Summing Over Bordisms In TQFT: déjà vu

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Work with Anindya Banerjee

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Comments On Summing Over Bordisms In TQFT

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I. Motivation

A longstanding problem in quantum gravity:

Probability amplitudes are computed by ``summing''
(as in a path integral) over metrics on some spacetime Y

$$\exp\{-\frac{G_N}{16\pi^2}\int_Y \mathcal{R}(g)vol(g)+\ldots\}$$

If we sum over metrics, should we
also sum over topologies?

Summing Over Topologies In AdS/CFT:

Hawing-Page transition as
Confinement/Deconfinement in N=4 SYM

Farey Tail Story

Recent work of Jafferis, Rozenberg, & Wong

Recent understanding of the ``Page curve'' and
(no) ``information loss'' via dominance of different
topologies related to BHs.

Puzzles In AdS/CFT

There are hyperbolic Y where ∂Y has multiple connected components.

$\Rightarrow$ Puzzling aspects of the AdS/CFT correspondence - the ``factorization problem'' [Yau & Witten 1999; Maldacena & Maoz 2004]

Saad-Shenker-Stanford [1903.11115] identifies sum of topologies in``JT gravity'' with a matrix model:
Raises conceptual questions about whether string theory should be dual to an ***ensemble*** of QFTs.

Motivated by these issues, and the recent vigorous discussion in the quantum gravity community,
D. Marolf and H. Maxfield [2002.08950] considered a curious ``topological model of 2d gravity.''

My project with Anindya Banerjee was motivated by the desire to understand the MM model in terms of the functorial approach to QFT

II. Reminders On TQFT

Definition of a ``bordism''

Let X_{in}, X_{out} be smooth, compact manifolds of dimension $d-1$.

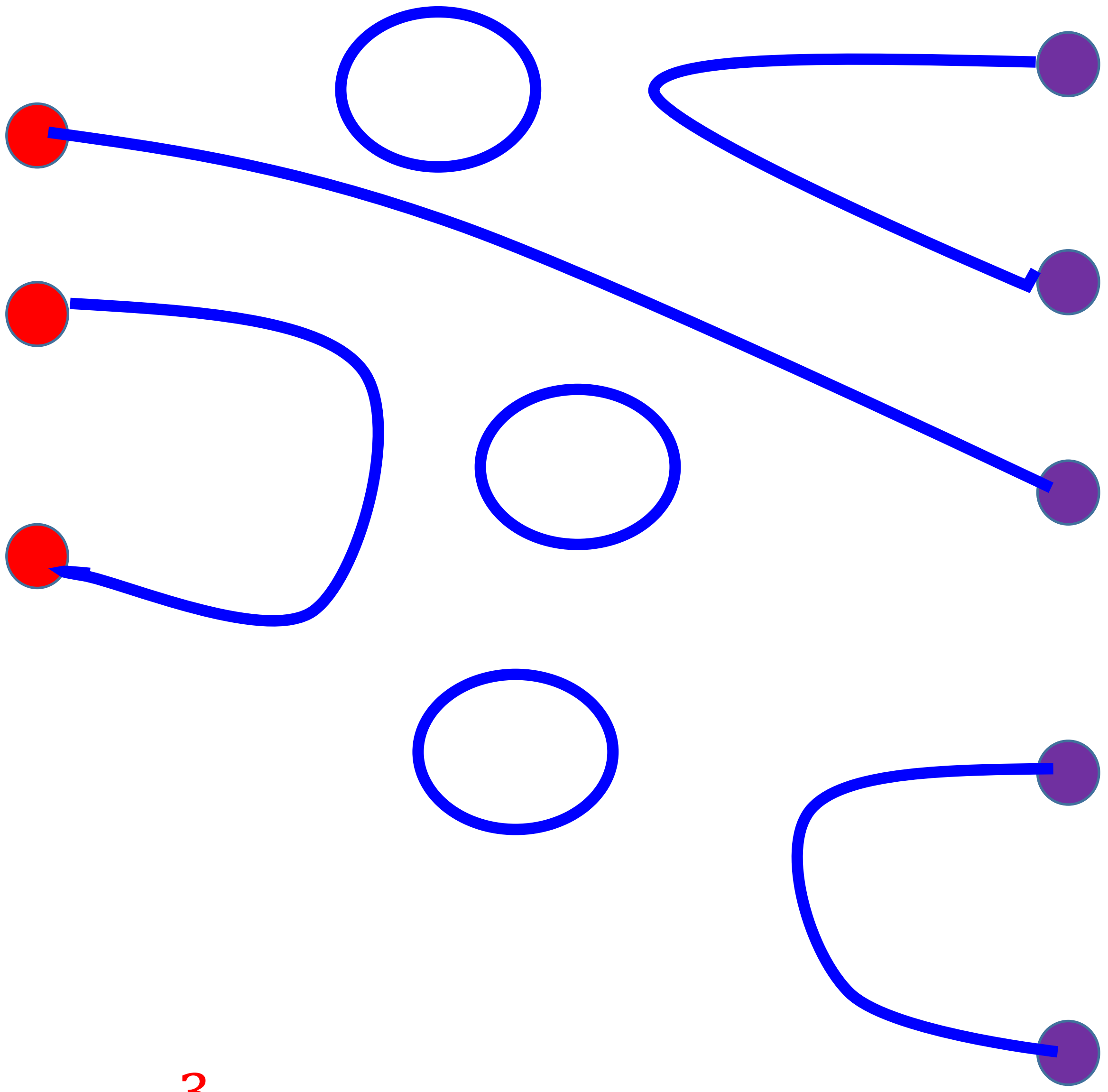
A ***bordism*** $\mathcal{Y}: X_{in} \to X_{out}$ is:

A d-manifold Y together with a disjoint decomposition $\partial Y = (\partial Y)_{in} \coprod (\partial Y)_{out}$

Diffeomorphisms $(\partial Y)_{in} \cong X_{in}$ & $(\partial Y)_{out} \cong X_{out}$

Embeddings $X_{in} \times [0,1) \to Y$ & $X_{out} \times (-1,0] \to Y$

which reduce to the specified diffeos on the boundary of Y



There are 105 such pictures.

+ infinitely many more including disjoint unions with circles....

$$Y: \quad X_{in} = \coprod_{1}^{3} pt \quad \rightarrow \quad X_{out} = \coprod_{1}^{5} pt$$

Bordisms/~ are morphisms in a category $\mathfrak{Bord}_{\langle d,d-1\rangle}$

A TQFT (in this talk) is a monoidal functor $\mathcal{Z}$ to the category $VECT_{\kappa}$ of f.d. vector spaces over a field κ

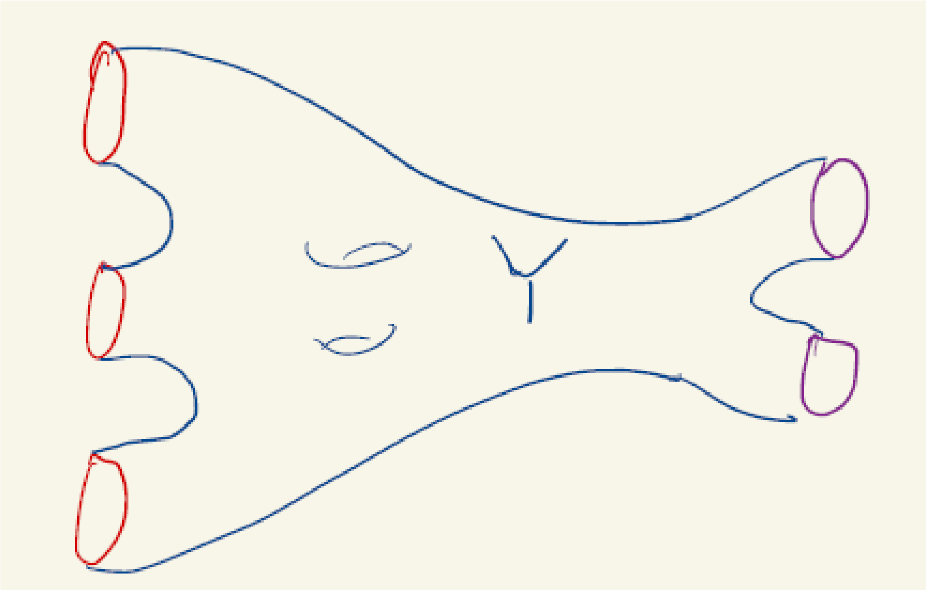
$\mathcal{Z}(X)$: Vector space of ``states'' for spatial manifold X

$$\mathcal{Z}(X_1 \coprod X_2) \cong \mathcal{Z}(X_1) \otimes \mathcal{Z}(X_2)$$

$$Y: X_{in} \to X_{out}$$

$$\mathcal{Z}(Y) \in Hom\big(\mathcal{Z}(X_{in}), \mathcal{Z}(X_{out})\big)$$

$$\mathcal{Z}(Y_1 \circ Y_2) = \mathcal{Z}(Y_1) \circ \mathcal{Z}(Y_2)$$



$$Y: \; S^1 \coprod S^1 \coprod S^1 \to S^1 \coprod S^1$$

$$\mathcal{C} := Z(S^1)$$

$$Z(Y): \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$$

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2 Reminders On TQFT

3 **Summed & Total Amplitudes: Splitting Property**

4 Example: $d = 1$

5 Example: $d = 2$, closed

6 Example: $d = 2$, open-closed

7 Coupling To 2d YM

8 $d \geq 3$: Comments

9 Summary & Some Open Problems

III. Summed & Total Amplitudes: Splitting Property

We can have different bordisms between fixed X_{in} and X_{out}

Given a TQFT $\mathcal{Z}$ (the ``seed TQFT'') define the ``summed amplitude''

$$\mathcal{A}(X_{in}, X_{out}) := \sum_{Y: X_{in} \to X_{out}} w(Y) \mathcal{Z}(Y)$$

$w(Y) = |Aut(Y)|^{-1}$, $Aut(Y)$ = Automorphism group of diffeomorphism type restricting to the *identity* on the boundary.

