F, A

[1] $I_L = \frac{V}{Z_L}$, $Z_L = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2} = \frac{C_1 + C_2}{j\omega C_1 C_2}$

$$V_1 = I_L Z_2 = V \frac{j\omega C_1 C_2}{C_1 + C_2} \frac{1}{j\omega C_2} = V \frac{C_1}{C_1 + C_2}$$

Similarly, $V_2 = V \frac{C_3}{C_3 + C_4}$

$$V_1 = V_2 \Rightarrow \frac{C_1}{C_1 + C_2} = \frac{C_3}{C_3 + C_4} \Rightarrow C_1 C_4 = C_2 C_3$$

[2] (a) If there were no hollow region, $\vec{E}_1 = \frac{\rho}{3\varepsilon_0} \vec{x}$

If only the hollow region has charge density ρ,

$$\vec{E}_2 = \frac{\rho}{3\varepsilon_0} \vec{x}'$$

F, A

Thus, $\vec{E} = \vec{E}_1 - \vec{E}_2 = \frac{\rho}{3\varepsilon_0} (\vec{x} - \vec{x}') = \frac{\rho}{3\varepsilon_0} \vec{a}$

This includes the center of the hollow.

(b) Consider a sphere of radius R with a uniform charge density ρ,

$$\vec{E}(\vec{r}) = \frac{\rho}{3\varepsilon_0} \vec{r} \quad , \; r < R$$

$$= \frac{\rho R^3}{3\varepsilon_0 r^3} \vec{r} \quad , \; r > R$$

Then, the potential at any point inside the sphere is

$$V = \left(\int_r^R + \int_R^\infty \right) \vec{E} \cdot d\vec{r} = \frac{\rho}{6\varepsilon_0} (3R^2 - r^2) \quad (\text{Here, } V(\infty) \equiv 0)$$

Thus, the potential at the center of the hollow is

$$V = V_1 - V_2 = \frac{\rho}{6\varepsilon_0} (3r_1^2 - a^2) - \frac{\rho}{6\varepsilon_0} (3r_2^2 - 0)$$

$$= \frac{\rho}{6\varepsilon_0} \left[3(r_1^2 - r_2^2) - a^2 \right]$$

EM - PS

Rutgers - Physics Graduate Qualifying Exam
Electricity and Magnetism: August ??, 2007

EC1

EC1

Consider a system in which n free electrons, each with charge, $-e$, and mass, m, are initially distributed uniformly throughout a spherical volume of radius, R. Also present in the same volume is an equal uniform density of ions, each with charge, $+e$. Assume that the ions are massive enough that they can be considered to remain permanently at rest.

1. Find the electric field intensity, $\vec{\mathbf{E}}$, immediately after all of the free electrons are given infinitesimal radial outward displacements proportional to the initial distances of the electrons from the center of the sphere.

2. Make appropriate approximations to show that, if the electrons are simultaneously released, they will all oscillate radially with the same frequency.

3. Find the value of this frequency.

Solution: EC1

Define the ion charge density as, $\rho = \frac{3ne}{4\pi R^3}$, and $k = \frac{1}{4\pi\epsilon_0}$. The electric field for the ions is given by

$$\vec{E_+} = \frac{kQ_{inside}(r)}{r^2} = \frac{k\rho(\frac{4}{3})\pi r^3}{r^2} = +\frac{kner}{R^3} = \vec{E} \tag{1}$$

and the electric field for the electrons *before* their displacement is

$$\vec{E_-} = -\frac{kner}{R^3} = -\vec{E_+} \tag{2}$$

and the total electric field before displacement is zero. After displacement of the electons, their contribution becomes

$$\vec{E_-}' = -\frac{kner}{R'^3} \tag{3}$$

and the net change in the overall electric field is

$$\vec{E_-}' = +\frac{3kner}{R'^3}\frac{dR}{R} \approx \frac{3knerx}{R^3} = 3\vec{E}x \tag{4}$$

