# d=4 $\mathcal{N}$=2 Field Theory and Physical Mathematics

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Rutgers University

Yale, Jan. 23, 2017

# Phys-i-cal Math-e-ma-tics, n.

**Pronunciation:** Brit. /ˈfɪzɨkl ˌmaθ(ə)ˈmatɪks / , U.S. /ˈfɪzək(ə)l ˌmæθ(ə)ˈmædɪks/

**Frequency (in current use):**

**1.** Physical mathematics is a fusion of mathematical and physical ideas, motivated by the dual, but equally central, goals of elucidating the laws of nature at their most fundamental level, together with discovering deep mathematical truths.

2014 G. Moore *Physical Mathematics and the Future,* http://www.physics.rutgers.edu/~gmoore

…….

1573 *Life Virgil* in T. Phaer & T. Twyne tr. Virgil *Whole .xii. Bks. Æneidos* sig. Aiv$^{v}$, Amonge other studies ….. he cheefly applied himself to Physick and Mathematickes.

# Snapshots from the Great Debate over the relation between Mathematics and Physics

Kepler

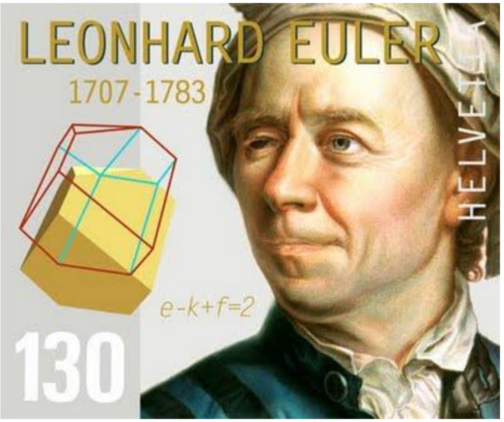
Galileo

Newton

Leibniz

# When did Natural Philosophers become either Physicists or Mathematicians?

Even around the turn of the 19th century …






But 60 years later … we read in volume 2 of Nature ….

# 1869: Sylvester’s Challenge

## A pure mathematician speaks:

of physical philosophy; the one here in print,” says Professor Sylvester, “is an attempted faint adumbration of the nature of mathematical science in the abstract. What is wanting (like a fourth sphere resting on three others in contact) to build up the ideal pyramid is a discourse on the relation of the two branches (mathematics and physics) to, and their action and reaction upon, one another—a magnificent theme, with which it is to be hoped that some future president of Section A will crown the edifice, and make the tetralogy (symbolisable by A + A′, A, A′, AA′) complete.”

# 1870: Maxwell's Answer

An undoubted physicist responds,

SECTIONAL PROCEEDINGS

SECTION A.—*Mathematical and Physical Science.*—President, Prof. J. Clerk Maxwell, F.R.S.

The president delivered the following address :—

Maxwell recommends his somewhat-neglected dynamical theory of the electromagnetic field to the mathematical community:

phenomena must be studied in order to be appreciated. Another theory of electricity which I prefer denies action at a distance and attributes electric action to tensions and pressures in an all-pervading medium, these stresses being the same in kind with those familiar to engineers, and the medium being identical with that in which light is supposed to be propagated."

# 1900: The Second ICM

Hilbert announced his famous 23 problems for the 20th century, on August 8, 1900

**Mathematische Probleme.**

Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900.

Von

**D. Hilbert.**

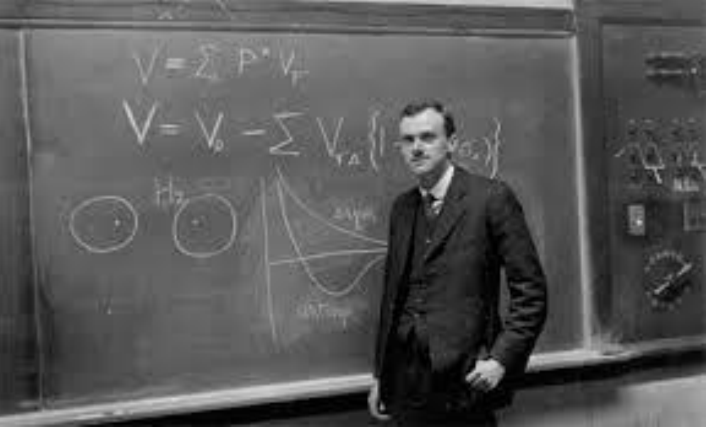
Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development ….

# 1900: Hilbert's $6^{th}$ Problem

To treat [...] by means of axioms, those physical sciences in which mathematics plays an important part [...]

October 7, 1900: Planck's formula, leading to h.

Prerequisite: 750:502 Quantum Mechanics, or equivalent. Lorentz group; relativistic wave-equations; second quantization; global and local symmetries; QED and gauge invariance; spontaneous symmetry breaking; nonabelian gauge theories; Standard Model; Feynman diagrams; cross sections, decay rates; renormalization group.



# 1931: Dirac's Paper on Monopoles

## Quantised Singularities in the Electromagnetic Field

P.A.M. Dirac

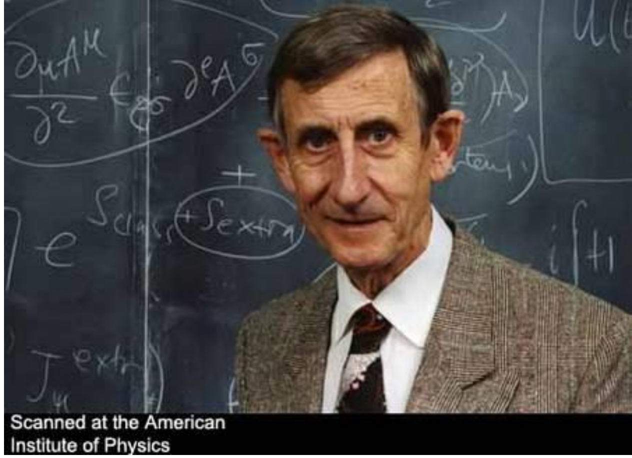
Received May 29, 1931

### § *1. Introduction*

The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers

for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a

# 1972: Dyson's Announcement




## MISSED OPPORTUNITIES[1]

BY FREEMAN J. DYSON

> It is important for him who wants to discover not to confine himself to one chapter of science, but to keep in touch with various others.
> JACQUES HADAMARD

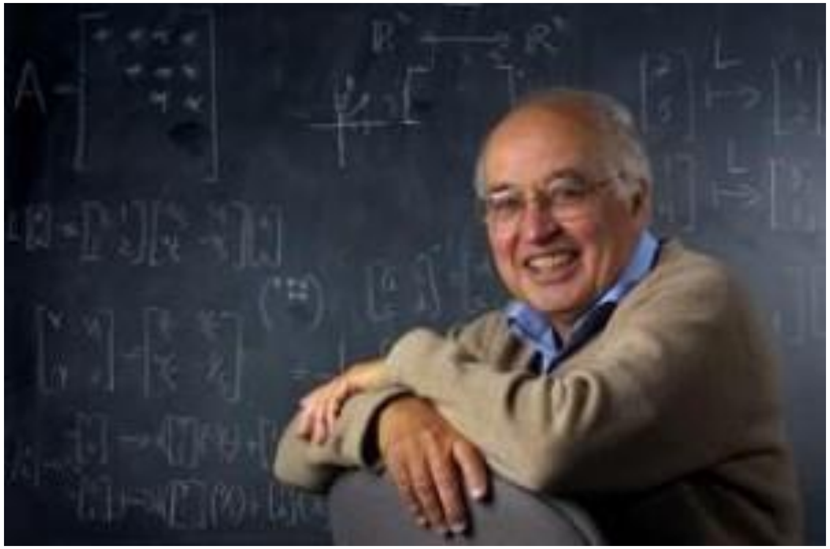
1. **Introduction.** The purpose of the Gibbs lectures is officially defined as "to enable the public and the academic community to become aware of the contribution that mathematics is making to present-day thinking and to modern civilization." This puts me in a difficult position. I happen to be a physicist who started life as a mathematician. As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce. Discussing this divorce, the

Well, I am happy to report that Mathematics and Physics have remarried!