The Total Amplitude

Consider all summed amplitudes simultaneously as a linear transformation on the tensor algebra:

$$\mathcal{A} \in End\left(T^*\left(\oplus_X \mathcal{Z}(X)\right)\right)$$

$\oplus_X$: Direct sum over all diffeo classes of smooth connected (d-1)-manifolds: countable sum

The summed amplitudes descend to

$$\bar{\mathcal{A}} \in End\left(S^*\left(\oplus_X \mathcal{Z}(X)\right)\right) := End(Fock(\mathcal{Z}))$$

Some Questions:

1. Does $\bar{\mathcal{A}}$ exist?

2. Is it computable?

3. Extension to the fully local TQFT ?

4. Is the weighting $w(Y) = 1/|Aut(Y)|$ well justified ?

5. What properties does it have ?

Some Answers:

1. It exists for d=1,2 and does not exist for $d \geq 3$, at least not in the most naïve sense…

2. Yes, when it exists.

3. Yes!

For d=2, this is the extension to open-closed TQFT.

Two Comments On The Weight Factor: $w(Y)$

1. Kontsevich suggests it is better to use $\chi(BDiff(Y,\partial Y))$.

Very nontrivially, for $\dim Y = 3$, a longstanding conjecture of Kontsevich that $BDiff(Y,\partial Y)$ is homotopy equivalent to a finite CW complex has been proven: Hatcher-McCullough; Nariman; Boyd, Bregman, Steinbrunner

Global structure of euclidean quantum gravity

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Abstract

Euclidean quantum gravity (EQG) separates into a local theory and a global theory. The local theory operates in every compact d-manifold with boundary to produce a state on the boundary. The global theory then sums these boundary states over the diffeomorphism classes of d-manifolds with boundary to make the Hartle-Hawking state. Global EQG is formulated here as classical statistical physics. The Hartle-Hawking state is the probability measure of a mathematically natural classical statistical system, analogous to the functional measure of euclidean quantum field theory. General principles of global EQG determine the numerical weights $w(M)$ in the sum over diffeomorphism classes M.

Derives conditions for $w(Y)$ from physical principles and reproduces $\frac{1}{|Aut(Y)|}$ at least for $d \leq 2$ dimensions.

This talk: Focus on the Splitting Property of $\bar{\mathcal{A}}$

For $\kappa = \mathbb{C}$ we can put an inner product structure on $Fock(\mathcal{Z})$

SP: $\exists$ an inner product space $\mathcal{W}$ such that

$$\Phi: Fock(\mathcal{Z}) \to \mathcal{W}$$

$$\bar{\mathcal{A}} = \Phi\Phi^*$$

Our Convention:

$$Hom(V_1, V_2) \cong \; V_1^\vee \otimes V_2$$

$$T_{12} \in Hom(V_1, V_2) \qquad\qquad T_{23} \in Hom(V_2, V_3)$$

$$T_{12}T_{23} \; \in Hom(V_1, V_3)$$

$$T_{12} \otimes T_{23} \; \in V_1^\vee \otimes V_2 \otimes V_2^\vee \otimes V_3 \; \mapsto \; T_{12}T_{23} \in V_1^\vee \otimes V_3$$

$$\bar{\mathcal{A}} = \Phi\Phi^*$$

1. $\bar{\mathcal{A}}$ need not be positive definite.

2. Even if existence is trivial , explicitly finding $\mathcal{W}$ and Φ in examples seems to be slightly nontrivial.

3. $\mathcal{W}$ is not unique: $\mathcal{W} \to \oplus_\alpha \mathcal{W}_\alpha$

$$\Phi \to \oplus_\alpha \sqrt{p_\alpha}\Phi_\alpha \qquad \sum p_\alpha = 1$$

4. There might be a ``minimal'' $\mathcal{W}_m$

$$
\begin{tikzcd}
 & \mathcal{W} \arrow[dr, "\Phi^*"] & \\
Fock(\mathcal{Z}) \arrow[ur, "\Phi"] \arrow[dr, "\Phi_{\mathcal{W}_m}"'] & & Fock(\mathcal{Z}) \\
 & \mathcal{W}_m \arrow[u, dashed] \arrow[ur, "\Phi^*_{\mathcal{W}_m}"'] &
\end{tikzcd}
$$

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3. Summed & Total Amplitudes: Splitting Property
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5. Example: $d = 2$, closed
6. Example: $d = 2$, open-closed
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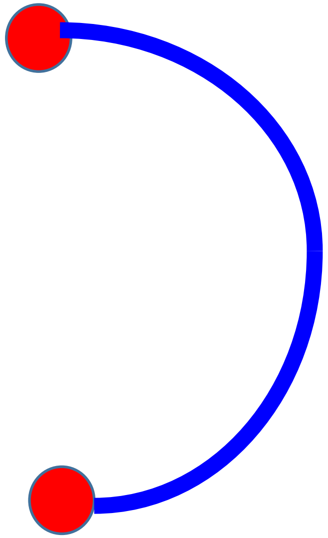
IV. Example: d=1, unoriented

$\mathcal{Z}$ is determined by a f.d. vector space $V = \mathcal{Z}(pt)$ and a symmetric nondegenerate bilinear form $b: V \otimes V \to \kappa$



$$Fock(\mathcal{Z}) = Fock(V) = S^*V = \kappa \oplus V \oplus S^2V \oplus \cdots$$

Start with $X_{in} = X_{out} = \emptyset$ $\qquad$ $\mathcal{Z}(S^1) = \dim_\kappa V$

$$w(\coprod_1^n S^1) = \frac{1}{n!} \quad \Rightarrow$$

$$\mathcal{A}(\emptyset, \emptyset) = \exp \dim_\kappa V$$

Hartle-Hawking Vector & Covector

Def: ***HH vector***: The sum of nothing to anything:

$$\kappa \hookrightarrow Fock(\mathcal{Z}) \overset{\bar{\mathcal{A}}}{\to} Fock(\mathcal{Z}) : 1 \mapsto \Psi_{HH} \in Fock(\mathcal{Z})$$

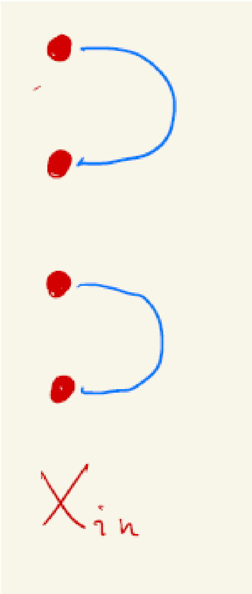
Def: ***HH covector***: The sum of anything to nothing:

$$Fock(\mathcal{Z}) \overset{\bar{\mathcal{A}}}{\to} Fock(\mathcal{Z}) \to \kappa : \quad \Psi_{HH}^{\vee} \in Hom(Fock(\mathcal{Z}), \kappa)$$

Simplest example: Suppose $\dim V = 1$

Take $\kappa = \mathbb{C}$ and choose v with $b(v,v) = 1$

$$\Psi_{HH}^{\vee} = \exp(1) \sum_{n=0}^{\infty} \frac{(2n)!}{n!\,2^n} (v^{\vee})^{2n} \in S^* V^{\vee}$$



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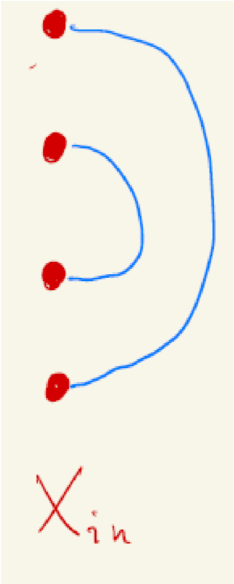