where the fractional displacement is $\frac{dR}{R} = \frac{dr}{r} = x$. The radial position of an electron can be written as $r' = r(1+x)$. The force on an electron at radius, r, with a fractional displacement, x, can be written:

$$m\frac{d^2r}{dt^2} = mr\frac{d^2x}{dt^2} = \vec{F}(r,x) = -e\vec{E_-}' = -\frac{3kne^2rx}{R^3} \tag{5}$$

which gives a differential equation in x, independent of r

$$m\frac{d^2x}{dt^2} = -\frac{3kne^2x}{R^3} = -\omega^2 x \tag{6}$$

and yields the angular velocity, ω, in radians/second and the frequency, f in Hz:

$$\omega = \sqrt{\frac{3kne^2}{R^3}}; \qquad f = \frac{\omega}{2\pi} \tag{7}$$

EC2

EC2

1. Define the magnetic dipole moment of an arbitrary current distribution.
2. What is the magnetic dipole moment of a ring of radius, a, carrying a current, i?
3. Suppose you have a particle with charge, q, moving in a circular orbit. Compute the dipole moment for this system.
4. Find the precession frequency of the orbit in a magnetic field, $\vec{B}$. (Express your answer in terms of q the particle's mass, m, and $\vec{B}$.
5. Discuss what is meant by diamagnetism. Give an example of a diamagnetic substance.
6. Discuss what is meant by paramagnetism. Give an example of a paramagnetic substance.

Solution: EC2

The magnetic dipole moment of a current distribution is:

$$\vec{m} = \frac{1}{2}\int_V \vec{r} \times \vec{j} dV = \int_A i d\vec{A} \tag{8}$$

For a ring of radius, a, the moment is: $|m| = \pi a^2 i$, perpendicular to the ring in the direction if the right-hand rule determined by the current.
For a charged particle with an angular momentum, $\vec{L}$, in a circular orbit with speed, v,

$$\vec{m} = \frac{e}{2mc}\vec{L} \tag{9}$$

$$\vec{L} = m\vec{r} \times \vec{v} \tag{10}$$

If this is in a magnetic field, $\vec{B}$, the torque is

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{m} \times \vec{B} = \frac{e}{2mc}\vec{L} \times \vec{B} \tag{11}$$

Adopt a Cartesian coordinate system, (x,y,z), with unit vectors $(\hat{i}, \hat{j}, \hat{k})$ and relate them to the angles, (θ, ϕ) in a spherical coordinate system We choose our directions to put $\vec{B}$ in the z-direction, and recognize that $\frac{d\theta}{dt} = 0$. Then the quantities above can be written:

$$\vec{L} = \hat{i}L\sin\theta\cos\phi + \hat{j}L\sin\theta\sin\phi + \hat{k}L\cos\theta \tag{12}$$

$$\vec{B} = \hat{k}B \tag{13}$$

$$\frac{d\vec{L}}{dt} = (-\hat{i}L\sin\theta\sin\phi + \hat{j}L\sin\theta\cos\phi)\frac{d\phi}{dt} \tag{14}$$

$$\vec{L} \times \vec{B} = \hat{i}LB\sin\theta\sin\phi - \hat{j}LB\sin\theta\cos\phi \tag{15}$$

Equating components yields the precession angular velocity , ω in radians/sec, and the frequency, f, in Hz:

$$\omega = \frac{d\phi}{dt} = \frac{e}{2mc}B; \qquad f = \frac{\omega}{2\pi} \tag{16}$$

For diamagnetism, the magnetic susceptibility, μ, has the property

$$\mu = 1 + \chi_m < 1, \quad \text{or} \quad \chi_m < 0 \tag{17}$$

This results from the fact that the magnetic field causes additional centripetal force on orbital electrons, yielding a frequency change, and an induced magnetic moment. By Lenz's law, the induced moment must oppose the field. Hence the negative sign for χ_m. Examples are water, and some metals (Hg, Ag, Pb, Cu) outside the transition regions in the periodic table.