But, the relationship has altered somewhat…

A sea change began in the 1970’s …..

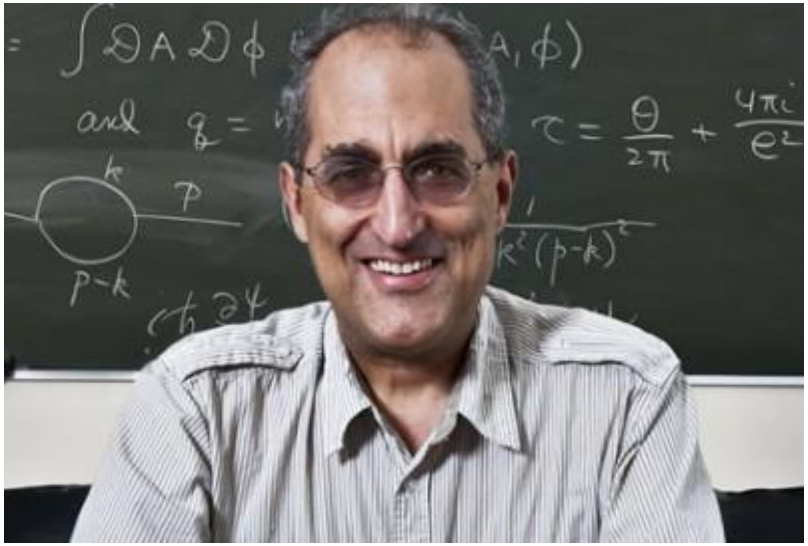
A number of great mathematicians got interested in the physics of gauge theory and string theory, among them,



Sir Michael Atiyah

$$+ \cdots$$

And at the same time a number of great physicists started producing results requiring ever increasing mathematical sophistication, among them



Edward Witten

$$+\cdots$$

# Physical Mathematics

With a great boost from string theory, after 40 years of intellectual ferment a new field has emerged with its own distinctive character, its own aims and values, its own standards of proof.

One of the guiding principles is certainly Hilbert's 6$^{th}$ Problem (generously interpreted): Discover the ultimate foundations of physics.

As predicted by Dirac, this quest has led to ever more sophisticated mathematics…

But getting there is more than half the fun: If a physical insight leads to an important new result in mathematics – that is considered a great success.

It is a success just as profound and notable as an experimental confirmation of a theoretical prediction.

1 What can d=4,$\mathcal{N}$=2 do for you?

# 2 Review: d=4, $\mathcal{N}$=2 field theory

3 Wall Crossing 101

4 Conclusion

# Two Types Of Physical Problems

Type 1: Given a QFT find the spectrum of the Hamiltonian, and compute forces, scattering amplitudes, expectation values of operators ….

**Algebraic & Quantum**

Type 2: Find solutions of Einstein's equations, and solve Yang-Mills equations on those Einstein manifolds.

**Geometrical & Classical**

# Exact Analytic Results

They are important

Where would we be without the harmonic oscillator?

Onsager's solution of the 2d Ising model
in zero magnetic field (Yale, 1944)

$\Rightarrow$ Modern theory of phase transitions and RG.

QFT's with ``extended supersymmetry'' in spacetime dimensions $\leq 6$ have led to many results answering questions of both type 1 & 2.

QFT's with ``extended supersymmetry'' in spacetime dimensions $\leq 6$ have led to many results answering questions of both types 1 & 2.

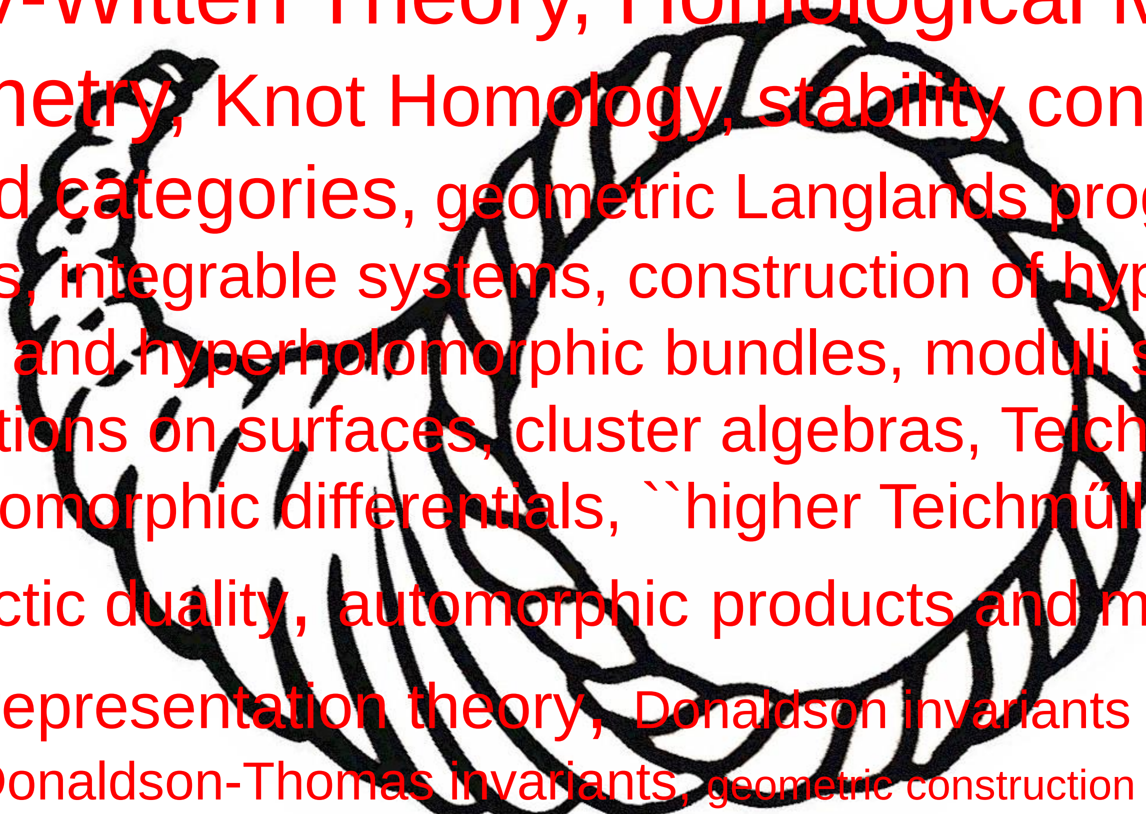
Surprise: There can be very close relations between questions of types 1 & 2

We found ways of computing the exact (BPS) spectrum of many quantum Hamiltonians via solving Einstein and Yang-Mills-type equations.

Another surprise: In deriving exact results about d=4 QFT it turns out that interacting QFT in **SIX** spacetime dimensions plays a crucial role!

# Cornucopia For Mathematicians

## Provides a rich and deep mathematical structure.



Gromov-Witten Theory, Homological Mirror Symmetry, Knot Homology, stability conditions on derived categories, geometric Langlands program, Hitchin systems, integrable systems, construction of hyperkähler metrics and hyperholomorphic bundles, moduli spaces of flat connections on surfaces, cluster algebras, Teichműller theory and holomorphic differentials, ``higher Teichműller theory,'' symplectic duality, automorphic products and modular forms, quiver representation theory, Donaldson invariants & four-manifolds, motivic Donaldson-Thomas invariants, geometric construction of affine Lie algebras, McKay correspondence, ……….

# The Importance Of BPS States

Much progress has been driven by trying to understand a portion of the spectrum of the Hamiltonian – the ``BPS spectrum'' –

***BPS states*** are special quantum states in a supersymmetric theory for which we can compute the energy exactly.

So today we will just focus on the BPS spectrum in d=4, $\mathcal{N}$=2 field theory.

# Added Motivation For BPS-ology

Counting BPS states is also crucial to the string-theoretic explanation of Beckenstein-Hawking black hole entropy in terms of microstates. (Another story, for another time.)

**Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon***

M. K. Prasad and Charles M. Sommerfield

*Sloane Physics Laboratory, Yale University, New Haven, Connecticut 06520*

(Received 16 June 1975)

Solving for the BPS spectrum is a glorious thing for God, for Country, and for Yale.

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1 What can d=4,$\mathcal{N}$=2 do for you?

# 2 Review: d=4, $\mathcal{N}$=2 field theory

2A *Definition, Representations, Hamiltonians*

2B *The Vacuum And Spontaneous Symmetry Breaking*

2C *BPS States: Monopoles & Dyons*

2D *Seiberg-Witten Theory*

2E *Unfinished Business*

# Definition Of d=4, $\mathcal{N}$=2 Field Theory

This is a special kind of four-dimensional quantum field theory with <u>*supersymmetry*</u>

**<u>Definition:</u>** A d=4, $\mathcal{N} = 2$ theory is a four-dimensional QFT such that the Hilbert space of states is a representation of

*The d=4, $\mathcal{N}$=2 super-Poincare algebra !*

OK…..

….. So what is the

d=4, $\mathcal{N}$=2 super-Poincare algebra??

# d=4, $\mathcal{N}$=2 Poincaré Superalgebra

*(For mathematicians)*

Super Lie algebra $\mathfrak{s} = \mathfrak{s}^0 \oplus \mathfrak{s}^1$

$$\mathfrak{s}^0 = \mathrm{poin}(1,3) \oplus \mathfrak{u}(2)_R \oplus \mathbb{R}^2_{\mathrm{central}}$$

$$\mathbb{R}^2_{\mathrm{central}} \cong \mathbb{C}$$

Generator Z = ``$\mathcal{N}$=2 central charge''

$$\mathfrak{s}^1 = [(2;2)_{+1} \oplus (2^*;2)_{-1}]_{\mathbb{R}}$$

$$\mathrm{Sym}^2\mathfrak{s}^1 \to \mathrm{transl} \oplus \mathbb{R}^2_{\mathrm{central}} \subset \mathfrak{s}^0$$

# d=4, $\mathcal{N}$=2 Poincaré Superalgebra

(For physicists)

$\mathcal{N}$=1 Supersymmetry:

There is an operator Q on the Hilbert space $\mathcal{H}$

$$\{Q, Q^\dagger\} = 2H$$

$\mathcal{N}$=2 Supersymmetry:

There are *two* operators $Q_1$, $Q_2$ on the Hilbert space

$$\{Q_i, Q_j^\dagger\} = 2\delta_{i,j} H$$

$$\{Q_1, Q_2\} = 2Z$$

# The Power Of $\mathcal{N} = 2$ Supersymmetry

## Representation theory:

*Field and particle multiplets*

## Hamiltonians:

*Typically depend on very few parameters for a given field content.*

## BPS Spectrum:

*Special subspace in the Hilbert space of states*

# Important Example Of An $\mathcal{N} = 2$ Theory

## $\mathcal{N} = 2$ supersymmetric version of Yang-Mills Theory

### Recall plain vanilla Yang-Mills Theory:

Recall Maxwell's theory of a vector-potential = gauge field: $A_\mu$

In Maxwell's theory electric & magnetic fields are encoded in $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$

Yang-Mills theory also describes physics of a vector-potential = gauge field: $A_\mu$

But now $A_\mu$ are MATRICES and the electric and magnetic fields are encoded in

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

# $\mathcal{N}$=2 Super-Yang-Mills For U(K)

Gauge fields:

Doublet of gluinos:

Complex scalars (Higgs fields):

$$\begin{tikzcd} & A_\mu \arrow[dl, "Q_1"'] \arrow[dr, "Q_2"] & \\ \psi^1_\alpha \arrow[dr, "Q_2"'] & & \psi^2_\alpha \arrow[dl, "Q_1"] \\ & \varphi & \end{tikzcd}$$

All are K x K matrices

Gauge transformations: $\varphi \to g^{-1} \varphi g$

# Hamiltonian Of $\mathcal{N}$=2 U(K) SYM

The Hamiltonian is completely determined, up to a choice of Yang-Mills coupling $e_0^2$

$$H = e_0^{-2} \int_{\mathbb{R}^3} \mathrm{Tr}\left(\vec{E}^2 + \vec{B}^2 + |\vec{D}\varphi|^2\right) + e_0^{-2} \int_{\mathbb{R}^3} \mathrm{Tr}\left([\varphi, \varphi^\dagger]^2\right)$$

Energy is a sum of squares.

Energy bounded below by zero.

1 What can d=4,$\mathcal{N}$=2 do for you?

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2E *Unfinished Business*

# Classical Vacua

Classical Vacua: Zero energy field configurations.

$$H = e_0^{-2} \int_{\mathbb{R}^3} \mathrm{Tr} \left( \vec{E}^2 + \vec{B}^2 + |\vec{D}\varphi|^2 \right)$$

$$+ e_0^{-2} \int_{\mathbb{R}^3} \mathrm{Tr} \left( [\varphi, \varphi^\dagger]^2 \right)$$

$$\vec{E} = \vec{B} = 0 \qquad \varphi = cnst.$$

$$[\varphi, \varphi^\dagger] = 0 \quad \Rightarrow$$

$$\varphi = \mathrm{Diag}\{a_1, \dots, a_K\}$$

***<u>Any</u>*** choice of $a_1, \dots a_K$ gives a vacuum!

# Quantum Moduli Space of Vacua

The continuous vacuum degeneracy is an exact property of the quantum theory:

$$\langle \mathrm{Vac}|\varphi|\mathrm{Vac}\rangle = \mathrm{Diag}\{a_1, \ldots, a_K\}$$

The quantum vacuum is not unique!

Manifold of quantum vacua $\mathcal{B}$

Parametrized by the complex numbers $a_1, \ldots, a_K$

# Gauge Invariant Vacuum Parameters

$$u_s := \langle \mathrm{Vac}(u) | \mathrm{Tr}(\varphi^s) | \mathrm{Vac}(u) \rangle$$

$$\mathcal{B} := \{ u := (u_1, \dots, u_K) \}$$

Physical properties depend on the choice of vacuum u in $\mathcal{B}$.

We will illustrate this by studying the properties of ``dyonic particles'' as a function of u.

# Spontaneous Symmetry Breaking

$$\langle \mathrm{Vac}(u)|\varphi|\mathrm{Vac}(u)\rangle = \mathrm{Diag}\{a_1, \dots, a_K\}$$

broken to:

$$U(K) \Longrightarrow U(1)^K$$

(For mathematicians)

$\varphi$ is in the adjoint of $U(K)$ : Stabilizer of a generic $\varphi \in u(K)$ is a Cartan torus

# Physics At Low Energy Scales: LEET

Only one kind of light comes out of the flashlights from the hardware store....

Most physics experiments are described very accurately by using (quantum) Maxwell theory (QED). The gauge group is U(1).

The true gauge group of electroweak forces is SU(2) x U(1)

The Higgs vev sets a scale: $\langle \varphi \rangle = 246 \mathrm{GeV}$

The subgroup preserving $\langle \varphi \rangle$ is U(1) of E&M.

