B - 1/05

Solution to Problem 1.

a) For $r \ll a$, $U(r) \approx -U_0 a/r$. In this Coulomb potential the wave functions of the low lying states are localized on distances of the order $r_n \sim a_0 n^2$, where $a_0 = \hbar^2/(maU_0)$. If $r_n \ll a$, i.e. $\xi \equiv ma^2U_0/\hbar^2 \gg n^2$, in zeroth order approximation the eigenstates and energies are those of a particle in the Coulomb potential. The perturbation is $V(r) = U(r) + U_0 a/r$. Because angular momentum is conserved in a central potential, this perturbation is diagonal in the basis of eigenstates $|n_r lm\rangle$.

The first order correction is

$$E_{nl}^{(1)} = \langle |n_r lm|V(r)|n_r lm\rangle$$

Expanding the perturbation to first order in r/a

$$V(r) = -U_0[-1/2 + r/(12a)]$$

and using the expression for $\langle n_r lm|\hat{r}|n_r lm\rangle$, we obtain

$$E_{nl}/U_0 = -\frac{\xi}{2n^2} + \frac{1}{2} - \frac{3n^2 - l(l+1)}{24\xi} \qquad n = n_r + l + 1$$

where $\xi = ma^2U_0/\hbar^2$

The three lowest lying states correspond to $n = 1; l = 0$ and $n = 2; l = 0, 1$.

$$E_{10} = -\xi/2 + 1/2 - 1/(8\xi) \quad E_{20} = -\xi/8 + 1/2 - 1/(2\xi) \quad E_{21} = -\xi/8 + 1/2 - 5/(12\xi)$$

b) Levels are degenerate with respect to the z-projection of the angular momentum m. The degree of the degeneracy is $N = 2l + 1$. In the unperturbed problem there is an additional ("accidental") degeneracy with respect to the angular momentum. The degree of the degeneracy is $N_0 = \sum_{l=0}^{n-1}(2l+1) = n^2$. We have

$$E_{10}: \quad N = N_0 = 1 \qquad E_{20}: \quad N = 1 \quad N_0 = 4 \qquad E_{21}: \quad N = 3 \quad N_0 = 4$$

Solution to Problem 2.

a) We have

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_0(x)}{dx^2} - \alpha\delta(x)\psi_0(x) = E\psi_0(x)$$

where $E < 0$ corresponds to the discrete part of the spectrum. The solution is

$$\psi_0(x) = Ae^{-\kappa x} \quad x > 0$$

$$\psi_0(x) = Be^{\kappa x} \quad x < 0$$

where $\kappa = \left(\frac{2m|E|}{\hbar^2}\right)^{1/2}$.

Integrating the Shrödinger equation, we obtain

$$\psi_0'(+0) - \psi'(-0) = -\frac{2m\alpha}{\hbar^2}\psi_0(0)$$

In addition we have

$$\psi_0(+0) = \psi(-0)$$

Using these two equations and a normalization $\int |\psi_0(x)|^2 dx = 1$, we find

$$\psi_0(x) = \sqrt{\kappa_0}e^{-\kappa_0|x|} \qquad E_0 = -\frac{m\alpha^2}{2\hbar^2}$$

b) We have

$$(\delta x)^2 = \int_{-\infty}^{+\infty} x^2|\psi_0(x)|^2 dx = \frac{1}{2\kappa_0^2}$$

To find $(\delta p)^2)$, note that

$$\frac{<\hat{p}^2>}{2m} + <U(x)>= E$$

Using $<U(x)>= -\alpha|\psi_0(x)|^2 = -m\alpha^2/\hbar^2$, we obtain

$$(\delta p)^2 = \hbar^2\kappa_0^2$$

Thus, $\delta x \delta p = \hbar/\sqrt{2}$. This does not minimize the uncertainty relationship because the minimum possible value of $\delta x \delta p$ is $\hbar/2$.

