The differential path length is $ds = dx\sqrt{1+(y_x)^2+(z_x)^2}$, and the differential time interval is $dT = dsn(x, y, z)/c$. In this formulation,

$$T = \frac{1}{c}\int_{x_A}^{x_B} dxn(x, y, z)\sqrt{1+(y_x)^2+(z_x)^2}.$$

The formulation above fails if the path proves to be re-entrant in x, in which case y and/or z would not be a unique function of x. A symmetric formulation that avoids this problem expresses all three coordinates as functions of a parameter q, which varies from 0 to 1, say, as the light propagates from A to B. In this case,

$$T = \frac{1}{c}\int_{0}^{1} dqn(x, y, z)\sqrt{(x_q)^2+(y_q)^2+(z_q)^2}.$$

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c) (3 points) Fermat's principle requires T to be stationary under variations in the ray path. The integrals in part b) are of the generic form

$$T = \int_{w_1}^{w_2} dwI(x, x_w, y, y_w, z, z_w).$$

Under a variation $x(w) \to x(w) + \delta x(w)$,

$$\delta T = \int_{w_1}^{w_2} dw\left[\delta x\frac{\partial I}{\partial x} + \delta x_w\frac{\partial I}{\partial x_w}\right] = \int_{w_1}^{w_2} dw\delta x\left[\frac{\partial I}{\partial x} - \frac{d}{dw}\frac{\partial I}{\partial x_w}\right],$$

after integration by parts. Requiring this to vanish for arbitrary $\delta x(w)$ leads to Euler's partial differential equations, the fundamental equations of the calculus of variations:

$$\frac{\partial I}{\partial x} - \frac{d}{dw}\frac{\partial I}{\partial x_w} = 0,$$

with analogous equations for the other coordinates.

For the case of parametrization by x in part b), this leads to two differential equations:

$$\sqrt{1+(y_x)^2+(z_x)^2}\frac{\partial n}{\partial y} - \frac{d}{dx}\left[n\frac{y_x}{\sqrt{1+(y_x)^2+(z_x)^2}}\right] = 0,$$

and a similar equation for the z component. For the case of parametrization by q, we get three differential equations:

$$\sqrt{(x_q)^2+(y_q)^2+(z_q)^2}\frac{\partial n}{\partial x} - \frac{d}{dq}\left[n\frac{x_q}{\sqrt{(x_q)^2+(y_q)^2+(z_q)^2}}\right] = 0,$$

and similar equations for the y and z components. Note that the unit vector $\mathbf{t}$ tangent to the trajectory has components $(x_q, y_q, z_q)/\sqrt{(x_q)^2 + (y_q)^2 + (z_q)^2}$. The three equations can be written concisely as

$$\sqrt{(x_q)^2 + (y_q)^2 + (z_q)^2}\nabla n = \frac{d}{dq}[n\mathbf{t}].$$

Finally, since $ds = dq\sqrt{(x_q)^2 + (y_q)^2 + (z_q)^2}$, if we use path length as the parameter, the chain rules gives the most elegant formulation of the three PDE for the ray trajectory:

$$\nabla n = \frac{d}{ds}[n\mathbf{t}].$$

[However, parametrization by path length introduces a constraint, $1 = (dx/ds)^2 + (dy/ds)^2 + (dz/ds)^2$, which is not present for other parametrizations, and which may limit the usefulness of the "elegant" formulation.]