At energies << 246 GeV we can describe physics using Maxwell's equations + small corrections:

# $\mathcal{N}$=2 Low Energy U(1)$^K$ Gauge Theory

Low energy effective theory (LEET) is described by an $\mathcal{N}$=2 extension of Maxwell's theory with gauge group U(1)$^K$

K different ``electric'' and K different ``magnetic'' fields:

$$\vec{E}^I \quad \vec{B}^I \qquad I = 1, \ldots, K$$

& their $\mathcal{N}$=2 superpartners

1 What can d=4, $\mathcal{N}$=2 do for you?

2 
# Review: d=4, $\mathcal{N}$=2 field theory

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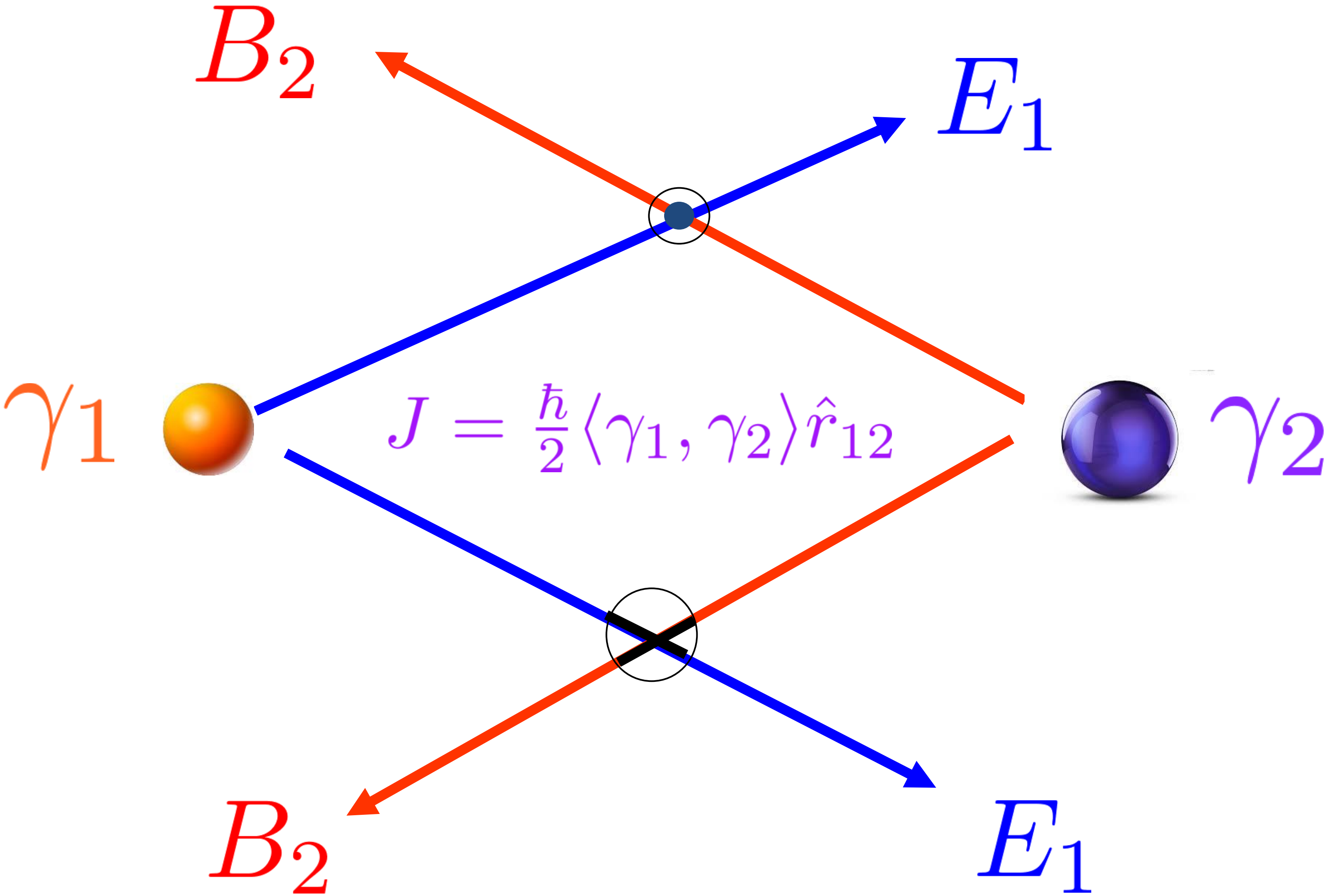
# Electro-magnetic Charges

The theory will also contain ``dyonic particles'' – particles with electric and magnetic charges for the fields $\vec{E}^I \quad \vec{B}^I \quad I = 1, \dots, K$

(Magnetic, Electric) Charges:

$$\gamma = (p^I, q_I)$$

Dirac quantization: *On general principles, the vectors* $\gamma$ *are in a* ***symplectic*** *lattice* $\Gamma$.




$$\langle\gamma_1, \gamma_2\rangle = p_1^I q_{2,I} - p_2^I q_{1,I} \in \mathbb{Z}$$

# BPS States: The Definition

Charge sectors: $\mathcal{H} = \oplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$

In the sector $\mathcal{H}_\gamma$ the ***operator***

Z is just a *c-number* $Z_\gamma \in \mathbb{C}$

Bogomolny bound: In sector $\mathcal{H}_\gamma$

$$E \geq |Z_\gamma|$$

$$\boxed{\mathcal{H}_\gamma^{\mathrm{BPS}} := \{\psi : E\psi = |Z_\gamma| \psi\}}$$

# The Central Charge Function

The $\mathcal{N} = 2$ ``central charge'' $Z_\gamma$ depends on $\gamma$:

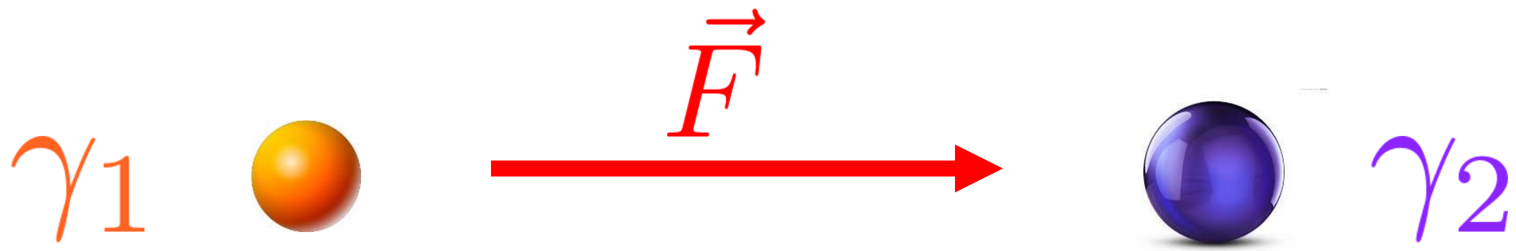
$$Z_{\gamma_1+\gamma_2} = Z_{\gamma_1} + Z_{\gamma_2}$$

This linear function is *<u>also</u>* a function of u $\in \mathcal{B}$:

On $\mathcal{H}_\gamma^{\mathrm{BPS}}$ $\quad E = |Z_\gamma(u)|$

So the mass of BPS particles depends on u $\in \mathcal{B}$.

# Coulomb Force Between Dyons




$\overrightarrow{F(u)}$ is a nontrivial function of u $\in \mathcal{B}$

***<u>It can be computed from $Z_\gamma(u)$</u>***

Computing $Z_\gamma(u)$ *allows us to determine the entire* LEET!

1 What can d=4,$\mathcal{N}$=2 do for you?

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So far, everything I’ve said follows easily from general principles

