Solutions C1, C2, C4, C6

C1

1/05

Number of Phonons

We will use the Bose-Einstein distribution function, Eq. I.6.22. The chemical potential is $\mu = 0$. For each branch s the integration is over a finite range of energies, from $\mathcal{E} = 0$ to $\mathcal{E} = \hbar c_s k_D$. We obtain the number of phonons in each mode s to be

$$N_s = 3N_a \left(\frac{k_B T}{\hbar c_s k_D}\right)^3 \int_0^{x_0} \frac{x^2}{e^x - 1}\, dx\,,$$

where k_D is the Debye wavenumber, N_a is the number of atoms in the solid and the upper limit of the integration is $x_0 = \hbar c_s k_D / k_B T$. The total number of phonons is $N = N_1 + N_2 + N_3$.

For low temperatures, the upper limit of the integral diverges, and the integral converges to a constant, independent of s. Therefore the number of phonons varies as

$$N \propto \left(\frac{1}{c_1^3} + \frac{1}{c_2^3} + \frac{1}{c_3^3}\right)\left(\frac{k_B}{\hbar k_D}\right)^3 T^3\,.$$

For high temperatures, x remains small within the range of integration and the distribution function can be expanded into a Taylor series. The integral can be performed, yielding

$$N \propto \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}\right)\frac{k_B}{\hbar k_D}\, T\,.$$

Note that the coefficient in front of the temperature dependent term contains an "averaged" sound velocity, but the average is different at low and at high temperatures. If we take the three sound velocities to be approximately equal, the expressions are further simplified to $N \propto (T/\theta_D)^3$ at low T, and $N \propto (T/\theta_D)$ at high T. Here $\theta_D = \hbar c k_D / k_B$ is the Debye temperature. For this particular case the results of the numerical integration of Equation II.6.5 is shown in Figure II.6.2.

$\lambda_n = \frac{2L}{n}$

$k_n = \frac{2\pi}{\lambda_n} = \frac{2\pi \cdot n}{2L} = \frac{\pi}{L} n$

$\lambda \nu = v$

$\nu = \frac{v k}{2\pi}$

$V\, d^3k$

one for each vol.

$\left(\frac{\pi}{L}\right)^3 = \frac{\pi^3}{V}$ in k

$\frac{d^3k}{\pi^3 / V}$

1st oct

$\frac{V d^3k}{(2\pi)^3}$ all k's

$= \frac{4\pi k^2 dk\, V}{(2\pi)^3}$

But $\frac{k}{2\pi} = \frac{\nu}{v}$

$\rightarrow 4\pi \frac{\nu^2 d\nu}{v^3}\, dl$

Solution C2

(a) For very large r_0 the electronic wave functions of the different atoms do not overlap at all and each level is N-fold degenerate; equivalent to the electron being on any one of ~~the~~ N atom ~~electrons~~. As r_0 decreases the outer orbitals begin to overlap and one must think of the electron as ~~being~~ having wave functions spread over all atoms. These come in various versions with slightly diff. energies so the line of degen. levels becomes a band of N levels. For the "inner" electrons ~~the~~ (1s here) the overlap does not occur until quite small r.

(b) At large r each level is completely full, or empty, so as r decreases the ~~2s~~ 1s & 2s bands are filled with 2N electrons. For $r = a$ these bands do not overlap with higher bands; there is a band gap between the full 2s & empty 2p band, so it would take a considerable amount of energy to excite an electron. Therefore at a the solid is an insulator.

At b the 2p band overlaps the 2s so the combined band is not full & the solid is a conductor.

(c) $1s^2 2s^2 2p^1$: A conductor for both ~~at~~ $r = a$ & $r = b$ since electrons can be excited by infin.

$1s^2 2s^2 2p^6$ Here the $2p$ band is full so insulator at both a & b.

C3

It is useful at low energies to do a partial wave analysis to understand the two-nucleon system.

a) [3 points] Write down the allowed quantum mechanical states of two nucleons, using the spectroscopic notation ${}^{2S+1}l_J$, for $J = 0$, 1, and 2. Use $l = s, p, \ldots$ For each state, specify also the parity (+ or -), and isospin T (0 or 1) allowed.

See Table 1.

Table 1: Two-nucleon states.

