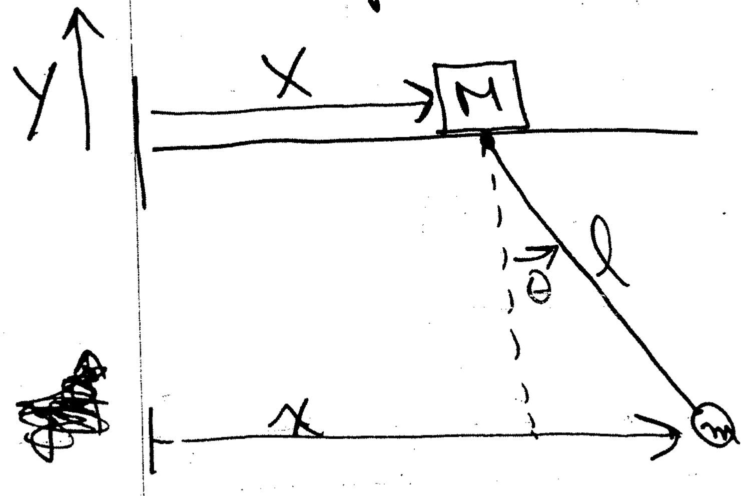
1/05

A1

A mass M is free to slide on a frictionless air track; suspended by a pivot and a light rod of length l is another mass m.



(a) Construct the Lagrangian for this system and derive the equations of motion.
(b) Determine the angular frequency when the system undergoes small oscillations.

Soln

(a) Constraints are: $Y = 0$
$[(x - X)^2 + y^2]^{1/2}$

Choose generalized co-ordinates: θ and X

The location of m is: $x = X + l \sin$
$y = -l \cos$

$$K.E. = \tfrac{1}{2} M \dot{X}^2 + \tfrac{1}{2} m [(\dot{X}^2 + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}$$
$U = -mgl\cos\theta$, taking zero at

Thus, $\mathcal{L} = K.E. - U$
$$= \tfrac{1}{2} M\dot{X}^2 + \tfrac{1}{2} m [(\dot{X}^2 + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2] + m$$

A1 cont

Now, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$.

With $q = X$, we get

① $\frac{d}{dt}\left[M\dot{X} + m(\dot{X} + l\dot{\theta}\cos\theta)\right] = 0$

With $q = \theta$, we get

② $\frac{d}{dt}\left(ml^2\dot{\theta} + ml\dot{X}\cos\theta\right) = -mgl\sin\theta - ml\dot{X}\dot{\theta}$

(b) Using ①, since $\dot{X}$ and $\dot{\theta}$ are zero initially (we release m from rest)

$$\dot{X} = \frac{-ml\dot{\theta}\cos\theta}{m+M}$$

Now, eliminate $\dot{X}$ in ②, and we get:

$$\frac{d}{dt}\left(\dot{\theta} - \frac{m}{m+M}\dot{\theta}\cos^2\theta\right) = -\frac{g}{l}\sin\theta + \frac{m}{m+M}\dot{\theta}^2\sin\theta\cos$$

For small oscillation, $\sin\theta \sim \theta$, $\cos\theta \sim 1$ and the second term on right is ≈ 0.

Thus, $\frac{d\dot{\theta}}{dt} = -\frac{g}{l}\frac{m+M}{M}\theta$.

and $\omega = \sqrt{\frac{g}{l}\frac{m+M}{M}}$

which approaches $(g/l)^{1/2}$ as M increases (as it should, since it becomes a simple pendulum).

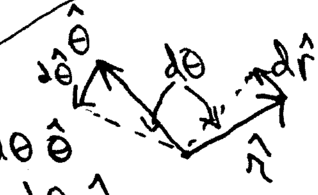
(A2)

2 pts

③ a Show that for a particle moving in a plane, the components of acceleration are given by:

$$a_r = \ddot{r} - r\dot{\theta}^2$$

and $$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$



soln a

$$d\hat{r} = d\theta\,\hat{\theta}$$
$$d\hat{\theta} = -d\theta\,\hat{r}$$

Since, $\vec{r} = r\hat{r}$,

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

ⓑ Using this, show that for $\vec{F} = \vec{F}(r)$, $r^2\dot{\theta} = \text{constant} = h$.

2 pts
soln b

$$\frac{F_\theta}{m} = 0 = 2\dot{r}\dot{\theta} + r\ddot{\theta}.$$

But $\frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta}$

This is $\frac{F_\theta}{m} r \to 0$.