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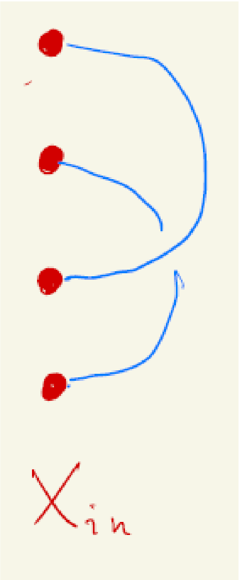
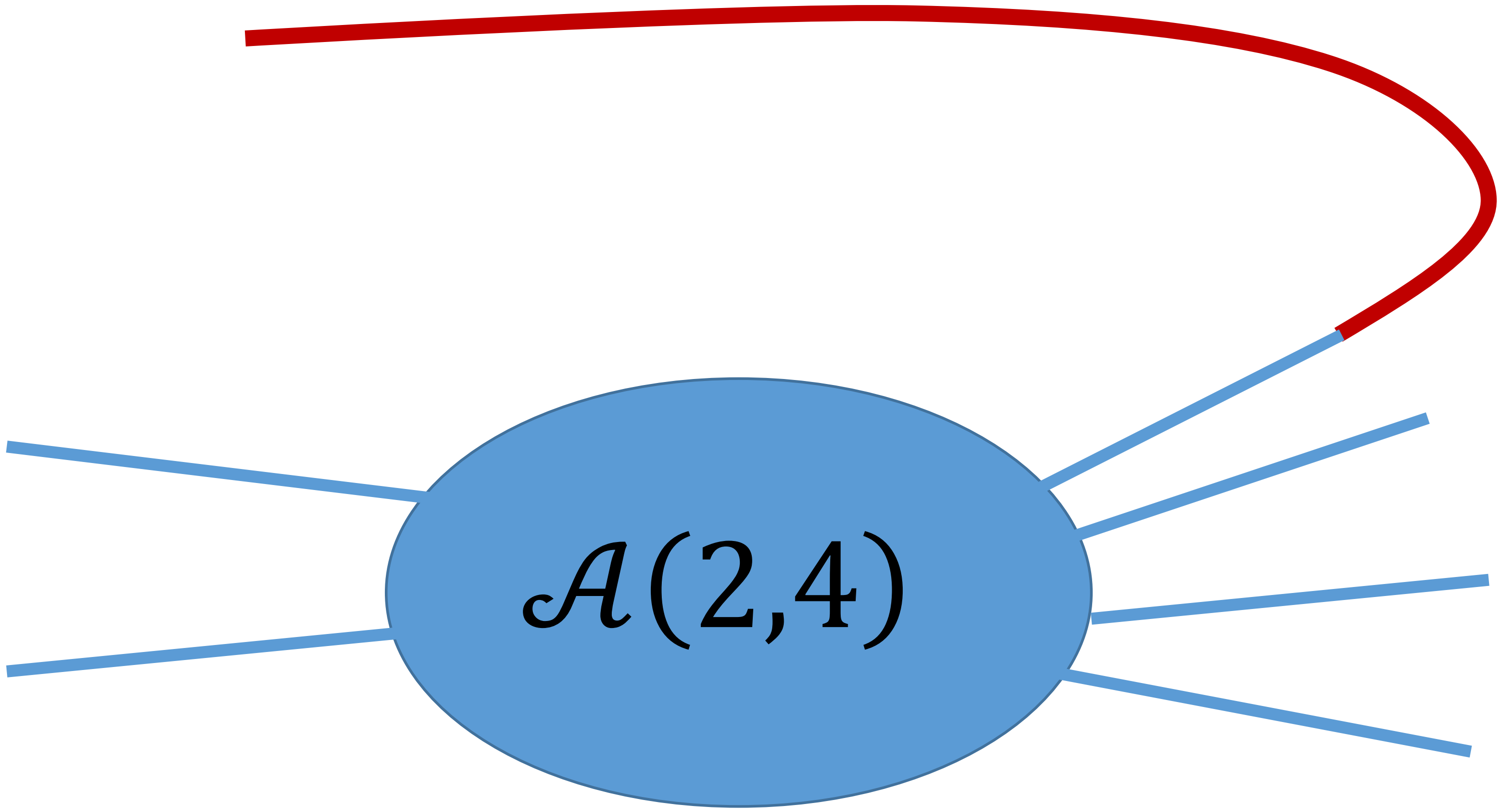


Nondegenerate $b \Rightarrow$ canonical isomorphisms

$$b^{\vee}: V \to V^{\vee} \qquad\qquad b_{\vee}: V^{\vee} \to V$$

$$b^{\vee} \circ \mathcal{A}(n_i, n_o) = \mathcal{A}(n_i + 1\,, n_0 - 1\,)$$



$$\bar{\mathcal{A}} = \exp(1) \sum_{n_{in}+n_{out}=2n}^{\infty} \frac{(2n)!}{n!\,2^n} (v^\vee)^{n_{in}} \otimes v^{n_{out}}$$

$$\in S^*V^\vee \otimes S^*V \cong End\big(Fock(V)\big)$$

Splitting

Wick's theorem: $$\frac{(2n)!}{n!\,2^n} = \int \frac{dh}{\sqrt{2\pi}}\, h^{2n}\, e^{-\frac{1}{2}h^2}$$

$$\psi_n = (2\pi)^{-\frac{1}{4}} h^n e^{-\frac{1}{4}h^2} \in L^2(\mathbb{R}) = \mathcal{W}$$

$$\langle \psi_n, \psi_m \rangle = \delta_{n+m=0(2)} \frac{(2n+2m)!}{(n+m)!\,2^{n+m}}$$

$$\Phi = \exp\left(\frac{1}{2}\right) \sum_n (v^\vee)^n \otimes \psi_n \in Hom(Fock(V), \mathcal{W})$$

Generalizes to $\dim V > 1$

$\Psi_{HH}^{\vee}$ is multilinear & totally symmetric $\Rightarrow$ determined by values on the exponentiated diagonal.

For $v \in V$ define $e^{v} = 1 + v + \frac{v \otimes v}{2!} + \cdots \in Fock(V)$

$$\Psi_{HH}^{\vee}(e^{v}) = \exp[\dim V + \frac{1}{2} b(v, v)]$$

Comments

1. $O(b) = Aut(V, b)$ acts on $Fock(V)$
   and intertwines with $\bar{\mathcal{A}}$.

2. Amplitudes can depend on continuous parameters

3. $(\Psi_{HH}, \Psi_{HH}^{\vee})$ is NOT $\mathcal{A}(\emptyset, \emptyset)$ In fact, it is divergent.

Friedan: $\Rightarrow$ $\Psi_{HH}^{\vee}$ should be interpreted as a generalized measure on $\bigoplus_X \mathcal{Z}(X)$

4. Results can be extended to oriented d=1 theory $\mathcal{Z}$ determined solely by a single vector space V.

$$\mathcal{Z}(pt_+) = V \qquad \mathcal{Z}(pt_-) = V^\vee$$

$$\mathcal{W} = L^2(T^2)$$

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V. Example: d=2 & Oriented

$\mathcal{Z}(S^1) = \mathcal{C}$: f.d. commutative Frobenius algebra

[Friedan, Dijkgraaf, Segal,…]

Complete proof of the sewing theorem (including equivariant case): Moore & Segal 2002

$$\mathcal{Z}(Disk): \quad \theta_{\mathcal{C}}: \mathcal{C} \rightarrow \kappa$$

$b(\phi_1, \phi_2) = \theta_{\mathcal{C}}(\phi_1 \phi_2)$: Symmetric nondegenerate form

Open-closed case discussed later.

	Semisimple	Non-semisimple
Closed	Yes	Examples
Open-closed	Yes	????

$\mathcal{Z}(S^1)$ Semisimple

$$\mathcal{C} = \bigoplus_{x \in \mathcal{X}} \mathcal{C}_x = \bigoplus_{x \in \mathcal{X}} \mathbb{C}\varepsilon_x$$

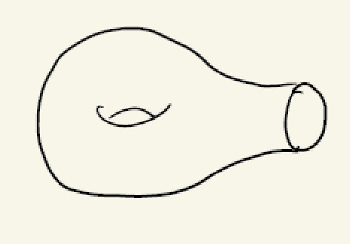
$$\varepsilon_x \varepsilon_y = \delta_{x,y}\varepsilon_x \qquad \theta(\varepsilon_x) = \theta_x \in \kappa^*$$

2d Topological String Theory with target space $\mathcal{X} = Spec(\mathcal{C})$ and dilaton $\theta_x = g_{string,x}^{-2}$

$$\bar{\mathcal{A}}(\emptyset,\emptyset) = \exp(\, \mathcal{Z}(Y_0) + \mathcal{Z}(Y_1) + \cdots)$$

$$= \exp(\, \theta_{\mathcal{C}}\left(\frac{1}{1-h}\right)) \quad = \exp \sum_{x \in \mathcal{X}} \lambda_x$$

$h \in \mathcal{C}$: Handle-adding element defined by the one-hole torus with one outgoing S^1



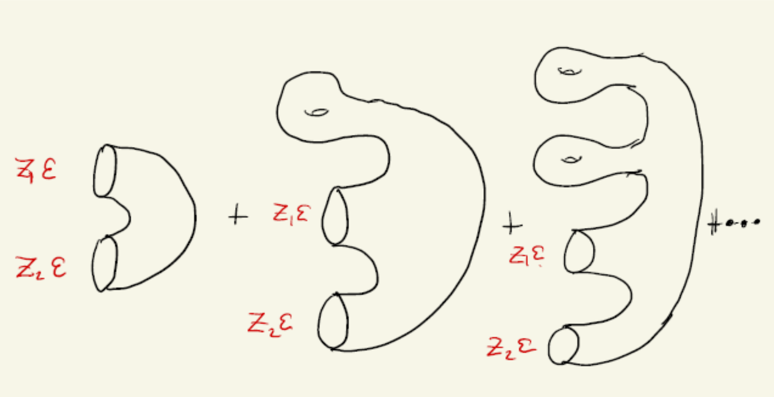
$$\lambda_x := \frac{\theta_x}{1-\theta_x^{-1}} = g_x^{-2} + 1 + g_x^2 + \cdots$$

