Solution for Problem A4

a) (2 points) The velocity of the pendulum mass perpendicular to the string is $\ell\dot{\theta}$. The velocity along the string is $\dot{\ell}$. Thus

$$T = \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}m\dot{\ell}^2.$$

b) (2 points) The exact potential energy is $mg\ell(1-\cos\theta)$, which is chosen to be zero when the mass is undiscplaced. The small angle approximation is $\cos\theta = 1-\theta^2/2$. Thus

$$V = \frac{1}{2}mg\ell\theta^2.$$

c) (2 points) The Lagrangian is $L = T - V$. We find $\partial L/\partial\dot{\theta} = m\ell^2\dot{\theta}$ and $\partial L/\partial\theta = -mgl\theta$. Thus the equation of motion is

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = \frac{d}{dt}[m\ell^2\dot{\theta}] + mg\ell\theta = m\ell^2\ddot{\theta} + mg\ell\theta + 2m\ell\dot{\ell}\dot{\theta}.$$

The last term drops out when $\dot{\ell} = 0$. The remaining terms give the equation of motion of a pendulum with fixed string length. The frequency of oscillation is $\omega = \sqrt{g/\ell}$.

d) (4 points)

$$\frac{d}{dt}(\omega^p E) = \left(\frac{d\omega^p}{dt}\right)E + \omega^p\left(\frac{dE}{dt}\right),$$

where $E = T + V$.

$$\frac{d\omega^p}{dt} = -\frac{p\dot{\ell}}{2\ell}\omega^p; \quad \frac{dE}{dt} = \dot{\theta}\left[m\ell^2\ddot{\theta} + mg\ell\theta\right] + \frac{\dot{\ell}}{\ell}\left[m\ell^2\dot{\theta}^2 + \frac{1}{2}mg\ell\theta^2 + m\ell\ddot{\ell}\right].$$

The second derivative can be reorganized to read

$$\frac{dE}{dt} = \dot{\theta}\left[m\ell^2\ddot{\theta} + mg\ell\theta + 2m\ell\dot{\ell}\dot{\theta}\right] + \frac{\dot{\ell}}{\ell}\left[-m\ell^2\dot{\theta}^2 + \frac{1}{2}mg\ell\theta^2 + m\ell\ddot{\ell}\right].$$

The first brace vanishes by the equation of motion, and the second terms can be re-expressed as

$$\frac{dE}{dt} = \frac{\dot{\ell}}{\ell}\left[-2T + V + m\ell\ddot{\ell}\right].$$