For paramagnetism we have

$$\mu = 1 + \chi_m > 1, \quad \text{or} \quad \chi_m > 0 \tag{18}$$

Some elements have natural magnetic moments due to unbalanced electrons spins or orbital moments. The axis precesses about the applied field and gives a net positive contribution to the flux. Examples are Al and some rare-earth elements.

Mechanics Problem 1.pdf

EM Qualifier.pdf

E01

1 Problem 1.

1.1 Problem

At time $t = 0$ non relativistic charged particle moves with velocity v perpendicular to magnetic field B. Find trajectory of the particle. Estimate the time scales (order of magnitude) characterizing this motion for the electron in Earth ionosphere. Give the condition on the density of the atoms needed to observe this motion and estimate order of magnitude for electron moving with velocity $0.1c$.

1.2 Solution.

In magnetic field particle experiences roughly circular motion with $\omega = qB/mc$. The radius of the trajectory is $R = v/\omega$. Because $R \ll \lambda = 2\pi c/\omega$ the particle emits radiation as a rotating dipole with dipole moment $p = qR$ so that

$$\omega^2 p = \omega v q = \left(\frac{q^2 B v}{mc}\right)$$

The radiation energy emitted by this dipole per unit time

$$\begin{aligned}\frac{dE}{dt} &= -\frac{2(\omega^2 p)^2}{3c^3} = -\frac{2}{3c^3}\left(\frac{q^2 B v}{mc}\right)^2 \\ &= -\frac{4}{3mc^3}\left(\frac{q^2 B}{mc}\right)^2 E\end{aligned}$$

As a result the energy of the particle decreases exponentially with the time constant

$$\gamma = \frac{4}{3}\frac{q^4 B^2}{(mc^2)^3} c$$

The trajectory of the particle will be concentric spiral

$$\begin{aligned}x &= \frac{v}{\omega}\cos\omega t \exp(-\gamma t) \\ x &= \frac{v}{\omega}\sin\omega t \exp(-\gamma t)\end{aligned}$$

Earth field is about $1Gs$, electron in such has a circular frequency $\omega \approx 2\ 10^7 s^{-1}$ and energy decay rate $\gamma \sim 410^{-9} s^{-1}$. In order to observe such motion the mean free path of the electron should be larger than $l = v/\gamma \approx 10^{18} cm$. The mean free path of the electron in a gas with density n is $l = 1/n\sigma > 10^{18} cm$ where σ is the cross-section that can be estimated by $\sigma \sim 10^{-16} cm^{-2}$. Thus, the condition for the density $n < 1/(l\sigma) \sim 10^{-2} cm^{-3}$.

F-02

2 Problem

2.1 Problem

Find the force between two permanent magnets that have the shape of the long cylinders (with length $L \gg R$) with magnetization M parallel to the axis of the cylinder and which are positioned coaxially one at the top of another at a distance $d \ll R$. Assume that ferromagnetic material has the pemeability of the vacuum.

2.2 Solution.

Free energy of the ferromagnet is

$$F = -\int MH - \frac{1}{8\pi}\int H^2$$

To find the magnetic field it convenient to introduce the scalar potential $H = \nabla\phi$ so that $\nabla B = \nabla(H + 4\pi M) = 0$ results in the equations similar to electrostatics $\nabla^2\phi = -4\pi\rho$ with $\rho = \nabla M$. Expressing the free energy through the 'charge' ρ we get

$$F = \frac{1}{2}\int \phi\rho$$

In the conditions of the problem the 'charge' density $\rho = \delta(z - z_0)M$ that gives the potential difference between the discs forming the edges $\Delta\phi = 4\pi Md$ and energy $F = 2\pi M^2 d\pi R^2$. Finally, differentiating the energy we get the force

$$f = 2\pi^2 M^2 R^2$$