Solution for Problem C4

a) (2 points) For free standing waves there can be no driving, which implies that the longitudinal component of the displacement D must be zero. Since $D \propto \epsilon E = 0$, and $E \neq 0$, we must have $\epsilon = 0$, implying that $\omega = \omega_{\mathrm{p}}$, with no dispersion with q in this model.

b) (2 points) For transverse waves, we have the usual $\omega^2 = c^2q^2/\epsilon$,so that $\omega^2 = \omega_{\mathrm{p}}^2 + c^2q^2$.

c) (2 points) Here ϵ is negative. This means that $qc = \pm\omega\sqrt{\epsilon} = \pm i\omega\sqrt{-\epsilon}$, giving a decay as stated with

$$d = \frac{c}{\omega\sqrt{-\epsilon}} = \frac{c}{\left(\omega_{\mathrm{p}}^2 - \omega^2\right)^{1/2}}.$$

d) (2 points) Note that ϵ is real in this region. This means that the polarization, given by $4\pi k_e P = (\epsilon - 1)E$, is in phase with E. The induced current, given by $j = (d/dt)P$, will therefore be $\pi/2$ out of phase with E. Plugging the numbers gives

$$j_x = -\frac{\omega_{\mathrm{p}}^2}{4\pi k_e \omega} E_x^0 \exp(-z/d) \sin\omega t.$$

This is just the free electron response, since $\omega_{\mathrm{p}}^2 = 4\pi k_e n e^2/m$, where n is the appropriate electron number density. The power dissipation will be a spatial integral of $j \cdot E$, averaged over a cycle. Because of the phase difference, this average vanishes. What is going on, of course, is that the incident wave is being reflected, in accordance with our everyday experience with visible light impinging on metals. The energy of the incident wave is not being transferred to the metal, but simply is being transferred to the reflected wave.

e) (2 points) In this limit $\epsilon = 1 + i\omega_{\mathrm{p}}^2\tau/\omega \approx i\omega_{\mathrm{p}}^2\tau/\omega$, so that $qc = \pm\omega\sqrt{\epsilon}$ gives

$$q = \pm(1+i)\frac{\omega_{\mathrm{p}}}{c}\sqrt{\omega\tau/2}.$$

Thus we have a damped wave with $\delta^{-1} = |\mathrm{Im}\, q|$. Since $\omega\tau \ll 1$, δ is much larger than the d of part c), where $d \sim c/\omega_{\mathrm{p}}$. Since $\epsilon - 1$ is pure imaginary in this limit, the current is in phase with the electric field, and the wave is damped by transferring its energy to the material.