Solution for Problem B3

a) (4 points) Standard time-independent perturbation theory gives, through second order,

$$\epsilon_i^{(2)} = \epsilon_i + \langle i|H_1|i\rangle + \sum_{i\neq j} \frac{\langle i|H_1|j\rangle\langle j|H_1|i\rangle}{\epsilon_i - \epsilon_j}.$$

This leads immediately to the expressions

$$\begin{aligned}\epsilon_A^{(2)} &= \epsilon_A + \frac{M^2}{\epsilon_A - \epsilon_B},\\ \epsilon_B^{(2)} &= \epsilon_B + \frac{M^2}{\epsilon_B - \epsilon_A} + \frac{M^2}{\epsilon_B - \epsilon_C},\\ \epsilon_C^{(2)} &= \epsilon_C + \frac{M^2}{\epsilon_C - \epsilon_B}.\end{aligned}$$

b) (3 points) Standard time-independent perturbation theory also gives

$$|i^{(1)}\rangle = |i\rangle + \sum_{i\neq j} \frac{|j\rangle\langle i|H_1|j\rangle}{\epsilon_i - \epsilon_j}.$$

This leads immediately to the expressions

$$\begin{aligned}|A^{(1)}\rangle &= |A\rangle + \frac{|B\rangle M}{\epsilon_A - \epsilon_B},\\ |B^{(1)}\rangle &= |B\rangle + \frac{|A\rangle M}{\epsilon_B - \epsilon_A} + \frac{|C\rangle M}{\epsilon_B - \epsilon_C},\\ |C^{(1)}\rangle &= |C\rangle + \frac{|B\rangle M}{\epsilon_C - \epsilon_B}.\end{aligned}$$

c) (3 points) We now have a problem in degenerate state perturbation theory. Represent the three states by the column vectors

$$|A\rangle \equiv \begin{pmatrix}1\\0\\0\end{pmatrix},\quad |B\rangle \equiv \begin{pmatrix}0\\1\\0\end{pmatrix},\quad |C\rangle \equiv \begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{3}$$

The first order energies and eigenstates can then be determined by solving the system

$$\begin{pmatrix}-\epsilon^{(1)} & M & 0\\ M & -\epsilon^{(1)} & M\\ 0 & M & -\epsilon^{(1)}\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = 0,$$