c) We have

$$\phi(x,t) = c_0 \exp\left(-\frac{iE_0 t}{\hbar}\right)\psi_0(x) + \int_0^\infty c(E)\exp\left(-\frac{iEt}{\hbar}\right)\psi_E(x)dE$$

where $\psi_E(x)$ are the eigenstates of the continuous part of the spectrum. The second term in this expression vanishes in the limit $t \to \infty$ due to rapid oscillations of the integrand. Therefore,

$$|\phi(x,t=\infty)|^2 = |c_0\psi_0(x)|^2 = \kappa_0|c_0|^2\exp(-2\kappa_0|x|)$$

where

$$c_0 = \int \phi(x)\psi^*(x)dx = \frac{2\sqrt{\kappa_0\beta}}{\kappa_0 + \beta}$$

Thus,

$$p(x) = \frac{4\beta\kappa_0^2}{(\beta+\kappa_0)^2}e^{-2\kappa_0|x|}$$

$$w = \int_{-\infty}^{+\infty} p(x)dx = \frac{4\beta\kappa_0}{(\beta+\kappa_0)^2} \leq 1$$

$w \neq 1$ because there is a finite probability $1 - w$ for the particle to escape to infinity.

Solution to Problem 3. $E(z=0^-) = E_0 \cos(\omega t + \phi)$, where ϕ is an arbitrary angle between 0 and 2π.

a) $<\cos^2\phi>= 1/2$, therefore intensity is reduced by 1/2 and $E_0 \to E_0/\sqrt{2}$, i.e.

$$\tilde{I}_0 = I_0/2 \qquad \tilde{E}_0 = E_0/\sqrt{2}$$

b) Linearly polarized light can be written as right and left circularly polarized light.

$$\vec{E}_R = \frac{E_0}{2}[\mathbf{i}\cos(k_R z - \omega t) + \mathbf{j}\sin(k_R z - \omega t)]$$

$$\vec{E}_L = \frac{E_0}{2}[\mathbf{i}\cos(k_R z - \omega t) - \mathbf{j}\sin(k_R z - \omega t)]$$

$\tilde{E}_0\mathbf{i} = \vec{E}_R + \vec{E}_L$ with $k_R = k_L$. Inside the material $k_R = n_R k_0$ and $k_L = n_L k_0$; ω is the same. Here

$$k_0 = \frac{2\pi}{\lambda}$$

$$\vec{E} = \frac{\tilde{E}_0}{2}\mathbf{i}\cos(n_R k_0 z - \omega t) + \frac{\tilde{E}_0}{2}\mathbf{j}\sin(n_R k_0 z - \omega t) + \frac{\tilde{E}_0}{2}\mathbf{i}\cos(n_L k_0 z - \omega t) - \frac{\tilde{E}_0}{2}\mathbf{j}\sin(n_L k_0 z - \omega t) =$$

$$\frac{\tilde{E}_0}{2}\mathbf{i}\{\cos(n_R k_0 z - \omega t) + \cos(n_L k_0 z - \omega t)\} + \frac{\tilde{E}_0}{2}\mathbf{j}\{\sin(n_R k_0 z - \omega t) - \sin(n_L k_0 z - \omega t)\}$$

Using $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$ and $\cos A + \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$, we obtain at $z = d$

$$\vec{E} = \tilde{E}_0 \cos\left[\frac{n_R + n_L}{2}k_0 d - \omega t\right]\left\{\mathbf{i}\cos\frac{n_R - n_L}{2}k_0 d + \mathbf{j}\sin\frac{n_R - n_L}{2}k_0 d\right\}$$

c) We see from the last expression that the amplitude of the electric field along the y-axis is

$$E_y = \tilde{E}_0 \sin\frac{n_R - n_L}{2}k_0 d$$

Therefore

$$I = \tilde{I}_0 \sin^2\frac{n_R - n_L}{2}k_0 d$$

Solution to Problem 4.