# General d=4, $\mathcal{N}$=2 Theories

1. A moduli space $\mathcal{B}$ of quantum vacua.

2. Low energy dynamics described by an effective $\mathcal{N}$=2 abelian gauge theory.

3. The Hilbert space is graded by a lattice of electric + magnetic charges, $\gamma \in \Gamma$ .

4. There is a BPS subsector with masses given exactly by $|Z_{\gamma}(u)|$.

But how do we compute

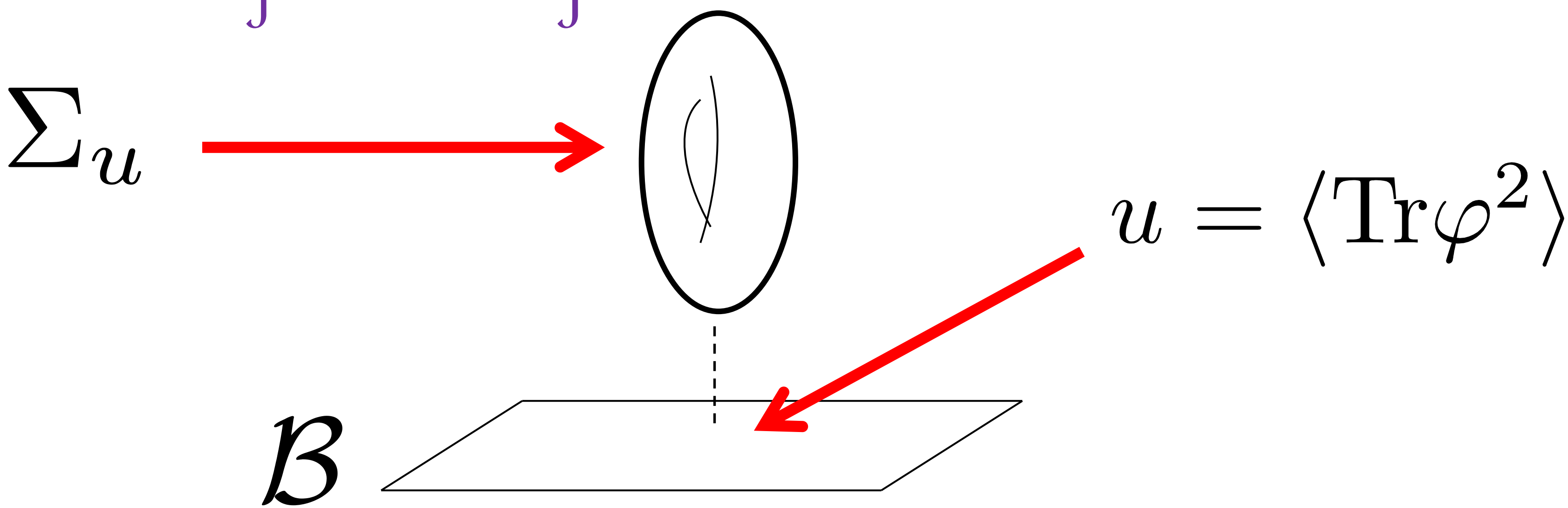
$$Z_\gamma(u)$$

as a function of $\gamma$ and $u$ ?

# Seiberg-Witten Paper

Seiberg & Witten (1994) found a way for the case of SU(2) SYM.

$Z_\gamma(u)$ can be computed in terms of the ***periods*** of a meromorphic differential form $\lambda$ on a Riemann surface $\Sigma$ both of which depend on $u$.

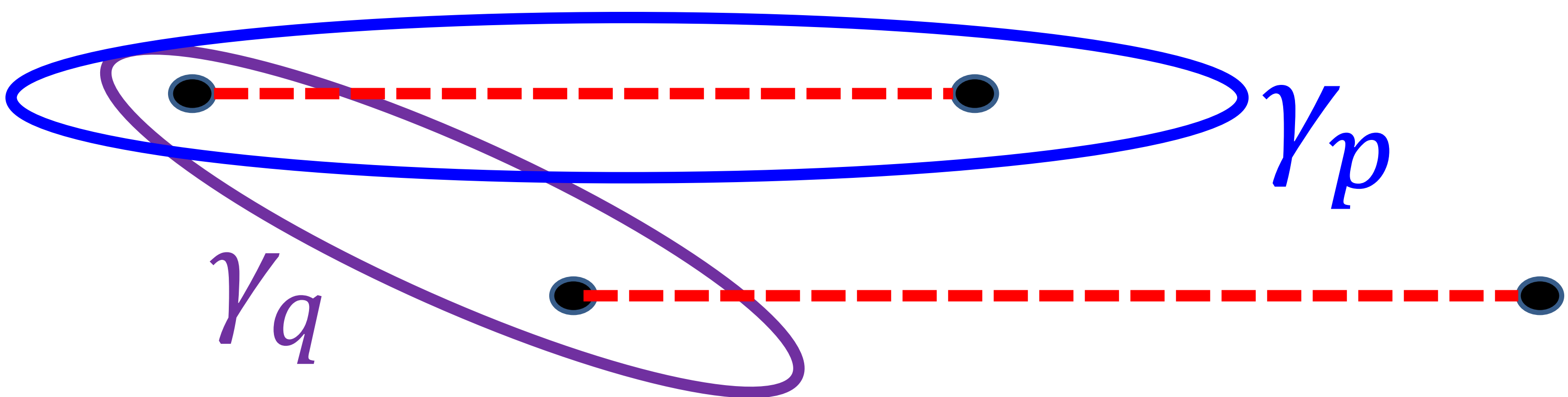


In more concrete terms: there is an integral formula like:

$$Z_\gamma(u) = \oint_\gamma \sqrt{\frac{1}{z^3} + \frac{2u}{z^2} + \frac{1}{z}}\,dz$$

$\gamma$ is a closed curve…

Up to continuous deformation there are only two basic curves and their deformation classes generate a lattice!



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# The Promise of Seiberg-Witten Theory: 1/2

Seiberg & Witten found the exact LEET for the *particular* case: G=SU(2) SYM.

They also gave cogent arguments for the exact BPS spectrum of this *particular* theory.

Their breakthrough raised the hope that for *general* d=4 $\mathcal{N}$=2 theories we could find many analogous exact results.

# The Promise of Seiberg-Witten Theory: 2/2

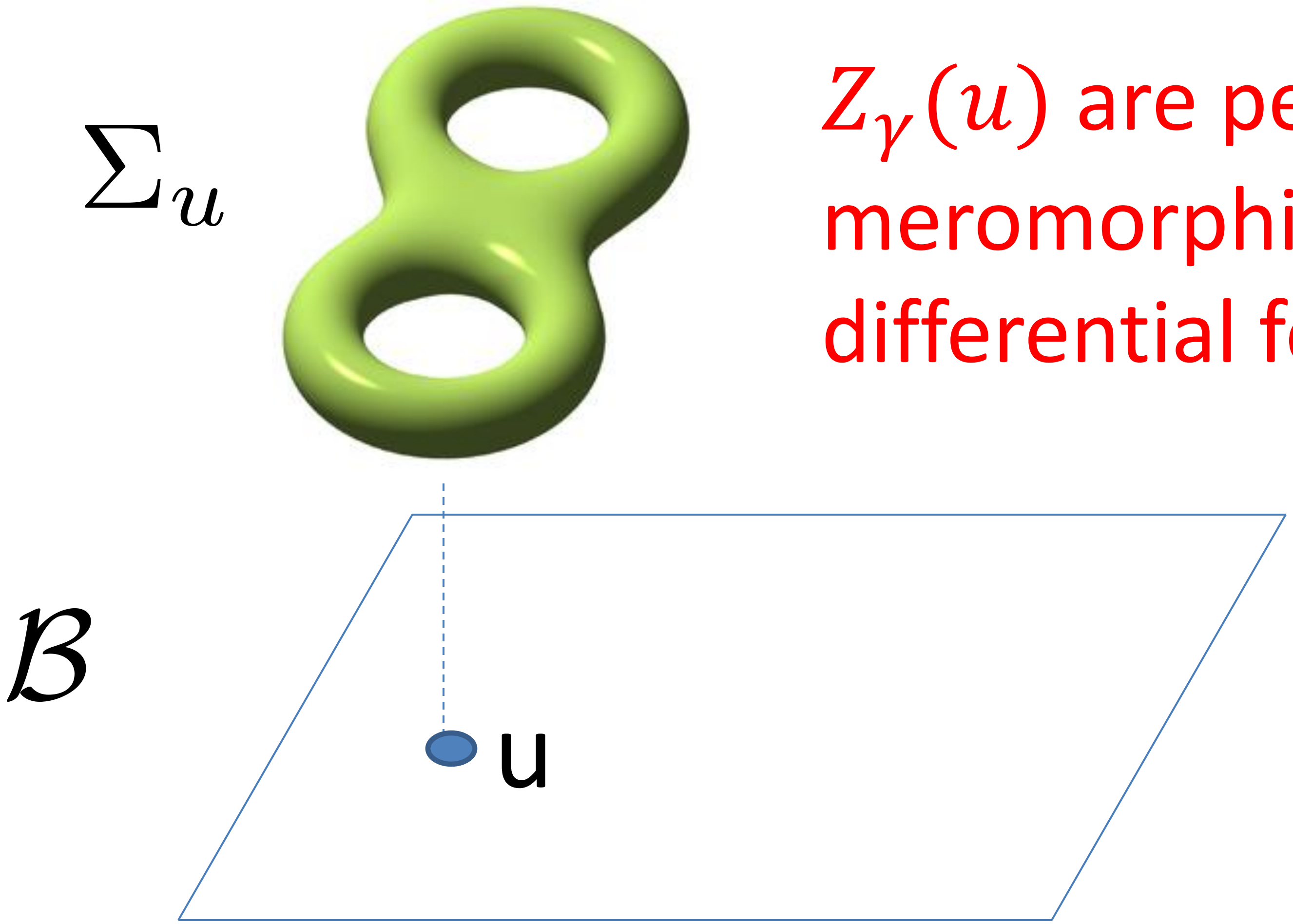
U.B. 1: Compute $Z_\gamma(u)$ for other theories.