L	S	J	$\pi = -1^L$	T (L+S+T odd)	${}^{2S+1}L_J$
0	0	0	+	1	${}^1S_0$
0	1	1	+	0	${}^3S_1$
1	0	1	-	0	${}^1P_1$
1	1	0	-	1	${}^3P_0$
1	1	1	-	1	${}^3P_1$
1	1	2	-	1	${}^3P_2$
2	0	2	+	1	${}^1D_2$
2	1	1	+	0	${}^3D_1$
2	1	2	+	0	${}^3D_2$
2	1	3	+	0	${}^3D_3$

The deuteron is a proton-neutron system bound by 2.225 MeV, and it is the only bound system of two nucleons. In a non-relativistic, nucleons-only model, you can determine the allowed ground state wave function configurations from the knowledge that it is spin 1, and that isobaric analogue states - states of different nuclei with the same isospin T but different T_z - have almost the same mass.

b) [1 points] Which of the above configurations are in the deuteron ground state?

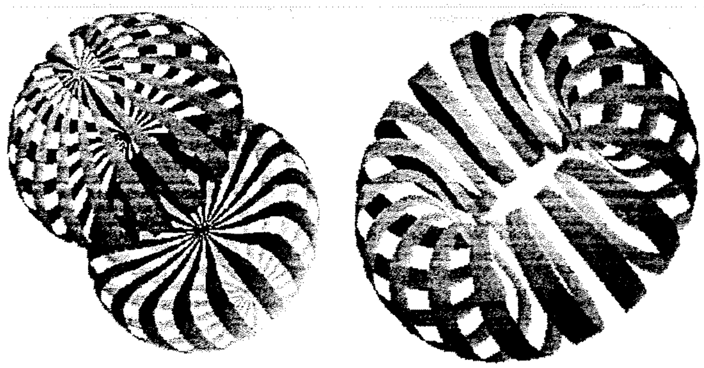
There are 4 states in the Table with $J = 1$. The $T = 1$ ${}^3P_1$ state is not allowed, since there then would be bound nn and pp systems as well. The ${}^3S_1$ and ${}^3D_1$ states have the same observable quantum numbers, as both are $J = 1$, $T = 0$, and + parity, while the ${}^1P_1$ is negative parity. The ground state is indeed positive parity, and is a mixture of the ${}^3S_1$ and ${}^3D_1$ states.

c) [2 points] The information above should also allow you to determine the most likely configuration for the just-unbound deuteron excited state. What is it?

The excited state should have just-unbound nn and pp analogs; it should be a $T = 1$ system. In the continuum, we expect that lower orbital angular momentum reflects lower momentum and lower energy. Thus, the unbound excited state is the ${}^1S_0$ system.

d) [2 points] If one allows non-nucleonic degrees of freedom in the deuteron, additional configurations are possible. Is there a ΔN component to the deuteron wave function? A $\Delta\Delta$ component? Explain.

The deuteron is $T = 0$. There is no way to make a total $T = 0$ state from a $T = 3/2$ Δ and a $T = 1/2$ nucleon. $\Delta\Delta$ states can however couple to $T = 0$, and are allowed.



e) [2 points] The deuteron density is calculated to be different between its $m = \pm 1$ and $m = 0$ sub-states, as indicated in the figure above. Even without knowing the radial densities, simple arguments about Y^l_m's should allow you to associate the left/right sides of the figure above with $m = \pm 1$ and $m = 0$. Explain which side is which m projection.

There is no ϕ dependence so the z axis goes up to the left / down to the right. Y^l_m's with $m = 0$ are maximum in the xy plane, while Y^l_m's with m not 0 put more strength near the z axis. Thus the left side is $m = \pm 1$, and the right side is $m = 0$.

C4

Particle Problem Solution Jan 2005

(a) The total spin 3/2 spin wave function is symmetric in the ~~three~~ particles (for example $\uparrow\uparrow\uparrow$, and the spatial wave function is also symmetric. The three particles mentioned each contain three ~~identical~~ quarks (uuu, ddd, or sss). If there were no color degree of freedom the overall wave function would therefore be *symmetric* under exchange of any two of three identical spin 1/2 particles (Fermions) instead of anti-symmetric as required by the Pauli principle. Adding a color degree of freedom solves this problem since the color singlet (colorless) color wave function is antisymmetric under exchange of any two quarks.

all of the same flavor.

