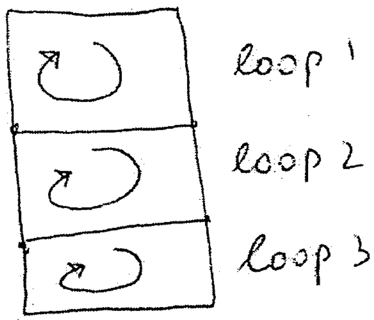
Problem EA



Kirchhoff's rules give

$$\begin{cases} I_1 + I_2 = I_3 & \\ E_1 - I_3 R_3 - I_2 R_2 = 0 & \text{loop 1} \\ E_2 - I_1 R_1 + I_2 R_2 = 0 & \text{loop 2} \end{cases}$$

Solving this system gives

$$I_1 = \frac{E_1}{R_3} - \frac{R_2 + R_3}{R_3} \left(\frac{R_1 E_1 - R_3 E_2}{R_1 R_3 + R_2 R_3 + R_1 R_2} \right) = 1.38\,A \quad \text{(down)}$$

$$I_2 = \frac{R_1 E_1 - R_3 E_2}{R_1 R_3 + R_2 R_3 + R_1 R_2} = -0.364\,A \quad \text{(to the right)}$$

$$I_3 = \frac{E_1}{R_3} - \frac{R_2}{R_3} \left(\frac{R_1 E_1 - R_3 E_2}{R_1 R_3 + R_2 R_3 + R_1 R_2} \right) = 1.02\,A \quad \text{(up)}$$

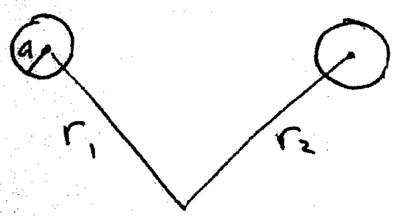
From loop 3, we get $\Delta V - E_2 - E_3 = 0$, where ΔV is the potential difference on the capacitor. The charge

$$Q = C\,\Delta V = C(E_2 + E_3) = 66\,\mu C$$

Problem E B

(a) From the Gauss' theorem, $\oint \vec{E} d\vec{s} = 4\pi Q$, the electric field of one wire is $E = \frac{2Q}{R\ell}$.

Since $\vec{E} = -\vec{\nabla}\varphi$, the potential of one wire is $\varphi = -\frac{2Q}{\ell}\ln R + \text{const}$. The potential produced by the two wires at any point is



$$\varphi = -\frac{2Q}{\ell}\ln r_1 + \frac{2Q}{\ell}\ln r_2 + \text{const}.$$

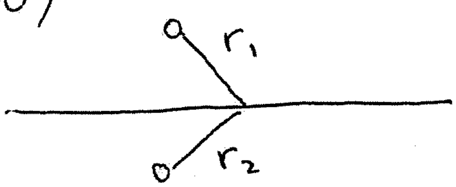
The potential difference between the wires is, therefore,
$$\Delta\varphi = \frac{2Q}{\ell}\left(-\ln a + \ln 2h - \ln a + \ln 2h\right) =$$
$$= \frac{4Q}{\ell}\ln\frac{2h}{a} \qquad (\text{we used } a \ll h).$$

The capacitance $\boxed{C = \frac{Q}{\Delta\varphi} = \frac{\ell}{4\ln\left(\frac{2h}{a}\right)}}$

In the SI units, this becomes $C = 4\pi\varepsilon_0 \frac{\ell}{4\ln\left(\frac{2h}{a}\right)}$.

$$C = \frac{\pi\varepsilon_0 \ell}{\ln\left(\frac{2h}{a}\right)}$$

B (b)



For the plane described in the problem, $r_1 = r_2$, and therefore $\varphi = \text{const}$. We can assume $\varphi = 0$.

Thus, the potential field between the wire and the ground in part (b) is the same as the potential field between the 2 wires of part (a).

From the symmetry of the problem, the potential difference between the plane and one wire is twice as small as the potential difference between the 2 wires of part (a). Therefore,

$$\boxed{C = \frac{l}{2 \ln\left(\frac{2h}{a}\right)}}$$

In the SI units,

$$C = 4\pi\varepsilon_0 \frac{l}{2\ln\left(\frac{2h}{a}\right)} .$$

$$= \frac{2\pi\varepsilon_0 l}{\ln\left(\frac{2h}{a}\right)}$$

Problem EC 1

The magnetic force on the carriers drifting with velocity v is balanced by the electric force qE_H, where E_H is the Hall electric field.