$\bar{\mathcal{A}}(S^1 \coprod S^1, \emptyset)(\phi_1, \phi_2) = ?$

For simplicity: Take $\dim \mathcal{C} = 1$ $\quad \phi_1 = z_1 \varepsilon, \quad \phi_2 = z_2 \varepsilon \in \mathcal{C}$

Bordisms with *one connected component*:

$$\phi_1 \otimes \phi_2 \mapsto z_1 z_2 \lambda$$

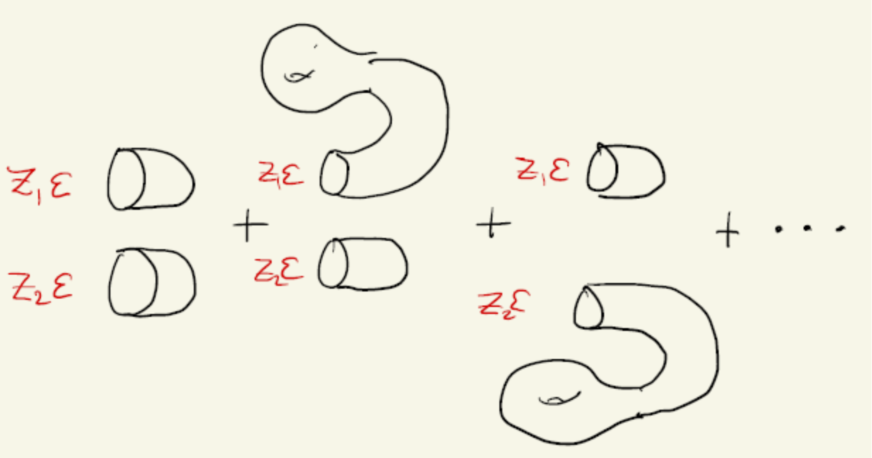


Bordisms with *one connected component* and n ingoing circles:

$$\phi_1 \otimes \cdots \otimes \phi_n \mapsto (z_1 \cdots z_n)\lambda$$

Returning to 2 ingoing circles: We can also have bordisms with two connected components:

$$\phi_1 \otimes \phi_2 \mapsto z_1 z_2 \lambda^2$$



Altogether: $\bar{\mathcal{A}}(2,0)(\phi_1, \phi_2) = z_1 z_2\, e^{\lambda}(\lambda + \lambda^2)$

Marolf-Maxfield recognize $B_2(\lambda)$ as a Bell polynomial

$$= z_1 z_2\, e^{\lambda} B_2(\lambda)$$

Bell Polynomials

$B_n(x_1, \dots, x_n)$: A polynomial that counts the ways a set of n elements can be partitioned

Coefficient of $x_1^{k_1} x_2^{k_2} \cdots$: counts disjoint decompositions with

k_1 subsets of cardinality 1

k_2 subsets of cardinality 2

Etc.

$$B_n(\lambda) \coloneqq B_n(\lambda, \lambda, \dots, \lambda)$$ (Touchard polynomials)

Dividing a bordism $\coprod_1^n S^1 \to \emptyset$ into connected components will have k_j connected components with j ingoing circles. Each such component, when summed over handles gives a factor of λ

Upshot is:

$$\Psi_{HH}^{\vee} = e^{\lambda} \sum_{n=0}^{\infty} B_n(\lambda)(\varepsilon^{\vee})^n$$

Generalizes to $\dim \mathcal{C} > 1$

$\Psi_{HH}^{\vee}$ is multilinear & totally symmetric $\Rightarrow$ determined by values on the exponentiated diagonal.

$$\Psi_{HH}^{\vee}\left(\exp[\sum_{x} z_x \varepsilon_x]\right) = \prod_{x \in X} \exp[\,\lambda_x \exp(z_x)\,]$$

Comments

1. Amplitudes are invariant under the action of the automorphism group of $\mathcal{C}$. (Product of symmetric groups.)

2. Amplitudes can depend on continuous parameters

3. $(\Psi_{HH}, \Psi_{HH}^{\vee})$ is NOT $\mathcal{A}(\emptyset, \emptyset)$ In fact, it is divergent.

Friedan: $\Rightarrow$ $\Psi_{HH}^{\vee}$ should be interpreted as a generalized measure on $\oplus_X \mathcal{Z}(X)$

(Returning to $\dim \mathcal{C} = 1$)

Applying $b_{\vee}$ ➡

$$\bar{\mathcal{A}} = e^{\lambda} \sum_{n_i, n_o} B_{n_i+n_o}(\lambda)(\varepsilon^{\vee})^{n_i} \otimes \left(\frac{\varepsilon}{\theta}\right)^{n_o}$$

In order to prove a splitting formula, use the identity:

$$e^{\lambda} B_n(\lambda) = \sum_{d=0}^{\infty} \frac{\lambda^d}{d!} d^n$$

$$\bar{\mathcal{A}} = e^{\lambda} \sum_{n_i, n_o} B_{n_i+n_o}(\lambda)(\varepsilon^{\vee})^{n_i} \otimes \left(\frac{\varepsilon}{\theta}\right)^{n_o}$$

$$e^{\lambda} B_n(\lambda) = \sum_{d=0}^{\infty} \frac{\lambda^d}{d!} d^n$$

$$\bar{\mathcal{A}} = \sum_{n_i, n_o \geq 0} (\varepsilon^{\vee})^{\otimes n_i} \; \left(\sum_{d=0}^{\infty} \frac{\lambda^d}{d!} \; d^{n_i+n_o} \right) \otimes \; \left(\frac{\varepsilon}{\theta}\right)^{\otimes n_o}$$

Frobenius structure gives canonical sesquilinear form

$$\varepsilon^* = \theta \varepsilon^{\vee} \qquad (\varepsilon^{\vee})^* = (\theta^*)^{-1} \varepsilon$$

For splitting formula take $\mathcal{W} = Fock(\mathcal{C})$

For θ real, but not necessarily positive,

$$\bar{\mathcal{A}} = \Phi\Phi^* \qquad \Phi = \sum_{\ell, d \in \mathbb{Z}_+} \sqrt{\frac{\lambda^d}{d!}} \, (d\, \varepsilon^{\vee})^{\ell} \otimes \left(\frac{\varepsilon}{\sqrt{\theta}}\right)^d$$

$$\Phi \otimes \Phi^* = \sum_{\ell_i, d_i, \ell_o, d_o} \sqrt{\frac{\lambda^{d_i}}{d_i!}} \sqrt{\frac{\bar{\lambda}^{d_o}}{d_o!}}$$

$$(d_i\, \varepsilon^\vee)^{\ell_i} \otimes \left(\frac{\varepsilon}{\sqrt{\theta}}\right)^{d_i} \otimes \left(\frac{\theta}{\sqrt{\bar{\theta}}}\varepsilon^\vee\right)^{d_o} \otimes \left(\frac{d_o}{\bar{\theta}}\varepsilon\right)^{\ell_o}$$

Contracting inner two factors puts $d_i = d_o$ and recovers $\bar{\mathcal{A}}$ if θ is real.

Relation to Coherent States

(Also used in MM, but in a different way.)

Relation To Coherent States - 1/2

$$[a, a^*] = 1 \qquad\qquad |d\rangle := \frac{1}{\sqrt{d!}}(a^*)^d|0\rangle$$

$$\Psi_\lambda := \exp\left(\sqrt{\lambda}\, a^*\right)|0\rangle$$

$$N := a^* a \qquad\qquad e^\lambda B_n(\lambda) = \langle \Psi_\lambda, N^n\, \Psi_\lambda \rangle$$

Relation To Coherent States - 2/2

$$Z_{ann}: \mathcal{W} \to \mathcal{C}^{\vee} \otimes \mathcal{W} \qquad |d\rangle \mapsto (d\,\varepsilon^{\vee}) \otimes |d\rangle$$

$$Z_{cr}: \mathcal{W}^{\vee} \to \mathcal{W}^{\vee} \otimes \mathcal{C} \qquad \langle d| \mapsto \langle d| \otimes \left(\frac{d}{\theta}\,\varepsilon\right)$$

$$\bar{\mathcal{A}} = \langle\, \Psi_{\lambda}\,, \frac{1}{1 - Z_{cr}}\,\frac{1}{1 - Z_{ann}} \Psi_{\lambda}\,\rangle$$

$$\in\; S^{*}\mathcal{C}^{\vee} \otimes S^{*}\mathcal{C} \;\cong\; End\big(Fock(\mathcal{C})\big)$$

In some sense $\mathcal{W}$ is the Hilbert space of a ``dual quantum mechanical system'' to the ``quantum gravity theory.''

$$\bar{\mathcal{A}} = \langle \frac{1}{1 - Z_{cr}} \Psi_\lambda , \frac{1}{1 - Z_{ann}} \Psi_\lambda \rangle \in End(S^* \mathcal{C})$$

$\mathcal{W}$ is not associated with any asymptotic boundary. So there is no factorization problem.

Digression: Formulae are reminiscent of

``quantum mechanics with noncommutative amplitudes''

Standard QM: $\langle \psi_1, \psi_2 \rangle \in \mathbb{C}$

QMNA: $\langle \Psi_1, \Psi_2 \rangle \in$ some $C^* -$algebra

i.e. Ψ in a Hilbert C* module $\mathcal{E}$

Born Rule ?