the quarks can have different colors, and in fact

(b) Each flavor i quark-antiquark pair gives a contribution to the cross section proportional to $3Q_i^2$, with Q_i the charge of the quark, because of the coupling to the intermediate virtual photon, and the factor of 3 from the 3 different colors possible for each flavor. The analogous factor for muon pair production is just e^2. The ratio R at an energy E is then simply

$$R = \frac{3\sum_i Q_i^2}{e^2},$$

where the sum is over all flavors which can be produced at that energy. Below the charm threshold these are u, d, and s with $Q_u = 2/3e$, $Q_d = -1/3e$, $Q_s = -1/3e$, giving $R = 2$. Just above the charm threshold (and thus below the bottom threshold) we have to add the charm contribution with $Q_c = 2/3e$ to the sum, increasing R to 10/3. Roughly this behavior is seen in experiments.

by $3\left(\frac{2}{3}\right)^2 = \frac{4}{3}$ to

ParticleColorSoln.tex November 21, 2004

C5

A neutrino of energy E scatters from a proton at rest, and a muon is detected at an angle θ relative to the incident neutrino, with energy E'. You may assume $E,\ E' >> m_\nu,\ m_\mu$. A leading-order Feynman diagram showing the process is below.

- a) [3 points] Label all the particles in the eight lines in the diagram. (If you do not understand the "blob", it indicates the three particles to its right are constituents of the particle to its left.)

  The plot is now labelled. If one takes seriously that time runs left to right, only a W^+ boson is labelled as the line is sloped. Otherwise a W^- is acceptable. The neutrino is a muon neutrino, as muon implies negative charge.

- b) [2 points] Assume the neutrino has initial momentum k and final momentum k'. Give the energy-momentum four vector of the transferred particle, $q = (\omega, \vec{q})$, and evaluate q^2.

  Neglecting the masses, $\omega = k - k'$, *and* $\vec{q} = \vec{k} - \vec{k}'$, *so* $q = (k - k',\ \vec{k} - \vec{k}')$. *Then* $q^2 = \omega^2 - \vec{q}^2 = -2kk' + 2\vec{k} \cdot \vec{k}' = -2kk'(1 - \cos\theta)$. *An alternate form is* $q^2 = -4kk' sin^2(\theta/2)$

- c) [2 points] What is the invariant mass squared $W^2 = E^2 - p^2$ of the undetected recoil system?

  Since W is the invariant mass, it is given by $W^2 = p'^2$ where $p' = p + q$ is the sum of the initial state proton and transfered q four vectors. Thus $W^2 = (M + \omega,\ \vec{q})^2 = M^2 + 2M\omega + \omega^2 - \vec{q}^2 = M^2 + 2M\omega + q^2$. *Of course, q^2 can be replaced from part b) above.*

- d) [3 points] If $E = 10$ GeV, $\theta = 10°$, and $E' = 6$ GeV, how many pions could be produced? You may use 0.94 (0.14) GeV for the proton (pion) mass.

$\omega = 4$ *GeV, and* $q^2 = 4*10*6*\sin^2(5) = 18.23$ *GeV*^{2}*. Then* $W^2 = 0.94^2 + 2*0.94*4 + 18.23 = 10.23$ *GeV*^{2} *and* $W = 3.20$ *GeV. This is 2.26 GeV more than the proton mass, enough energy for* $2.26/0.14 \approx 16$ *pions.*

Written Qual Jan 2005
Solution to C6

(a) (4 points) For hydrostatic equilibrium the pressure must increase with depth rapidly enough to support the weight of each spherical shell of the star. This requires

$$dP/dr = -\rho_0 GM(r)/r^2,$$

where $M(r) = (4/3)\pi r^3 \rho_0$ is the mass inside the radius r. Integrating this relation, and using $P(R) = 0$ gives the pressure at the center of the star

$$P(0) = (3/8\pi)GM^2/R^4.$$

(3 points) The ideal gas law can be written $P = nkT$, where $n$is the number of particles per unit volume and k is the Boltzmann constant. Since the electron is much lighter than the proton there are 2 particles for every mass m_p and therefore $n = 2\rho_0/m_p$, so the temperature at the center of the star is

$$T = m_p P(0)/(2k\rho_0) = GMm_p/(4kR).$$

(b) (3 points) Putting in the numbers for the sun one finds

$$P(0) = 1.34 \times 10^{14} N/m^2$$

and

$$T(0) = 5.77 \times 10^6 K$$

.

AstroPT.tex January 7, 2005