U.B. 2: Find the space of BPS states for other theories.

U.B. 3: Find exact results for path integrals – including insertions of ``defects'' such as ``line operators,'' ``surface operators'', …..

U.B. 1: The LEET: Compute $Z_\gamma(u)$.

Extensive subsequent work quickly showed that the SW picture indeed generalizes to all known d=4 , $\mathcal{N}$=2 field theories:




$Z_\gamma(u)$ are periods of a meromorphic differential form on $\Sigma_u$

But, to this day, there is no general algorithm for computing $\Sigma_u$ for a given d=4, $\mathcal{N}$=2 field theory.

# But what about U.B. 2: Find the BPS spectrum?

In the 1990's the BPS spectrum was *only* determined in a handful of cases…

( SU(2) with ($\mathcal{N}$=2 supersymmetric) quarks flavors: $N_f$ = 1,2,3,4, for special masses: Bilal & Ferrari)

Knowing the value of $Z_\gamma(u)$ in the sector $\mathcal{H}_\gamma$ does not tell us whether there are, or are not, BPS particles of charge $\gamma$. It does not tell us if $\mathcal{H}_\gamma^{\mathrm{BPS}}$ is zero or not.

In the past 10 years there has been a great deal of progress in understanding the BPS spectra in a large class of other $\mathcal{N}=2$ theories.

One key step in this progress has been a much-improved understanding of the ``*wall-crossing phenomenon*.''

1 What can d=4,$\mathcal{N}$=2 do for you?

2 Review: d=4, $\mathcal{N}$=2 field theory

3 Wall Crossing 101

4 Conclusion

Recall we want to compute the space of BPS states :

$$\mathcal{H}_{\gamma}^{\mathrm{BPS}} = \{\psi : E\psi = |Z_{\gamma}(u)|\psi\}$$

It is finite dimensional.

So let's compute the dimension.

A tiny change of couplings can raise the energy above the BPS bound:

The dimension can depend on u !


# Atiyah & Singer To The Rescue

Family of vector spaces dim $\mathcal{H}_{\mathrm{u}}$ jumps with $u$

But there is an operator $\mathcal{F}^2 = 1$

$$I(u) = Tr_{\mathcal{H}_u} \mathcal{F} = \dim(\mathcal{H}_u^+) - \dim(\mathcal{H}_u^-)$$

Much better behaved!

Much more computable!

Example: Index of elliptic operators.

# BPS Index

For $\mathcal{H}_u^{BPS}$ take $\mathcal{F} = (-1)^F$ (Witten index)

$$\Omega(\gamma) := \mathrm{Tr}_{\mathfrak{h}_\gamma^{\mathrm{BPS}}} (-1)^{2J_3}$$

$J_3$ is any generator of so(3)

Formal arguments prove: $\Omega(\gamma)$ is ***invariant*** under change of parameters such as the choice of u …

# Index Of An Operator: 1/4

*(For physicists)*

Suppose $T_u$ is a family of linear operators depending on parameters $u \in \mathcal{B}$

$$T_u : V \to W$$

If V and W are ***finite-dimensional*** Hilbert spaces then:

$$\dim(\ker T_u) - \dim(\ker T_u^\dagger) \; = \dim V - \dim W$$

***independent of the parameter u!***

# Index Of An Operator: 2/4

Example: Suppose V=W is one-dimensional.

$$T_u(\psi) = u\psi \qquad \psi \in V \qquad u \in \mathbb{C}$$

$$u \neq 0 \qquad \dim(\ker T_u) = \dim(\ker T_u^\dagger) = 0$$

$$u = 0 \qquad \dim(\ker T_u) = \dim(\ker T_u^\dagger) = 1$$

So if we take $\dim V = 3 \; and \; \dim W = 2$ and consider the index of

$$T_u = \begin{pmatrix} u & u & u^2 \\ \sin(u) & \sin(u) & \sin(u) \end{pmatrix} \qquad \mathrm{Ind}(T_u) = 3 - 2 = 1$$

# Index Of An Operator: 3/4

Now suppose $T_u$ is a family of linear operators between two *infinite-dimensional* Hilbert spaces

$$\dim(\ker T_u) - \dim(\ker T_u^{\dagger}) = \dim \mathcal{H}_1 - \dim \mathcal{H}_2$$

$$= \infty - \infty$$

Still the LHS makes sense for suitable (Fredholm) operators and is ***invariant*** under *continuous* changes of (Fredholm) operators.

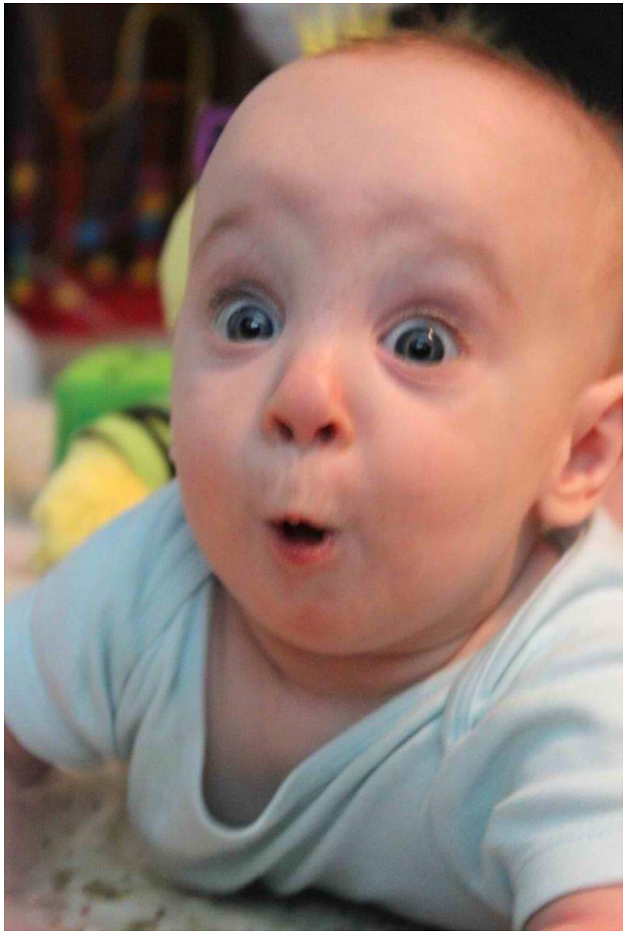
# Index Of An Operator: 4/4

The BPS index $\Omega(\gamma)$ *is* the index of the supersymmetry operator Q on Hilbert space.

(In the weak-coupling limit it is literally the index of a Dirac operator on a moduli space of magnetic monopoles.)