Consider ``adjointable operators'' $T : \mathcal{E} \to \mathcal{E}$

$$(\Psi_1, T\Psi_2)_{\mathfrak{A}} = (T^* \Psi_1, \Psi_2)_{\mathfrak{A}}$$

The adjointable operators $\mathfrak{B}$ are another C* algebra.

Definition: *QMNA observables* are self-adjoint elements of $\mathfrak{B}$

Definition: A ***QMNA state*** is a completely positive unital map $\varphi : \mathfrak{B} \to \mathfrak{A}$

QMNA Born Rule

Main insight is that we should regard the Born Rule as a map

$$BR : \mathcal{S}^{\mathrm{QMNA}} \times \mathcal{O}^{\mathrm{QMNA}} \times \mathcal{S}(\mathfrak{A}) \to \mathcal{P}$$

For general $\mathfrak{A}$ the datum $\omega \in \mathcal{S}(\mathfrak{A})$ together with complete positivity of φ give just the right information to state a Born rule in general:

$$BR(\varphi, T, \omega) \in \mathcal{P}$$

Family of quantum systems over a noncommutative space.

Reinterpret various constructions in quantum information theory in terms of noncommutative geometry

END OF DIGRESSION

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VI. d=2 Open-Closed: Oriented, Semi-simple

In/out manifolds are disjoint unions of circles and oriented intervals

The intervals are 1-morphisms in a category (of manifolds with corners)

b

$\uparrow$

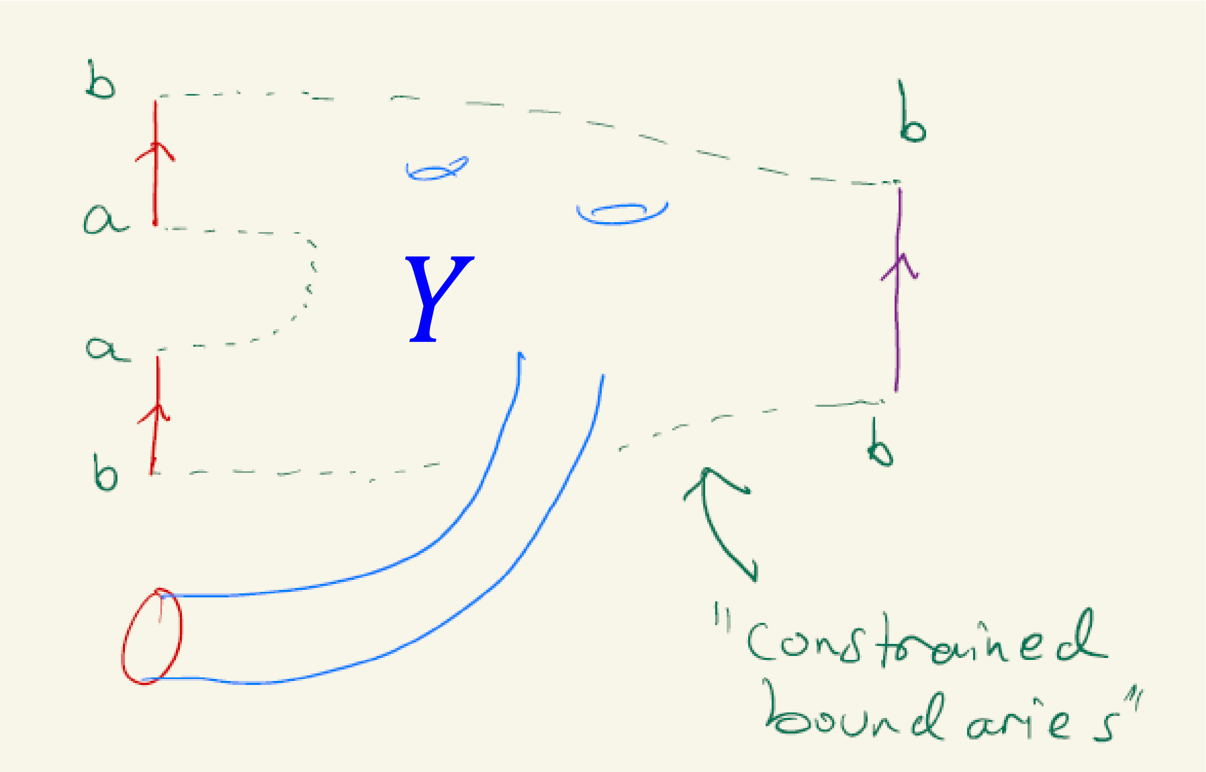
a

a, b are objects in a category of boundary conditions.

[Moore & Segal]

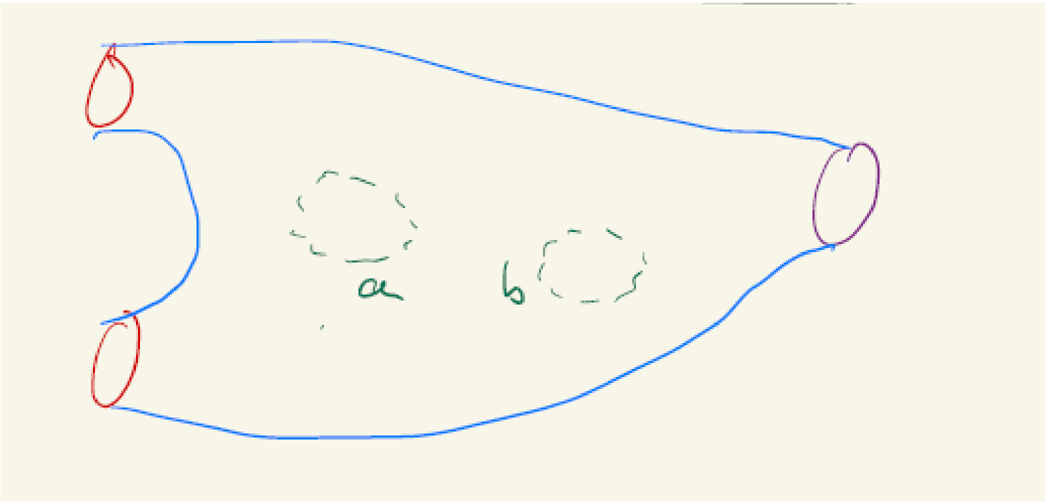
$$\mathcal{Z}(I_{ab}) = Hom(a,b) \coloneqq \mathcal{O}_{ab}$$

The surfaces are now 2-morphisms in a 2-category



$$\partial Y = (\partial Y)_{in} \coprod (\partial Y)_{out} \coprod (\partial Y)_{constrained}$$

We can also have closed constrained boundaries



Three conceptually distinct kinds of boundaries

1. Ingoing/outgoing circles & intervals

2. Constrained boundaries connecting in/out endpoints to in and/or out endpoints of intervals

3. Closed constrained boundaries

Side Remark On Marolf-Maxfield Model

MM define their model by summing over surfaces Y with boundary, with the weighting factor

$$\exp\{S_0\chi(Y) + S_\partial|\pi_0(\partial Y)|\}$$

$S_\partial|\pi_0(\partial Y)|$ is not a local term in the action

Resolution: When one is careful about the interpretations of the circles S_∂ is a parameter that need not be interpreted as a part of the action

There are different interpretations depending on whether we take the boundary circles to be in/out going or constrained boundaries.

In our language MM consider $\dim \mathcal{C} = 1$

(Generalizing their story to $\dim \mathcal{C} > 1$: Gardiner-Megas)

$$\Psi^{\vee}_{HH}(\exp(\tilde{u}\varepsilon)) = \exp[\,\lambda\, e^{\tilde{u}}]$$

$$\tilde{u} = u_{MM}\, e^{S_\partial - S_0}$$

Or, if we consider their boundaries to be closed constrained boundaries then e^{S_∂} is a fugacity

Splitting Formula - Simplest Case

For simplicity (we can relax all these conditions):

1. $\dim \mathcal{C} = 1$

2. All constrained boundaries are labeled with single b.c. a with $Hom(a,a) = \mathcal{O}_{aa}$

3. No closed constrained boundaries

4. All in/out manifolds are intervals I_{aa}

μ^{-1} = open string coupling: $\mu^2 = \theta$

For μ real, $\theta > 0$ $\bar{\mathcal{A}} = \Phi\Phi^*$

$$\Phi: \; S^*\mathcal{O}_{aa} \to L^2(\mathcal{E}_{N_a}) = \mathcal{W}$$

``Cardy condition'' implies $\mathcal{O}_{aa} \cong Mat_{N_a \times N_a}(\mathbb{C})$ [Moore&Segal]

$\mathcal{E}_{N_a} =$ vector space of $N_a \times N_a$ Hermitian matrices

$$\Phi = \sum_{n} \sum_{S=\{i_1 j_1, i_2 j_2, \ldots i_n j_n\}} \prod_{a=1}^{n} e^{\vee}_{i_a j_a} \int_{\mathcal{E}_{N_a}} [dH]\, e^{-\frac{1}{2}U(H)} \prod_{a=1} H_{j_a i_a} \;\langle\, H \,|$$

e_{ij} : Basis of matrix units for $\mathcal{O}_{aa}$; $e^{\vee}_{ij}$ is the dual basis

$$e^{-U(H)} = \int_{\sqrt{-1}\mathcal{E}_{N_a}} [dS] \exp\left(\left(\frac{\lambda}{\det(1-S)^{\frac{1}{\mu}}}\right) - Tr(SH)\right)$$

Corollary:

$$\Psi^{\vee}_{HH}(e^{T}) = \exp[\,\lambda/(\det(1-T)^{\frac{1}{\mu}}\,)]$$

Related formula in Gardiner-Megas

G- Equivariant Generalization: Finite Groups

1. Bordism category: Principal G-bundles

2. Replace Frobenius algebra by Turaev algebra $\mathcal{C}$

$$\mathcal{C} = \oplus_{g \in G} \ \mathcal{C}_g \qquad \text{with } \alpha: G \to Aut(\mathcal{C})$$

3. Sum over bordisms splits into two parts:

Sum over bundles, then sum over 2d surfaces

G- Equivariant Generalization: Finite Groups

4. Sum over bundles:

a.) Fix G-bundle on boundary (rules for computation: M&S 2002)

b.) Sum over all G-bundles, including on the boundary

4a: Replace $\mathcal{C}$ by Turaev algebra $\oplus_g \;\; \mathcal{Z}(P_g \to S^1)$

$\bar{\mathcal{A}}$ has NOT been worked out

4b $\Rightarrow$ gauging G. It simply replaces $\mathcal{C}$ by ss Frobenius algebra $\mathcal{C}^{orb}$ and we are back in the previous case.

