Solution to Problem B4

a) (3 points)

$$H = \frac{L_z^2}{2I} = -\frac{\hbar^2}{md^2}\frac{\partial^2}{\partial\phi^2},$$

where

$$I \equiv 2m\left(\frac{d}{2}\right)^2, \quad L_z \equiv \frac{\hbar}{i}\frac{\partial}{\partial\phi}.$$

b) (3 points) The initial wave function can be written

$$\psi(\phi, 0) = \frac{A}{4}(e^{2i\phi} + e^{-2i\phi} + 2),$$

which is a superposition of eigenstates of $L_z = \pm 2\hbar, 0$ respectively, with the relative probability amplitudes $1:1:2$, and, therefore, the relative probabilities $1:1:4$, so

$$P(L_z = 2\hbar) = P(L_z = -2\hbar) = \frac{1}{6}, \text{ and } P(L_z = 0) = \frac{2}{3},$$

where $P(L_z = x)$ stands for "probability". Then

$$\langle L_z^2\rangle = \sum_i P(i)i^2 = \frac{1}{3} \times 4\hbar^2 + \frac{2}{3} \times 0 = \frac{4}{3}\hbar^2.$$

Similarly,

$$\langle E\rangle = \frac{\langle L_z^2\rangle}{2I} = \frac{4\hbar^2}{3md^2}.$$

c) (2 points) Energy eigenstates ψ_i of energy E_i have the time d ependence $\psi_i(\phi, t) = e^{-i\omega_i t}\psi_i(\phi, 0)$, where $\omega_i \equiv E/\hbar$. So

$$\psi(\phi, t) = \frac{A}{4}\left[e^{-i\omega_1 t}(e^{2i\phi} + e^{-2i\phi}) + 2\right],$$

which clearly has the angular frequency

$$\omega_1 = \frac{4\hbar}{md^2}.$$

d) (1 point) If we suddenly decrease the separation of the two particles by a factor of two, we do nothing to the angular momentum (either classically or quantum mechanically), while the moment of inertia, I increases by a factor of 4, so the answers to the questions of part b) remain unchanged, except for the expectation value of the energy which increases by a factor of 4.

e) (1 point) The angular frequency of the $L_z = \pm 2\hbar$ *energy* eigenstates increases by the same factor, so $\omega_2/\omega_1 = 4$.