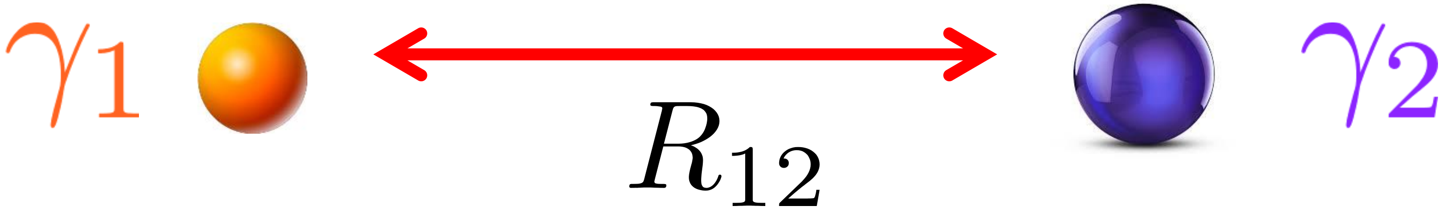
# The Wall-Crossing Phenomenon

But even the ***<u>index</u>*** can depend on u !!



How can that be ?

BPS particles can form bound states which are themselves BPS!



# Denef's Boundstate Radius Formula

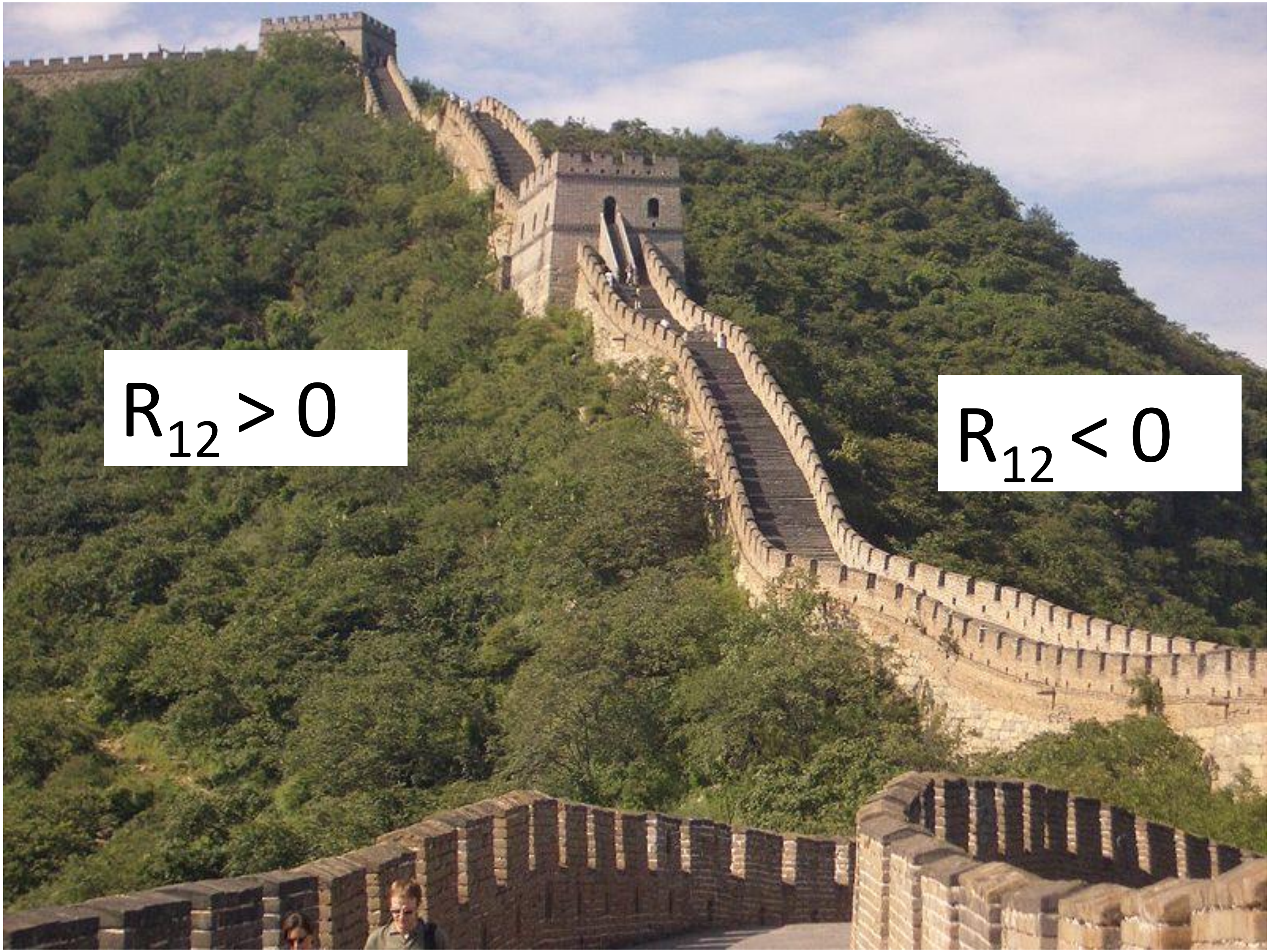
$$R_{12}(u) = \langle \gamma_1, \gamma_2 \rangle \frac{|Z_{\gamma_1}(u) + Z_{\gamma_2}(u)|}{2\mathrm{Im}\left(Z_{\gamma_1}(u) Z_{\gamma_2}(u)^*\right)}$$

The $Z_\gamma$'s are functions of the moduli $u \in \mathcal{B}$

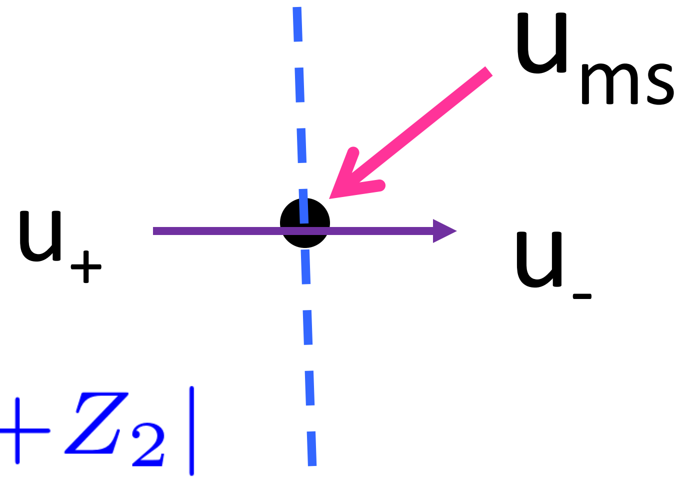
So the moduli space of vacua $\mathcal{B}$

is divided into two regions:

$\mathrm{Im}(Z_1 Z_2^*) > 0$ ***OR*** $\mathrm{Im}(Z_1 Z_2^*) < 0$


$R_{12} > 0$
$R_{12} < 0$

# Wall of Marginal Stability




Consider a path of vacua crossing the wall:

$$R_{12} = \langle \gamma_1, \gamma_2 \rangle \frac{|Z_1 + Z_2|}{2\mathrm{Im}(Z_1 Z_2^*)}$$

Crossing the wall: $\mathrm{Im}(Z_1 Z_2^*) \to 0$

# The Primitive Wall-Crossing Formula

(Denef & Moore, 2007)

$$R_{12} = \langle \gamma_1, \gamma_2 \rangle \frac{|Z_1 + Z_2|}{2\mathrm{Im}(Z_1 Z_2^*)}$$