a) Since $N_e = N_p = N$ (electric neutrality) and

$$g_s \frac{V \frac{4\pi}{3} p_F^3}{h^3} = N$$

we have

$$p_{F,e} = p_{F,p} = \left(h^3 \frac{3}{4\pi} \frac{N}{V} \right)^{1/3}$$

when $p_F c \sim m_e c^2$ electrons will be relativistic

$$\left(h^3 \frac{3}{4\pi} \frac{N}{V} \right)^{1/3} = m_e c$$

$$\frac{N}{V} = (m_e c)^3 \frac{4\pi}{3h} = (m_e c^2)3 \frac{4\pi}{3(2\pi)^3(\hbar c)^3}$$

$$E_{F,e} = \frac{p_F^2}{2m_e} = \frac{p_F^2}{2m_p} \frac{m_p}{m_e} = \frac{m_p}{m_e} E_{F,p} \qquad E_{F,e} \gg E_{F,p}$$

This is also true ultra-relativistically. In this case

$$E_{F,e} \sim p_{F,e} c = p_{F,p} c$$

$$E_{F,p} = \frac{p_{F,p}^2}{2m_p} = \frac{(p_{F,e}c)(p_{F,p}c)}{2m_p c^2} = E_{F,e} \frac{p_{F,p}c}{2m_p c^2}, \quad \text{but } p_{F,p} \ll m_p c$$

b) $P = 2E/3V$, therefore $P_e \geq P_p$. This is true relativistically since $P \sim E/V$.

c)

$$E_e = \int g_s \frac{V d^3 p}{h^3} \sqrt{(pc)^2 + (mc^2)^2} = g_s \frac{V 4\pi}{h^3} \int_0^{p_F} p^2 dp \sqrt{(pc)^2 + (m_e c^2)^2} =$$

$$g_s V \frac{4\pi}{h^3} m_e c^2 (m_e c)^3 \int_0^{x_F} dx x^2 \sqrt{x^2 + 1}$$

$x_F \ll 1$, $\sqrt{1 + x^2} \approx 1 + x^2/2$.

$$E_e = g_s V \frac{4\pi}{h^3} m_e^4 c^5 \left(\frac{x_F^3}{3} + \frac{x_F^5}{10} \right) = \text{rest mass term} + \text{kinetic energy term}$$

$x_F \ll 1$, $\sqrt{1 + x^2} \approx x + 1/(2x^2)$.

$$E_e = g_s V \frac{4\pi}{h^3} m_e^4 c^5 \frac{x_F^4}{4}$$

1. Non-relativistic

$$E_e = g_s V \frac{4\pi}{h^3} m_e^4 c^5 \left(\frac{1}{3} \frac{p_F^3 c^3}{(m_e c)^3} + \frac{1}{10} \frac{p_F^5 c^5}{(m_e c)^5} \right)$$

Expect

$$\frac{E_e}{N} = m_e c^2 + \frac{3}{5} E_F = m_e c^2 + \frac{3}{5}\frac{p_F^2}{2m_e} = m_e + \frac{3}{5}\frac{E_k}{N}$$

This checks.

2. Ultra-relativistic $E_e = g_s V \frac{4\pi}{h^3}\frac{p_F^4 c}{4}$ Expect

$$\frac{E_e}{N} = \frac{\int_0^{p_F} g_s \frac{V d^3p}{h^3}(pc)}{\int_0^{p_F} g_s \frac{V d^3p}{h^3}} = \frac{c\int p^3 dp}{\int p^2 dp} = \frac{3}{4} p_F c$$

This checks.

d)

$$P = -\frac{\partial E}{\partial V} = -\frac{\partial E/N}{\partial V/N} = \rho^2 \frac{\partial E/N}{\partial \rho}$$

1. non-relativistic

$$E \sim p_F^2 \sim \rho^{2/3}$$

$$P \sim \rho^2 \frac{\partial \rho^{2/3}}{\partial \rho} \sim \rho^{5/3}$$

i.e. $\alpha = 5/3$.