$$qvB = qE_H$$

The Hall voltage $V = E_H d = vBd \qquad (*)$

Since $I = nqAv = nq\,dt\,v$,

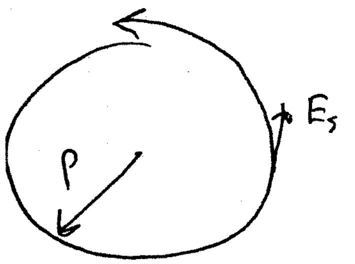
$$\boxed{V = \frac{IB}{nqt}}$$

For part (b), equation $(*)$ gives

$$\boxed{v = \frac{V}{Bd}}$$

Problem EC-2

Time-dependent magnetic field generates an electric field. Consider the projection of this field E_s on the trajectory of the particle,



and assume it constant for one loop. Then,

$$\left| \oint E_s \, ds \right| = \left| \frac{1}{c} \frac{d\Phi}{dt} \right| = \frac{\pi \rho^2}{c} \frac{dB}{dt}.$$

$$|E_s| = \frac{\rho}{2c} \frac{dB}{dt} = \frac{mv}{2eB} \left| \frac{dB}{dt} \right|$$

The equation of motion $m\dot{v} = eE_s$ becomes

$$m \frac{dv}{dt} = \frac{mv}{2B} \frac{dB}{dt} \quad \text{giving}$$

$$\boxed{\frac{v^2}{B} = \text{const}}$$

(b) Here $v_{perp} = v \sin\alpha$. Using $\frac{v_{perp}^2}{B} = \text{const}$,

$$\frac{\sin^2\alpha}{B} = \frac{\sin^2\alpha_0}{B_0}.$$ The particle gets reflected when $\sin\alpha = 1$, that is at $\boxed{B = \frac{B_0}{\sin^2\alpha_0}}$

Problem ED-1

Maxwell Equations with no displacement current give (using $\vec{B}=\mu\vec{H}$ and $\vec{j}=\lambda\vec{E}$):

$$\begin{cases} \vec{\nabla}\times\vec{E} = -\frac{\mu}{c}\frac{\partial \vec{H}}{\partial t} \\ \vec{\nabla}\times\vec{H} = \frac{4\pi\lambda}{c}\vec{E} \end{cases}$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\frac{\mu}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{H}\right) = -\frac{4\pi\mu\lambda}{c^2}\frac{\partial \vec{E}}{\partial t}$$

On the other hand, $\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}$

In the absence of volume charge $(\rho=0)$ $\vec{\nabla}\cdot\vec{E}=0$.

Thus, $\nabla^2\vec{E} = \frac{4\pi\lambda}{c^2}\frac{\partial \vec{E}}{\partial t}$. $(*)$

For a plane wave, $\vec{E} = \vec{E}_0(x,y,z)\,e^{i\omega t}$

Equation $(*)$ then gives

$$\nabla^2\vec{E}_0 = \frac{4\pi\mu\lambda i\omega}{c^2}\vec{E}_0 = 2i\,p^2\,\vec{E}_0$$

where we define $p^2 \equiv \frac{2\pi\mu\lambda\omega}{c^2}$.

In our problem, the wave is propagating along z. Let $\vec{E}_0$ be in the x direction. Then inside the conductor

$$\nabla^2 E_{0x} = \frac{\partial^2 E_x}{\partial z^2} = 2i\rho^2 E_{0x}$$

$$E_{0x} = A e^{kz} + B e^{-kz} \qquad \text{where } k^2 \equiv 2ip^2,$$

i.e. $k = p\sqrt{2i} = p(i+1)$

Thus $E_{0x} = A e^{pz} e^{ipz} + B e^{-pz} e^{-ipz}$.

Clearly, $A = 0$ (or $E \to \infty$ as the wave propagates inside the metal).

$$E_x = E_{0x} e^{i\omega t} = B e^{-pz} e^{i(\omega t - pz)}$$

or $\quad E_x = B e^{-pz} \cos(\omega t - pz)$.

The amplitude of the electric field is decaying exponentially with the characteristic penetration depth

$$\boxed{p = \sqrt{\frac{2\pi \mu \lambda \omega}{c^2}}}$$

In the SI units, $p = \sqrt{\dfrac{\lambda \mu \mu_0 \omega}{2}}$.

For $\omega = 2\pi f = 2\pi \cdot 10^5 \ \frac{1}{s}$, $\lambda = 5.9 \cdot 10^7 \ \frac{1}{\Omega m}$, $\mu = 1$, $\mu_0 = 4\pi \cdot 10^7$,

this gives $p = 0.2$ mm.