Topological Gravity Coupled To 2D YM

G: Compact connected simple Lie group

Morphisms are surfaces $Y(\vec{A})$ with area ``vector'' ,
$\vec{A} = (A_c)_{c \in \pi_0(Y)}$ which is additive under gluing

Seed QFT: $\mathcal{Z}(S^1) = L^2(G)^G \otimes \mathcal{C}$

ON basis: $\chi_R \otimes \varepsilon/\sqrt{\theta}$ $R \in G^\vee$:

$G^\vee$: Unitary dual: Irreps of G

Measure for bordisms?

Guess: $$w\left(Y(\vec{A})\right) = \prod_c \frac{dA_c}{A_c} A_c^p \times Z_{inv}\left(Y(\vec{A})\right) \times w_{top}(Y)$$

It would be very nice to derive it from general physical principles a la Friedan.

$$\log \mathcal{A}(\emptyset, \emptyset) = \int_0^\infty \frac{dA}{A} A^p \sum_{g=0}^{\infty} \sum_{R \in \mathcal{G}^\vee} (\theta \dim(R)^2)^{1-g} e^{-A\left(g_{ym}^{-2} C_2(R) + \Lambda_0\right)}$$

Convergence @ $A \to 0 \Rightarrow p > 0$

Convergence @ $A \to \infty$: $\Rightarrow$

$\Lambda_0 > 0$ Cosmological constant

$$\log \mathcal{A}(\emptyset,\emptyset) = \int_0^\infty \frac{dA}{A}\ A^p \sum_{g=0}^{\infty} \sum_{R\in \mathcal{G}^\vee} (\theta \dim(R)^2)^{1-g} e^{-A(\mu C_2(R)+\mu_0)}$$

$$= \sum_{R\in G^\vee} \lambda_R \qquad \lambda_R = \frac{\theta(\dim(R)^2)}{1-(\theta(\dim(R))^2)^{-1}} \ \frac{\Gamma(p)}{\left(g_{ym}^{-2} C_2(R) + \Lambda_0\right)^p}$$

Converges for $Re(p) > |\Delta_+(\mathfrak{g})| + \frac{1}{2}$

Expected to admit analytic continuation in p

$$\Psi^{\vee}_{HH}(\exp[\sum_{R} z_R \chi_R \otimes \varepsilon])$$

$$= \prod_{R \in G^{\vee}} \exp[\ \lambda_R \exp(\ z_R)]$$

Compare with result for semisimple 2d closed case:

$$\Psi^{\vee}_{HH}(e^{z_x \varepsilon_x}) = \prod_{x \in \mathcal{X}} \exp[\ \lambda_x \exp(\ z_x)\ ]$$

strongly suggests there will be factorization with

$$\mathcal{W} = Fock\big(\mathcal{Z}(S^1)\big) = S^*(\, L^2(G)^G \otimes \mathcal{C})$$

1. Motivation
2. Reminders On TQFT
3. Summed & Total Amplitudes: Splitting Property
4. Example: $d = 1$
5. Example: $d = 2$, closed
6. Example: $d = 2$, open-closed
7. Coupling To 2d YM
8. $d \geq 3$: Comments
9. Summary & Some Open Problems

VIII. Comments On $d \geq 3$

Can we extend these ideas to d=3 TQFT ?

Classification of manifolds is <u>***MUCH***</u> more difficult !!

$$\mathcal{A}(\emptyset, \emptyset) = \exp\Big(\sum_{Y} \mathcal{Z}(Y) \Big)$$

Sum over closed connected 3-folds Y

That includes the sum over $Y = S^1 \times \Sigma_g$

$$\mathcal{Z}(Y) = \dim \mathcal{Z}(\Sigma_g)$$

For standard fully local TQFT, $\dim \mathcal{Z}(\Sigma_g)$ grows with g

The sum is irretrievably divergent.

Can we have $\dim \mathcal{Z}(\Sigma_g) = 0$ for sufficiently large g ?

Sergei Gukov: No!

Cut along the boundary of a handlebody for any g

If $\dim \mathcal{Z}(\Sigma_g) = 0$ for <u>any</u> g then all amplitudes vanish!!

$\Rightarrow$ Change domain and/or codomain

Couple to some gravity theory suppressing ``complicated'' manifolds

Possibly we should allow for asymptotic expansions:

$$\sum_Y \frac{Z(Y)}{|Aut(Y)|} \sim \sum_k e^{S(k)} \; Z_k$$

$$e^{S(k)} \sim k! \qquad Z_k \sim e^{-c\,k}$$

Problem of growing topological entropy (Carlip):

For $d \geq 4$, any finitely generated group is
π_1 of some d-manifold.

For $d \geq 4 \ \exists \ a, b$

$$a\, V \log V \ < \#\{ \, Hyperbolic\ M : vol(M) = V \} < \ b\, V \log V$$

[Burger, Gelander,Lubotzky,Mozes]

d=3 there can be countably infinite hyperbolic geometries of bounded hyperbolic volume.

IX. Summary And Open Problems

For suitable parameters of our TQFT, the total amplitude

$$\bar{\mathcal{A}} \in End(\otimes_X S^*(\mathcal{Z}(X))) := End\left(Fock(\mathcal{Z})\right)$$

Has a splitting: $\bar{\mathcal{A}} = \Phi\Phi^*$

$$\Phi: Fock(\mathcal{Z}) \rightarrow \mathcal{W}$$

Potential conceptual interpretations:

Holography & QMNA

Extensions of the d=2 results

1. The general non-semisimple case, closed, and open

2. Other tangential structures: Unorientable, (s)pin, …

3. Topological string theory: $\bar{\mathcal{A}} \in End\left(S^* H_q^*(\mathcal{X})\right)$

4. ∃ splitting formula for total amplitude of JT gravity?

Is the existence of a splitting formula deep or a trivial consequence of linear algebra ?

Rough idea: The total amplitude is symmetric under exchange of all in-going boundaries for all out-going boundaries.

But any symmetric (f.d. complex) matrix S can be written as $S = \Phi\Phi^{tr}$

If it doesn't just follow by linear algebra, is there an a priori reason why it should hold?

And what to do about $d \geq 3$???

Thanks for your attention!

SUPPLEMENT 1

MM construction of ``baby universe Hilbert space''

A sesquilinear form on $S^*\mathcal{C}$ is defined by

$$\langle \phi_1, \phi_2 \rangle = \Psi_{HH}^{\vee}(K(\phi_1)\phi_2)$$

$$\phi = \sum_{n=0}^{\infty} c_n \varepsilon^n \;\rightarrow\; f_\phi(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$\langle \phi_1, \phi_2 \rangle = \sum_{d=0}^{\infty} \frac{\lambda^d}{d!} \left(f_1(d)\right)^* f_2(d)$$

$Ann(\langle \cdot,\cdot \rangle) \cong$ A vector space of order ≤ 1 entire functions that vanish on $\mathbb{Z}_+$

$S^*\mathcal{C}$ is viewed as a $*$-algebra.

MM then imitate the GNS construction and define a ``baby universe Hilbert space''

$$\mathcal{H}_{BU} := S^*\mathcal{C}/Ann(\langle \cdot,\cdot \rangle)$$

$$\cong \{ (\xi_0, \xi_1, \dots) \in \mathbb{C}^\infty \mid \sum \frac{\lambda^d}{d!} |\xi_d|^2 < \infty \}$$

$(\lambda > 0)$ $\cong$ H.O. representation of Heisenberg algebra

Expectation values in a coherent state are then interpreted as stochastic expectations of a ``universe creation operator Z''

$\Psi_{HH}^{\vee}$ is viewed as defining an expectation value on polynomials in a stochastic variable Z on $S^*\mathcal{C}$ where $Z(\varepsilon)$ has the interpretation of the partition function of a 1d TQFT chosen from an ensemble with Poisson probability distribution

$$p(d) = e^{-\lambda} \frac{\lambda^d}{d!}$$

for an ensemble of 1d TQFTs with $\dim V = d$.