2. ultra-relativistic

$$E \sim p_F \sim \rho^{1/3}$$

$$P \sim \rho^2 \frac{\partial \rho^{1/3}}{\partial \rho} \sim \rho^{4/3}$$

i.e. $\alpha = 4/3$.

e) The mass comes from the protons $N_p = N_e$, $N_p = \frac{M}{m_p}$. 1. non-relativistic

$$P \sim \left(\frac{M}{R^3}\right)^{5/3} \sim \frac{M^{5/3}}{R^5}$$

$$\beta = 5/3 \qquad \gamma = 5$$

2. ultra-relativistic

$$P \sim \left(\frac{M}{R^3}\right)^{4/3} \sim \frac{M^{4/3}}{R^4}$$

$$\beta = 4/3 \qquad \gamma = 4$$

Solution to Problem 5.

a)

$$Z_N = \sum_{E_N} e^{-\beta E_N} \qquad E_N = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \ldots \frac{p_N^2}{2m}$$

$$Z_N \to \frac{1}{N!}\left(\sum_{\vec{p}_i} \exp(-\beta p_i^2/2m)\right)^N \to \frac{1}{N!}\left(\int \frac{V d^3\vec{p}}{h^3} g_s \exp(-\beta p^2/2m)\right)^N = \frac{1}{N!}\left(g_s \frac{V}{\lambda_T^3}\right)^N$$

where

$$\lambda_T = \frac{h^3}{(2\pi m k T)^{3/2}}$$

$$Z_{GC} = \sum e^{\beta\mu N} Z_N = \sum g_s \left(\frac{V}{\lambda_T^3}\right)^N \frac{1}{N!} e^{\beta\mu N} = \exp\left(g_s \frac{V}{\lambda_T^3} e^{\beta\mu}\right)$$

b)

$$< N >= \frac{\sum e^{\beta\mu N} Z_N N}{\sum e^{\beta\mu N} Z_N} = \frac{\sum e^{\beta\mu N} \frac{N}{N!}\left(g_s \frac{V}{\lambda_T^3}\right)^N}{\sum e^{\beta\mu N} \frac{1}{N!}\left(g_s \frac{V}{\lambda_T^3}\right)^N}$$

$$< N >= e^{\beta\mu} \frac{V g_s}{\lambda_T^3}\left(\sum e^{\beta\mu(N-1)}\left(g_s \frac{V}{\lambda_T^3}\right)^{N-1} \frac{1}{(N-1)!}\right) / Z_{GC} = e^{\beta\mu} \frac{V}{\lambda_T^3} g_s$$

c)

$$< N^2 > - < N >^2 = < N(N-1) > + < N > - < N >^2$$

$$< N(N-1) >= e^{2\beta\mu}\left(\frac{V}{\lambda_T^3}\right)^2 g_s^2 \frac{\sum e^{\beta\mu(N-2)}\left(\frac{V}{\lambda_T^3} g_s\right)^{N-2} \frac{1}{(N-2)!}}{Z_{GC}} = e^{2\beta\mu}\left(\frac{V}{\lambda_T^3}\right)^2 g_s^2 = < N >^2$$

$$< N^2 > - < N >^2 = < N >$$

$$\sqrt{\frac{< N^2 > - < N >^2}{< N >}} = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{V}\sqrt{\frac{N}{V}}}$$

d)

$$
\begin{aligned}
F &= -k_B T \ln Z_A \\
&= -k_B T \ln \left[g_s \left(\tfrac{V}{\lambda_T^3} \right)^N / N! \right] \\
&= -k_B T N \ln g_s \tfrac{V}{\lambda_T^3} + k_B T \ln N! \\
&= -k_B T N \ln g_s \tfrac{V}{\lambda_T^3} + k_B T (N \ln N - N) \\
&= -k_B T N \ln g_s \tfrac{V}{\lambda_T^3} + k_B T (N \ln N - N \ln e) \\
&= -k_B T N \ln g_s \tfrac{Ve}{N\lambda_T^3} \sim N
\end{aligned}
$$

Denote $\rho = N/V$

$$dF = -SdT - PdV + \mu dN$$

$$\left.\frac{\partial F}{\partial T}\right|_{V,N} = -S$$

$$F = -k_B T N \ln g_s \frac{Ve}{N} \frac{(2\pi m k_B T)^{3/2}}{h^3}$$

$$-S = -N k_B \ln g_s \frac{Ve}{N\lambda_T^3} - k_B T N \frac{3}{2T} = -N k_B \ln g_s \frac{V}{N\lambda_T^3} e^{5/2}$$

$$S = N k_B \ln g_s \frac{V}{N\lambda_T^3} e^{5/2} \sim N$$