Crossing the wall: $\mathrm{Im}(Z_1 Z_2^*) \to 0$

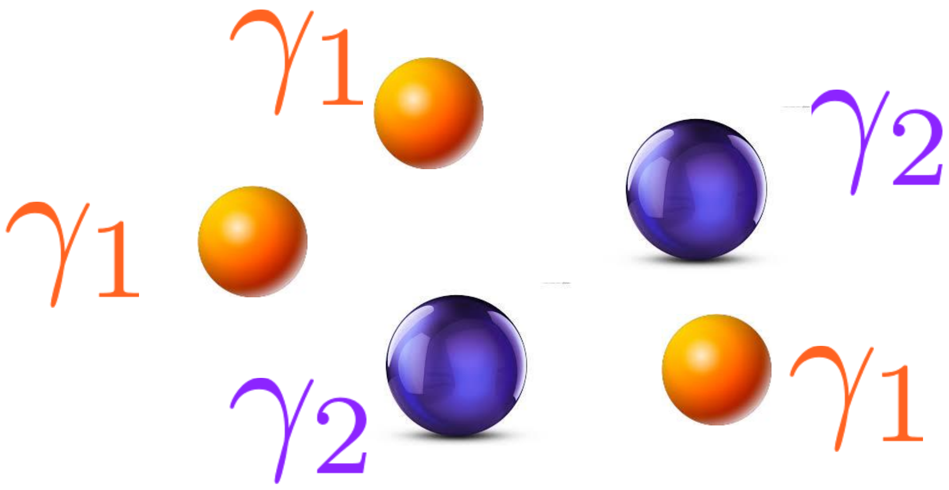

$$\boxed{\Delta\mathcal{H}^{BPS} = \mathcal{H}^{\mathrm{spin}}_{\mathrm{J}_{12}} \otimes \mathcal{H}^{BPS}_{\gamma_1} \otimes \mathcal{H}^{BPS}_{\gamma_2}}$$

$$2J_{12} + 1 = |\langle \gamma_1, \gamma_2 \rangle|$$

# Non-Primitive Bound States

But this is not the full story, since the *same* marginal stability wall holds for charges $N_1\gamma_1$ and $N_2\gamma_2$ for $N_1, N_2 > 0$

The primitive wall-crossing formula assumes the charge vectors $\gamma_1$ and $\gamma_2$ are *primitive vectors*.




?????

# Kontsevich-Soibelman WCF

In 2008 K & S wrote a wall-crossing formula for Donaldson-Thomas invariants of Calabi-Yau manifolds..

But their formula could in principle apply to ``BPS indices'' of general boundstates in more general situations.

We needed a physics argument for why their formula should apply to d=4, $\mathcal{N}$=2 field theories, in particular.

# We gave a physics derivation of the KSWCF

A key step used explicit constructions of hyperkahler metrics on moduli spaces of solutions to Hitchin's equations.

*Hyperkahler metrics* are solutions to Einstein's equations.

*Hitchin's equations* are special cases of Yang-Mills equations.

So Physics Questions of Type 1 and Type 2 become closely related here.

The explicit construction made use of techniques from the theory of integrable systems, in particular, a form of Zamolodchikov's Thermodynamic Bethe Ansatz

The explicit construction of HK metrics also made direct contact with the work of Fock & Goncharov on moduli spaces of flat conections on Riemann surfaces. (``Higher Teichmuller theory'')

# Wall-Crossing: Only half the battle…

The wall crossing formula only describes the CHANGE of the BPS spectrum across a wall of marginal stability.

It does NOT determine the BPS spectrum!

Further use of integrable systems techniques applied to Hitchin moduli spaces led to a solution of this problem for a infinite  class of d=4 $\mathcal{N}$=2 theories known as

``theories of class S''

1 What can d=4,$\mathcal{N}$=2 do for you?

2 Review: d=4, $\mathcal{N}$=2 field theory

3 Wall Crossing 101

4 Conclusion

# Conclusion For Physicists

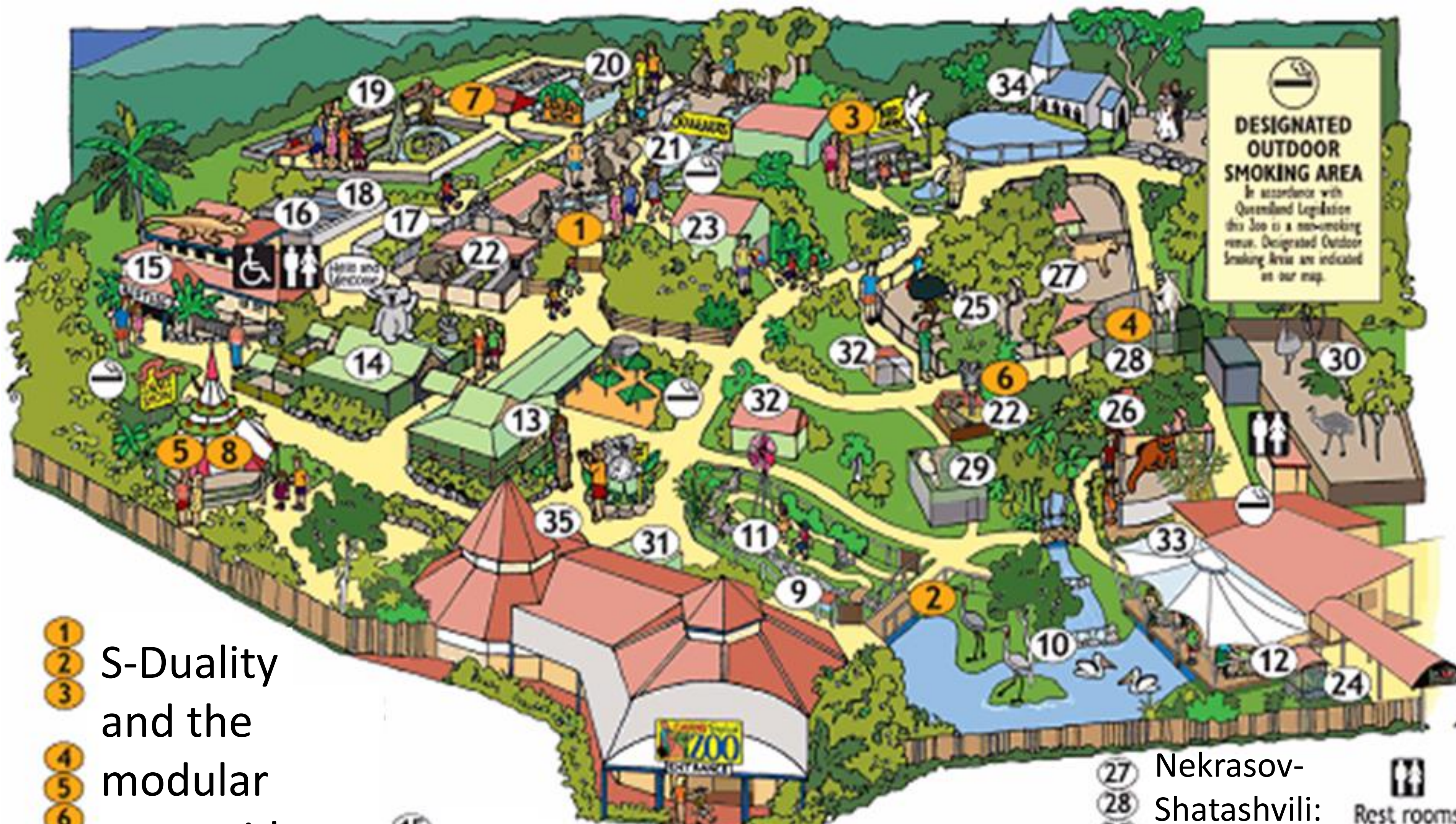
Seiberg and Witten's breakthrough in 1994, opened up many interesting problems. Some were quickly solved, but some remained stubbornly open.

But the past ten years has witnessed a renaissance of the subject, with a much deeper understanding of the BPS spectrum and the line and surface defects in these theories.

# Conclusion For Mathematicians

This progress has involved nontrivial and surprising connections to other aspects of Physical Mathematics:

Hyperkähler geometry, cluster algebras, moduli spaces of flat connections, Hitchin systems, integrable systems, Teichmüller theory, …




1 2 3 4 5 6 7 8 S-Duality and the modular groupoid

9 10 11 12 13 14 AGT: Liouville & Toda theory

15 16 Higgs_branches

17 18 Cluster algebras

19 20 Holographic duals

21 N=4 scattering

22 23 24 ␣-backgrounds, Nekrasov partition functions, Pestun localization.

25 26 $Z(S^3 \times S^1)$ Scfml indx

27 28 29 30 Nekrasov-Shatashvili: Quantum Integrable systems

31 32 33 34 35 Three dimensions, Chern-Simons, and mirror symmetry


NOT
That's all Folks!