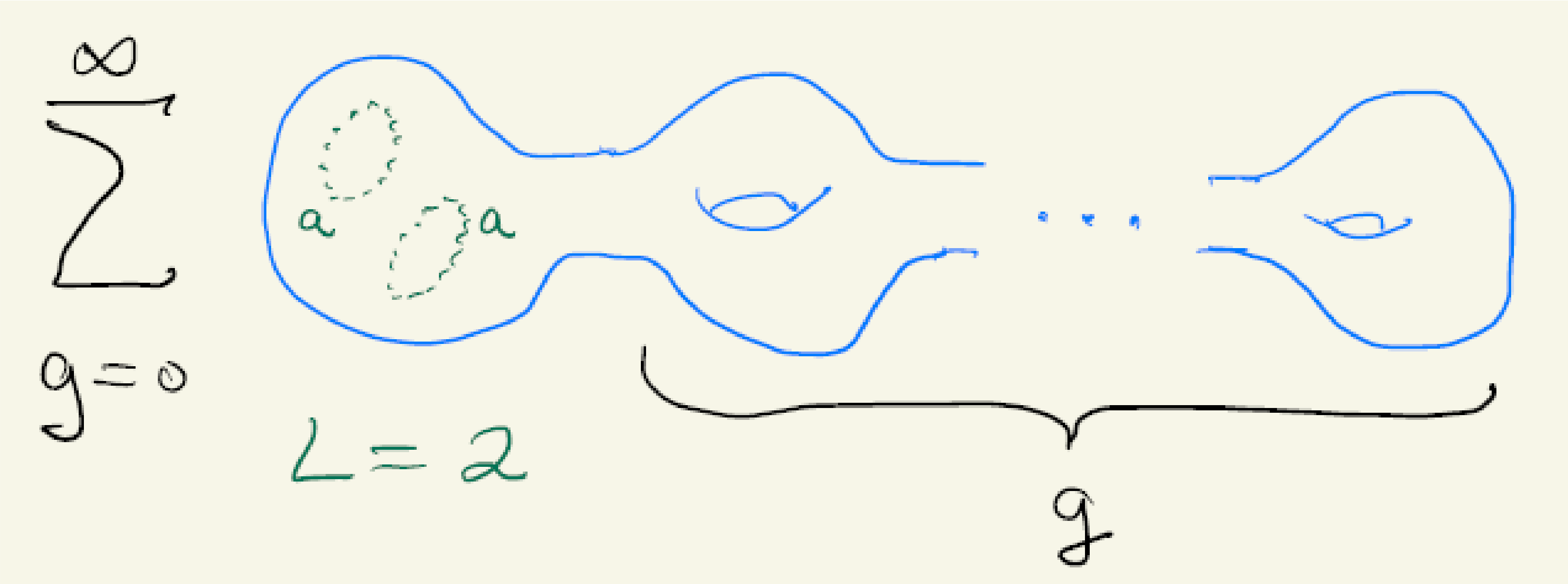
END OF SUPPLEMENT 1

BEGIN SUPPLEMENT 2

VII. Constrained Boundaries & An Ensemble Interpretation

MM paper aimed to give an interpretation of the 2d model in terms of an ensemble average of 1d models.

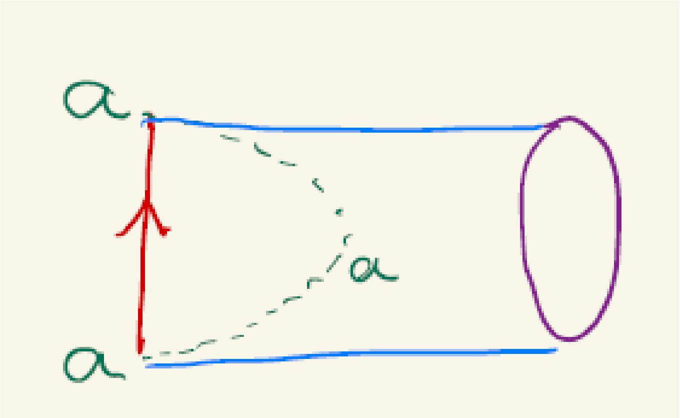
Sum over bordisms $\emptyset \to \emptyset$ with L constrained boundaries of type a



+ Disconnected surfaces

$$\frac{\bar{\mathcal{A}}_{\{L\}}(\emptyset,\emptyset)}{\bar{\mathcal{A}}(\emptyset,\emptyset)} = B_L(x_1,\ldots,x_L) \qquad x_j = \theta_{\mathcal{C}}\left(\frac{1}{1-h}B_a^j\right)$$

$$B_a := \iota_a\left(1_{\mathcal{O}_{aa}}\right) \in \mathcal{C}$$



$$\dim \mathcal{C} = 1: \qquad B_a = \frac{N_a}{\mu}\,\varepsilon$$

$$\frac{\bar{\mathcal{A}}_{\{L\}}(\emptyset, \emptyset)}{\bar{\mathcal{A}}(\emptyset, \emptyset)} = \left(\frac{1}{\mu}\right)^L \sum_{d=0}^{\infty} e^{-\lambda} \frac{\lambda^d}{d!} \, (dN_a)^L$$

$$= \left(\frac{1}{\mu}\right)^L \langle\, \mathcal{Z}(S^1)^L \rangle_{\mathcal{E}}$$

$\mathcal{Z}$ is a stochastic variable on an ensemble $\mathcal{E}$ of 1d oriented TQFT's $\mathcal{Z}_d$ labeled by $d \in \mathbb{Z}_+$ with

$$p(\mathcal{Z}_d) = e^{-\lambda} \frac{\lambda^d}{d!} \qquad\qquad \mathcal{Z}_d(S^1) = \dim V_d = d\, N_a$$

It would be interesting to give an ensemble interpretation to the full set of open/closed amplitudes.

This suggests it could be interesting to consider TQFT’s where the target category is the category of f.d. vector bundles over measure spaces as a way to model ensemble averages of field theories.

END OF SUPPLEMENT 2

BEGIN SUPPLEMENT 3

Quantum Systems

Set of physical ``states'' $\mathcal{S}$

Set of physical ``observables'' $\mathcal{O}$

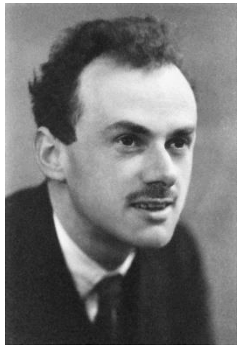
Born Rule: $BR : \mathcal{S} \times \mathcal{O} \to \mathcal{P}$

$\mathcal{P}$ Probability measures on $\mathbb{R}$.

$m \in \mathfrak{M}(\mathbb{R})$ ⟹ $0 \leq \wp(m) \leq 1$

$m = [r_1, r_2] \subset \mathbb{R}$ $\qquad BR(\mathrm{s}, \mathrm{O})([r_1, r_2])$

is the probability that a measurement of the observable O in the state s has value between r_1 and r_2 .



Dirac-von Neumann Axioms

$\mathcal{S}$ Density matrices ρ: Positive trace class operators on Hilbert space of trace =1

$\mathcal{O}$ Self-adjoint operators T on Hilbert space

Spectral Theorem: There is a one-one correspondence of self-adjoint operators T and projection valued measures:

Density matrices ρ: Positive trace class operators on Hilbert space of trace =1 $\rightarrow$ $P_T(m)$

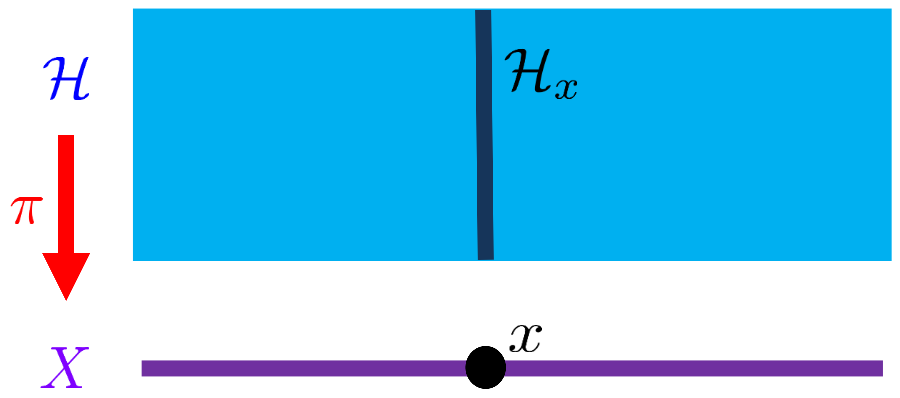
Example: $T = \sum_\lambda \lambda P_\lambda$ $\quad P_T([r_1, r_2]) = \sum_{r_1 \leq \lambda \leq r_2} P_\lambda$

Density matrices ρ: Positive trace class operators on Hilbert space of trace =1

$$BR(\rho, T)(m) = \mathrm{Tr}_{\mathcal{H}} \left(\rho P_T(m)\right)$$

Continuous Families Of Quantum Systems

Hilbert bundle over space X of control parameters.



For each x get a probability measure $\wp_x$:

$$m \in \mathfrak{M}(\mathbb{R}) \mapsto \wp_x(m) := \mathrm{Tr}_{\mathcal{H}_x}(\rho_x P_{T_x}(m))$$

$$BR : \mathcal{S} \times \mathcal{O} \times X \to \mathcal{P}$$

$$BR(\rho, T, x) = \wp_x$$

Noncommutative Control Parameters

We would like to define a family of quantum systems parametrized by a NC manifold whose ``algebra of functions'' is a general C* algebra $\mathfrak{A}$

What are observables?

