Solution for Problem A1

a) (5 points) The vector potential may be found from the integral (see, for instance, Marion and Heald, Classical Electromagnetic Radiation, Chapter 8):

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int_V dV' \int dt' \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\delta\left[t' + (|\vec{x}-\vec{x}'| - ct)/c\right]. \tag{1}$$

The current density may be written with a complex time dependence (taking the real part at the end of the calculation):

$$\vec{J}(\vec{x},t) = I_o\hat{z}\cos\left(\frac{2\pi z}{\lambda}\right)\delta(x)\delta(y)e^{i\omega t} \tag{2}$$

Substituting (2) into (1) and integrating over x' and y', we obtain

$$\vec{A}(\vec{x},t) \approx \frac{I_o\hat{z}}{cr}\int_{-\lambda/4}^{\lambda/4} dz' \cos\left(\frac{2\pi z'}{\lambda}\right) e^{i\omega(t-(1/c)\sqrt{x^2+y^2+(z-z')^2})}, \tag{3}$$

where we have used the assumption that we are in the radiation zone, so that

$$|\vec{x}-\vec{x}'| \approx r.$$

Expanding the square root in (3) to order z'/r, we find

$$\vec{A}(\vec{x},t) \approx \frac{I_o\hat{z}}{cr}e^{i(\omega t-kr)}\int_{-\lambda/4}^{\lambda/4} dz' \cos\left(\frac{2\pi z'}{\lambda}\right) e^{i(2\pi zz')/(\lambda r)}. \tag{4}$$

Letting z = r cos θ and performing the integral in (4) (write cos as sum of exponentials), we get

$$\vec{A}(\vec{x},t) \approx \frac{I_o\hat{z}}{cr}\frac{\lambda}{2\pi}\left[\frac{\sin\left((\pi/2)(1+\cos\theta)\right)}{1+\cos\theta} + \frac{\sin((\pi/2)(1-\cos\theta))}{1-\cos\theta}\right]e^{i(\omega t-kr)}. \tag{5}$$

b) (3 points) The electric field $\vec{E}$ in the radiation zone may be found directly from

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -i\frac{\omega}{c}\vec{A}$$

using (5). The magnetic induction $\vec{B}$ in the radiation zone is given by

$$\begin{aligned}\vec{B} &= -\frac{1}{c}\vec{n}\times\frac{\partial\vec{A}}{\partial t}\\ &= -ik(\vec{n}\times\vec{A}).\end{aligned}$$