Problem ED-2

Lorentz transformation of the field is

$$E_x = E_x' ; \quad E_y = \frac{E_y' + \frac{V}{c} H_z'}{\sqrt{1-\frac{V^2}{c^2}}} ; \quad E_z = \frac{E_z' - \frac{V}{c} H_y'}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (1)$$

$$H_x = H_x' ; \quad H_y = \frac{H_y' - \frac{V}{c} E_z'}{\sqrt{1-\frac{V^2}{c^2}}} ; \quad H_z = \frac{H_z' + \frac{V}{c} E_y'}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (2)$$

First, we calculate the fields produced by a moving charge. From $\vec{E}' = \frac{e\vec{R}'}{R'^3}$ and (1), we get

$$E_x = \frac{ex'}{R'^3} ; \quad E_y = \frac{ey'}{R'^3 \sqrt{1-\frac{V^2}{c^2}}} ; \quad E_z = \frac{ez'}{R'^3 \sqrt{1-\frac{V^2}{c^2}}} \qquad (3)$$

where $R'^2 = x'^2 + y'^2 + z'^2 = \dfrac{(x-Vt)^2 + (1-\frac{V^2}{c^2})(y^2+z^2)}{1-\frac{V^2}{c^2}}$.

Using the Lorentz equations for x', y', z', and introducing $\vec{R} = (x-vt; y; z)$ as the radius vector from the charge to the point (x, y, z), and also defining $R^{*2} \equiv (x-vt)^2 + \left(1-\frac{v^2}{c^2}\right)(y^2+z^2)$,

equations (3) can be written as

$$\vec{E} = \left(1-\frac{v^2}{c^2}\right)\frac{e\vec{R}}{R^{*3}}$$

If θ is the angle between $\vec{R}$ and $\vec{v}$, then

$y^2+z^2 = R^2\sin^2\theta$, and therefore $R^{*2} = R^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right)$.

Thus, $$\vec{E} = \frac{e\vec{R}}{R^3}\frac{1-\frac{v^2}{c^2}}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \qquad (4)$$

We also notice that because $\vec{H}'=0$, equations (1) and (2) give $$\vec{H} = \frac{1}{c}\vec{v}\times\vec{E} \qquad (5).$$

We now move to the main problem.

We determine the force $\vec{F}$ by computing the force acting on one of the charges (e_1) in the field produced by the other (e_2).

$$\vec{F} = e_1\vec{E_2} + \frac{e_1}{c}\vec{v}\times\vec{H_2}$$

Using (5), $\vec{F} = e_1\vec{E_2} + \frac{e_1}{c}\vec{v}\times\left(\frac{1}{c}\vec{v}\times\vec{E_2}\right) =$

$$= e_1\vec{E_2} + \frac{e_1}{c^2}\vec{v}(\vec{v}\cdot\vec{E_2}) - \frac{e_1}{c^2}\vec{E_2}v^2 = e_1\left(1-\frac{v^2}{c^2}\right)\vec{E_2} + \frac{e_1}{c^2}\vec{v}(\vec{v}\cdot\vec{E_2})$$

Using $\vec{R} = R(\cos\theta; \sin\theta; 0)$, $\vec{V} = (V;0;0)$ and (4), this gives

$$F_x = e_1\left(1-\frac{V^2}{c^2}\right)\frac{e_2}{R^3} R\cos\theta \,\frac{\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{V^2}{c^2}\sin^2\theta\right)^{3/2}} +$$

$$+ \frac{e_1}{c^2} V^2 \frac{e_2}{R^3} R\cos\theta \,\frac{\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{V^2}{c^2}\sin^2\theta\right)^{3/2}} = \frac{e_1 e_2}{R^2}\,\frac{\left(1-\frac{V^2}{c^2}\right)\cos\theta}{\left(1-\frac{V^2}{c^2}\sin^2\theta\right)^{3/2}}$$

$$F_y = e_1\left(1-\frac{V^2}{c^2}\right)\frac{e_2 R\sin\theta}{R^3}\,\frac{\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{V^2}{c^2}\sin^2\theta\right)^{3/2}}$$

$$F_z = 0$$

$$\boxed{\begin{aligned} F_x &= \frac{e_1 e_2}{R^2}\,\frac{\left(1-\frac{V^2}{c^2}\right)\cos\theta}{\left(1-\frac{V^2}{c^2}\sin^2\theta\right)^{3/2}} \\ F_y &= \frac{e_1 e_2}{R^2}\,\frac{\left(1-\frac{V^2}{c^2}\right)^2\sin\theta}{\left(1-\frac{V^2}{c^2}\sin^2\theta\right)^{3/2}} \\ F_z &= 0 \end{aligned}}$$