Thus $r^2\dot{\theta} = \text{constant} = h$, if $F_\theta = 0$

6 pts

ⓒ If $f(r) = -cr^n$ show that circular orbits are stable against small perturbations if $n > -3$.

soln c

$$f(r) = m\ddot{r} - mr\dot{\theta}^2 = m\ddot{r} - \frac{mh^2}{r^3}$$

For a circular orbit, $r = a$, and $f(a) = -\frac{mh^2}{a^3}$

A2 cont

Let $x = r - a$.

Then, $m\ddot{x} - mh^2 (x+a)^{-3} = f(x+a)$

Expand in a power series:

$m\ddot{x} - mh^2 a^{-3} \left(1 - 3\frac{x}{a} + \cdots\right) = f(a) + f'(a)$

$\therefore \quad m\ddot{x} + \left[-\frac{3}{a} f(a) - f'(a)\right] x = 0$

If [ ] is positive, we have SHM.

Thus, $\frac{3}{a} f(a) + f'(a) < 0$
for stability!

If $f(r) = -cr^n$,

then $-\frac{3}{a} c a^n - c n a^{n-1} < 0$

or $3c + cn > 0$,

so $n > -3$

A3

RAB 1: Jan 05

a) $\vec{F} = m\ddot{\vec{x}} \Rightarrow m\ddot{\vec{x}} = -\frac{e}{c}(\dot{\vec{x}} \times \vec{B}) - e\vec{E}$ ①

so with $\vec{B} = B_0\hat{z}$ we have $(\dot{\vec{x}} \times \vec{B}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & 0 \\ 0 & 0 & B_z \end{vmatrix}$

Let $\vec{x} = x\hat{x} + y\hat{y}$

$= B_z \dot{y}\hat{x} - B_z \dot{x}\hat{y}$

$\Rightarrow$

$$m\ddot{x} + \frac{eB_z}{c}\dot{y} + eE_{0x} = 0$$

$$m\ddot{y} - \frac{eB_z}{c}\dot{x} + eE_{0y} = 0$$

(b) Let $\vec{x} = x^{\pm}(\hat{x} \pm i\hat{y}) \Rightarrow \dot{\vec{x}} = \dot{x}^{\pm}(\hat{x} \pm i\hat{y})$

$\ddot{\vec{x}} = \ddot{x}^{\pm}(\hat{x} \pm i\hat{y})$

$$\Rightarrow (\dot{\vec{x}}^{\pm} \times \vec{B}) = \dot{x}^{\pm}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & \pm i & 0 \\ 0 & 0 & B_z \end{vmatrix} = \dot{x}^{\pm}B_z(\pm i\hat{x} - \hat{y})$$

$$= \pm i\dot{x}^{\pm}B_z(\hat{x} \pm i\hat{y})$$

so, with $\vec{E}^{\pm} = E_0(\hat{x} \pm i\hat{y})e^{-i\omega t}$ (i.e. $e^{ikz} \sim 1$)

$$m\ddot{x}^{\pm}(\hat{x} \pm i\hat{y}) + \frac{eB_z}{c}\dot{x}(\hat{x} \pm i\hat{y}) + eE_0(\hat{x} \pm i\hat{y})e^{-i\omega t} = 0$$

To solve, let $x^{\pm} = x_0^{\pm}e^{-i\omega t} \Rightarrow \dot{x}^{\pm} = -i\omega x^{\pm}$, $\ddot{x}^{\pm} = -\omega^2$

$$\Rightarrow -\omega^2 m x_0^{\pm} \pm i\frac{eB}{c}(-i\omega)x_0^{\pm} + eE_0 = 0 \Rightarrow -m\omega^2 x_0^{\pm} \pm \frac{eB}{c}x_0 = -$$

$$x_0^{\pm} = \frac{\frac{e}{m}E_0}{\omega(\omega \mp \frac{eB_z}{\cdots})}$$

(C) P = electric dipole moment / volume

$\Rightarrow P^{\pm} = -n e x^{\pm}$ where n = electron density, $-e$ = electron charge

$$\Rightarrow P^{\pm} = \frac{-\frac{ne^2}{m} E_0}{\omega\left(\omega \mp \frac{eB_z}{mc}\right)}$$

Now, $\vec{D} = E + 4\pi P$, we have $P \propto E \Rightarrow P = \chi E$

So ($D = E + 4\pi \chi E = (1 + 4\pi\chi) E = \epsilon E$

$$\Rightarrow \epsilon^{\pm} = 1 - \frac{4\pi n e^2/m}{\omega\left(\omega \mp \frac{eB_z}{mc}\right)}$$

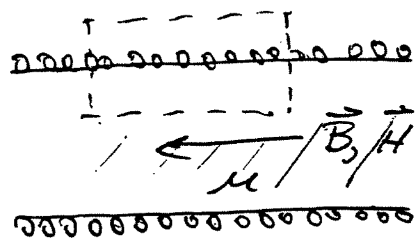
- For low frequencies $\omega^+ > 0$, $\epsilon^- < 0 \Rightarrow E^+$ propagates in medium while E^- decays + is evanescent $\Rightarrow E^-$ compone of incident wave is perfectly reflected.
- For high frequencies, $\epsilon^+ > 0$, $\epsilon^- > 0 \Rightarrow$ both modes propagate.

