eff}(r) = V_{eff}(r_0) + (r - r_0)V'_{eff}(r_0) + \frac{1}{2}(r - r_0)^2 V''_{eff}(r_0) + \cdots. \tag{10.98}$$

If we use this expansion in the Lagrangian (10.94) together with the condition (10.96), we find

$$L = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}(r - r_0)^2 V''_{eff}(r_0), \tag{10.99}$$

where we have dropped a constant term. This is just the Lagrangian for a simple harmonic oscillator, describing a particle undergoing radial oscillations with frequency

$$\omega^2 = \frac{1}{m}V''_{eff}(r_0). \tag{10.100}$$

Differentiating $V_{eff}(r)$ twice gives us

$$\frac{3l^2}{mr_0^4} - \frac{2k}{r_0^3} + b = m\omega^2. \tag{10.101}$$

We can eliminate l between equations (10.101) and (10.97) to give the frequency of radial oscillations:

$$\omega = \left(\frac{k}{mr_0^3} + \frac{4b}{m}\right)^{1/2}. \tag{10.102}$$

To find the rate of precession of the perihelion, we need to know the period of the orbit. From the definition of angular momentum l, equation (10.91), we have an equation for the orbital angular velocity ω_1,

$$\omega_1 \equiv \frac{d\theta}{dt} = \frac{l}{mr^2}. \tag{10.103}$$

Let us write $r(t) = r_0 + \epsilon(t)$, where $\epsilon(t)$ is sinusoidal with frequency ω and average value zero. We substitute $r(t)$ into equation (10.103) and expand in $\epsilon(t)$:

$$\frac{d\theta}{dt} = \frac{l}{mr_0^2}\left(1 - \frac{2\epsilon}{r_0} + \mathcal{O}(\epsilon^2)\right). \qquad (10.104)$$

To zeroth order in the small quantities br_0^3/k and ϵ/r_0, the period of the orbit T_1 is the same as the period of oscillations $T_2 = 2\pi/\omega$. Therefore we can average ϵ over T_1 rather than T_2 and still get zero, to within terms of second order, which we are neglecting. The average angular velocity is therefore

$$\bar{\omega_1} = \frac{2\pi}{T_1} \approx \frac{l}{mr_0^2} = \sqrt{\frac{k}{mr_0^3} + \frac{b}{m}}, \qquad (10.105)$$

where we have made use of (10.97).

Now consider one complete period of the radial oscillation. This takes place in time $T_2 = 2\pi/\omega$. In this time the particle travels along its orbit through an angle of

$$\theta = 2\pi\frac{\bar{\omega_1}}{\omega} = 2\pi\frac{\sqrt{k/mr_0^3 + b/m}}{\sqrt{k/mr_0^3 + 4b/m}}$$

$$\approx 2\pi\left(1 - \frac{3br_0^3}{2k}\right). \qquad (10.106)$$

In other words, the particle does not quite orbit through 2π before the radial oscillation is completed. Each time around the perihelion precesses backwards through an angle

$$\delta\theta = 3\pi\frac{br_0^3}{k}, \qquad (10.107)$$

and it gets around in time T_2, so the precession rate is

$$\alpha = \frac{\delta\theta}{T_2} = \frac{3\pi br_0^3}{k}\frac{\sqrt{k/mr_0^3 + 4b/m}}{2\pi}$$

$$\approx \frac{3b}{2}\sqrt{\frac{r_0^3}{mk}}. \qquad (10.108)$$

b) When r is large enough that $F_r \approx -br$, we see that the force is like that of a linear spring. In this case the planar motion of the orbit can be resolved into simple harmonic motion in each of its three cartesian components. Thus the orbits will in general be ellipses; however, in each case the sun will be at the *center* of the ellipse rather than at one of the foci (as is the case for Newtonian gravity).