What are states?

What is the Born rule?

What replaces the Hilbert bundle?

Noncommutative Hilbert Bundles

Definition: Hilbert C* module $\mathcal{E}$ over C*-algebra $\mathfrak{A}$.

Complex vector space $\mathcal{E}$ with a right-action of $\mathfrak{A}$
and an ``inner product'' valued in $\mathfrak{A}$

$$\Psi_1, \Psi_2 \in \mathcal{E} \qquad (\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A}$$

$$(\Psi_1, \Psi_2)^*_{\mathfrak{A}} = (\Psi_2, \Psi_1)_{\mathfrak{A}}$$

$$(\Psi, \Psi)_{\mathfrak{A}} \geq 0$$ (Positive element of the C* algebra.)

$$(\Psi_1, \Psi_2 a) = (\Psi_1, \Psi_2) a \quad .....$$

Like a Hilbert space, but ``overlaps'' are valued in a (possibly) noncommutative algebra.

Quantum Mechanics With Noncommutative Amplitudes

Basic idea: Replace the Hilbert space by a Hilbert C* module

$$\mathcal{H} \to \mathcal{E}$$

$$\Psi_1, \Psi_2 \in \mathcal{E} \qquad (\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A}$$

Overlaps are valued in a possibly noncommutative algebra.

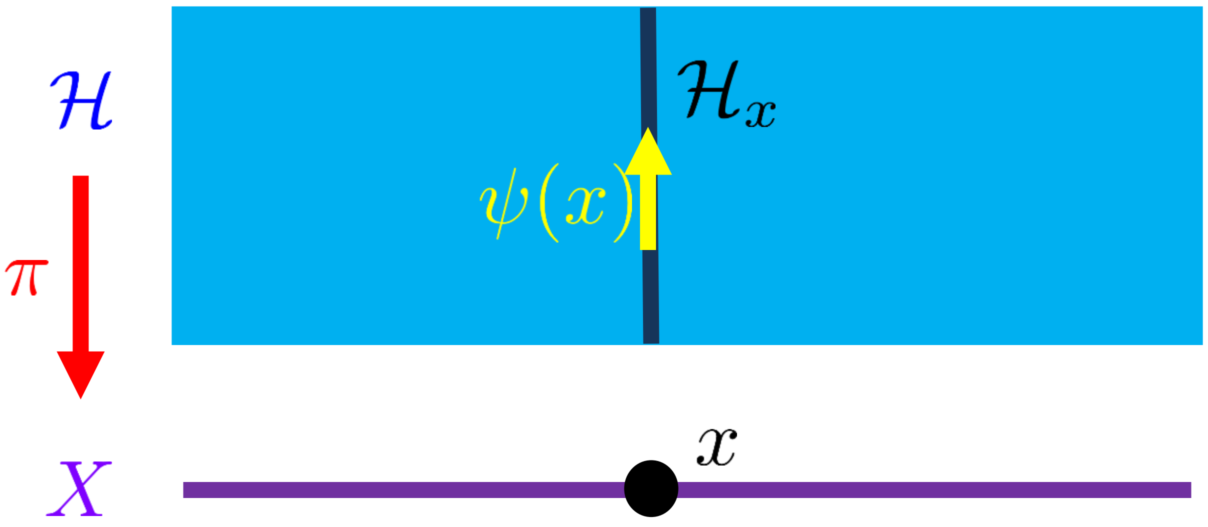
QM: $0 \leq \wp(\lambda) = (\psi_\lambda, \psi)(\psi_\lambda, \psi)^* \leq 1$

QMNA: $(\Psi_\lambda, \Psi)(\Psi_\lambda, \Psi)^* \in \mathfrak{A}$

Example 1: Hilbert Bundle Over A Commutative Manifold

$$\mathcal{E} = \Gamma[\mathcal{H} \to X] \qquad \mathfrak{A} = C(X)$$

$$\Psi : x \mapsto \psi(x) \in \mathcal{H}_x$$



$$(\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A} := C(X)$$

$$(\Psi_1, \Psi_2)_{\mathfrak{A}}(x) := (\psi_1(x), \psi_2(x))_{\mathcal{H}_x} \in \mathbb{C}$$

Example 2: Hilbert Bundle Over A Fuzzy Point

Def: ``fuzzy point'' has $\mathfrak{A} \cong \mathrm{Mat}_{a\times a}(\mathbb{C})$

$$\mathcal{E} = \mathrm{Mat}_{b\times a}(\mathbb{C})$$

$$(\Psi_1, \Psi_2)_{\mathfrak{A}} = \Psi_1^{\dagger}\Psi_2$$

Observables In QMNA

Consider ``adjointable operators'' $T : \mathcal{E} \to \mathcal{E}$

$$(\Psi_1, T\Psi_2)_{\mathfrak{A}} = (T^*\Psi_1, \Psi_2)_{\mathfrak{A}}$$

The adjointable operators
$\mathfrak{B}$ are another C* algebra.

Definition: *QMNA observables*
are self-adjoint elements of $\mathfrak{B}$

(Technical problem: There is no spectral theorem for self-adjoint elements of an abstract C* algebra.)

C* Algebra States

Definition: A <u>*C*-algebra state*</u> $\omega \in \mathcal{S}(\mathfrak{A})$ is a positive linear functional

$$\omega : \mathfrak{A} \to \mathbb{C} \qquad \omega(\mathbf{1}) = 1$$

$$\mathfrak{A} = C(X) \qquad \omega \in \mathcal{S}(\mathfrak{A})$$

$\omega(f) = \int_X f \, d\mu$ $d\mu$ = a positive measure on X:

$$\mathfrak{A} \cong \mathrm{Mat}_{a\times a}(\mathbb{C}) \qquad \omega \in \mathcal{S}(\mathfrak{A})$$

$\omega(T) = \mathrm{Tr}_{\mathcal{H}}(\rho T)$ ρ = a density matrix

QMNA States

Definition: A ***QMNA state*** is a completely positive unital map $\varphi : \mathfrak{B} \to \mathfrak{A}$

``Completely positive'' comes up naturally both in math and in quantum information theory.

Positive: $\varphi : \mathfrak{B}_{\geq 0} \to \mathfrak{A}_{\geq 0}$

Unital: $\varphi(1_{\mathfrak{B}}) = 1_{\mathfrak{A}}$

Completely positive

$$\varphi \otimes 1 : (\mathfrak{B} \otimes \mathrm{Mat}_n(\mathbb{C}))_{\geq 0} \to (\mathfrak{A} \otimes \mathrm{Mat}_n(\mathbb{C}))_{\geq 0}$$

QMNA Born Rule

Main insight is that we should regard the Born Rule as a map

$$BR : \mathcal{S}^{\mathrm{QMNA}} \times \mathcal{O}^{\mathrm{QMNA}} \times \mathcal{S}(\mathfrak{A}) \to \mathcal{P}$$

For general $\mathfrak{A}$ the datum $\omega \in \mathcal{S}(\mathfrak{A})$ together with complete positivity of φ give just the right information to state a Born rule in general:

$$BR(\varphi, T, \omega) \in \mathcal{P}$$

Family Of Quantum Systems Over A Fuzzy Point

$$\mathcal{E} = \mathrm{Mat}_{b\times a}(\mathbb{C}) = \mathbb{C}^b \otimes \mathbb{C}^a = \mathcal{H}_{\mathrm{Bob}} \otimes \mathcal{H}_{\mathrm{Alice}}$$

$$\mathfrak{A} = Mat_a(\mathbb{C}) = \mathrm{End}(\mathcal{H}_{\mathrm{Alice}})$$

$$\mathfrak{B} = Mat_b(\mathbb{C}) = \mathrm{End}(\mathcal{H}_{\mathrm{Bob}})$$

$$BR(\varphi, T, \omega)(m) = \mathrm{Tr}_{\mathcal{H}_A} \rho_A \varphi(P_T(m))$$

``A NC measure $\omega \in \mathcal{S}(\mathfrak{A})$'' is equivalent to a density matrix ρ_A on $\mathcal{H}_A$

QMNA state: $\varphi(T) = \sum_\alpha E_\alpha^\dagger T E_\alpha \quad \sum_\alpha E_\alpha^\dagger E_\alpha = 1$

Quantum Information Theory & Noncommutative Geometry

$$
\begin{aligned}
BR(\varphi, T, \omega)(m) &= \mathrm{Tr}_{\mathcal{H}_A} \rho_A \varphi(P_T(m)) \\
&= \textstyle\sum_\alpha \mathrm{Tr}_{\mathcal{H}_A} \rho_A E_\alpha^\dagger (P_T(m)) E_\alpha \\
&= \textstyle\sum_\alpha \mathrm{Tr}_{\mathcal{H}_B} E_\alpha \rho_A E_\alpha^\dagger P_T(m) \\
&= \mathrm{Tr}_{\mathcal{H}_B} \mathcal{E}(\rho_A) P_T(m)
\end{aligned}
$$

Last expression is the measurement by Bob of T in the state ρ_A prepared by Alice and sent to Bob through quantum channel $\mathcal{E}$.