[redacted]

A4

RAB2: Jan 05

a) Using Ampere's Law with a loop as shown in the diagram:



$$\int \vec{H}\cdot d\vec{l} = I \Rightarrow$$

$Hl = l n I_0$ where $n = \#\text{turns}/\text{length}$.

So $\boxed{H = n I_0}$

In medium, $B = \mu H \Rightarrow \boxed{B = \mu n I_0}$

b) To find fields in gap, use continuity relations:

B_{norm} continuous, H_{tan} continuous $\}$ at interface

Since $B \perp$ surface $\Rightarrow B_{gap} = B_{in} = \boxed{\mu n I_0 = B_{gap}}$

moreover, in vacuum $B = \mu_0 H$

$$\Rightarrow \boxed{H_{gap} = \frac{B_{gap}}{\mu_0} = \frac{\mu}{\mu_0} n I_0}$$

c) Energy density $= \frac{1}{2}\vec{B}\cdot\vec{H}$

$\Rightarrow$ in gap $u_g = \frac{1}{2}(\mu n I_0)\cdot(\frac{\mu}{\mu_0} n I_0) = \frac{\mu}{\mu_0}(\frac{1}{2}\mu n^2 I_0^2)$

in material $u_{in} = \frac{1}{2}(\mu n I_0)(n I_0) = \frac{1}{2}\mu n^2 I_0^2$

$\Rightarrow (u_g > u$

d) Since $U_{gap} > U_{in}$, requires work done by external agent to increase size of gap.

$$\Delta E = (U_{gap} - U_{material}) \cdot A \cdot \Delta x$$

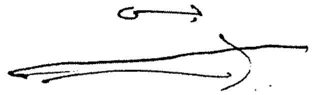
$$= \left(\frac{\mu}{\mu_0} - 1\right)\left(\tfrac{1}{2}\mu n^2 I_0^2\right)\pi R^2 \Delta x$$

$$\text{Force} = \frac{\Delta E}{\Delta x} = \left(\frac{\mu}{\mu_0} - 1\right)\left(\tfrac{1}{2}\mu n^2 I_0^2\right)\pi R^2$$

Tensile force applied by external agent.

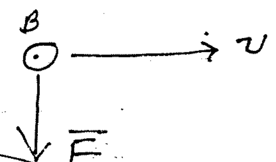
(note, independent of w if fringe fields neglected)

(A5)



(a) $B = \frac{\mu_0 I}{2\pi r}$ out of ampere's law
pak.

$$\vec{F} = q\vec{E} + q\,\vec{v}\times\vec{B}$$



$$\boxed{\vec{F} = \frac{\mu_0 I}{2\pi a} q v}$$ downward

(b) In this fram $\vec{v}' = 0$

so $\vec{F} = q\vec{E}'$

must be an electric field,

so a charge

$(\rho c, \vec{j})$ is a 4-vecto

so is $A(\rho c, \vec{j}) = [\lambda c, I]$

λ = charge/area

In the original frame $\lambda = 0$

a usual Lorentz transf

$$\lambda' c = -\gamma\beta I$$

$$\lambda' = -\gamma \frac{v}{c^2} I$$

$c^2 = \frac{1}{\mu\epsilon}$

$$E' = \frac{1}{2\pi\epsilon_0}\frac{\lambda'}{R} = \frac{\gamma v}{2\pi\epsilon_0 c^2}\frac{I}{R}$$

$$= \frac{\gamma \mu_0 v I/R}{2\pi} =$$

$$\boxed{\vec{F}' = \gamma \vec{F}}$$

This could also be obtained by field trans. (Jackson) $E'_\perp = \gamma \vec{v} \vec{B}$ or force transformation: see French pg 4.

A5 cont

(c) If the wire is neutral by carries a current in the original frame it will carry a negative charge in the rest frame of q. This produces a electric field pointing toward the wire & thus an inward force on q. $\vec{F}' = \gamma \vec{F}$ so non-rel. the force on q is the same.

(By field transformation the pure magnetic field in the original frame become a mix of magnetic field and electric field in the 2nd frame).