Finite Symmetries Of Field Theories From TQFT

Gregory Moore

Rutgers

Symmetry Seminar, May 23, 2023

TOPOLOGICAL SYMMETRY IN QUANTUM FIELD THEORY

DANIEL S. FREED, GREGORY W. MOORE, AND CONSTANTIN TELEMAN

In memory of Vaughan Jones

N.B. v3 is a significant upgrade

Abstract. We introduce a definition and framework for internal topological symmetries in quantum field theory, including "noninvertible symmetries" and "categorical symmetries". We outline a calculus of topological defects which takes advantage of well-developed theorems and techniques in topological field theory. Our discussion focuses on finite symmetries, and we give indications for a generalization to other symmetries. We treat quotients and quotient defects (often called "gauging" and "condensation defects"), finite electromagnetic duality, and duality defects, among other topics. We include an appendix on finite homotopy theories, which are often used to encode finite symmetries and for which computations can be carried out using methods of algebraic topology. Throughout we emphasize exposition and examples over a detailed technical treatment.

The study of symmetry in quantum field theory is longstanding with many points of view. For a relativistic field theory in Minkowski spacetime, the symmetry group of the theory is the domain of a homomorphism to the group of isometries of spacetime; the kernel consists of *internal* symmetries that do not move the points of spacetime. It is these internal symmetries—in Wick-rotated form—that are the subject of this paper. Higher groups, which have a more homotopical nature, appear in many recent papers and they are included in our treatment. The word 'symmetry' usually refers to invertible transformations that preserve structure, as in Felix Klein's *Erlangen program*, but one can also consider algebras of symmetries—e.g., the universal enveloping algebra of a Lie algebra acting on a representation of a Lie group—and in this sense symmetries can be non-invertible.

Quantum field theory affords new formulations of symmetry beyond what one usually encounters in geometry. If a Lie group G acts as symmetries of an n-dimensional field theory F, then one expresses the symmetry as a larger theory in which there is an additional background (nondynamical)

Many Many Antecedants

``Like every global symmetry on the brane this is a gauge symmetry in spacetime'' – N. Seiberg, hep-th/9608111

Theory of topological modes/singletons in AdS/CFT:
Witten 98: ``AdS/CFT Correspondence And Topological Field Theory,''

followed up c. 2004 by Belov & Moore, …

developed much further by Apruzzi, Bah, Bhardwaj, Bonetti, Garcia Etxebarria, Hosseini, Minasian, Schafer-Nameki, Tiwari,….

Many Many Antecedants

Open-Closed 2d TQFT: Moore & Segal, ….

Fuchs, Runkel, Schweigert, Valentino, … , Kapustin & Saulina, …

Gaiotto, Kapustin, Seiberg, Willet: Section 6 & 7.3, ...

Gaiotto-Kulp, 2008.05960

Kong & Zheng, 1705.01087

What we add: Systematic calculus of defects in TFT, especially, finite homotopy theories and how it ``implements symmetry.''

Previous Talks

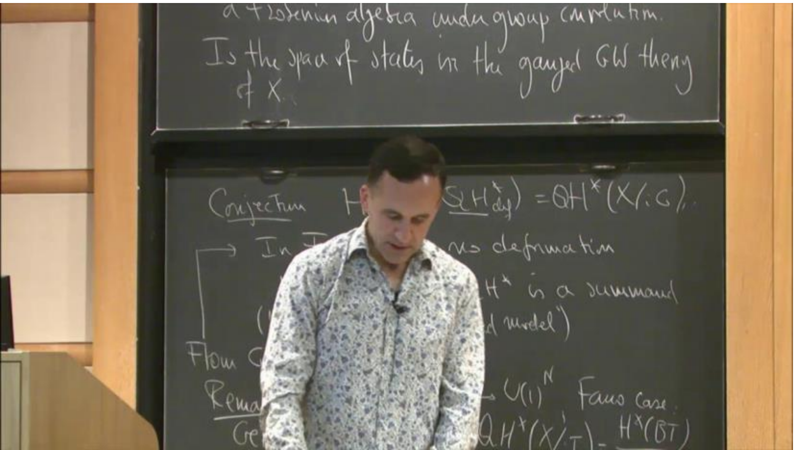
Perimeter Lectures (with lecture notes):

Finite Symmetry In QFT, PIRSA, June 13-17, 2022

StringMath 2022 & arXiv…

CMSA , Nov. 8, 2022

Simons Foundation, November 17, 2022



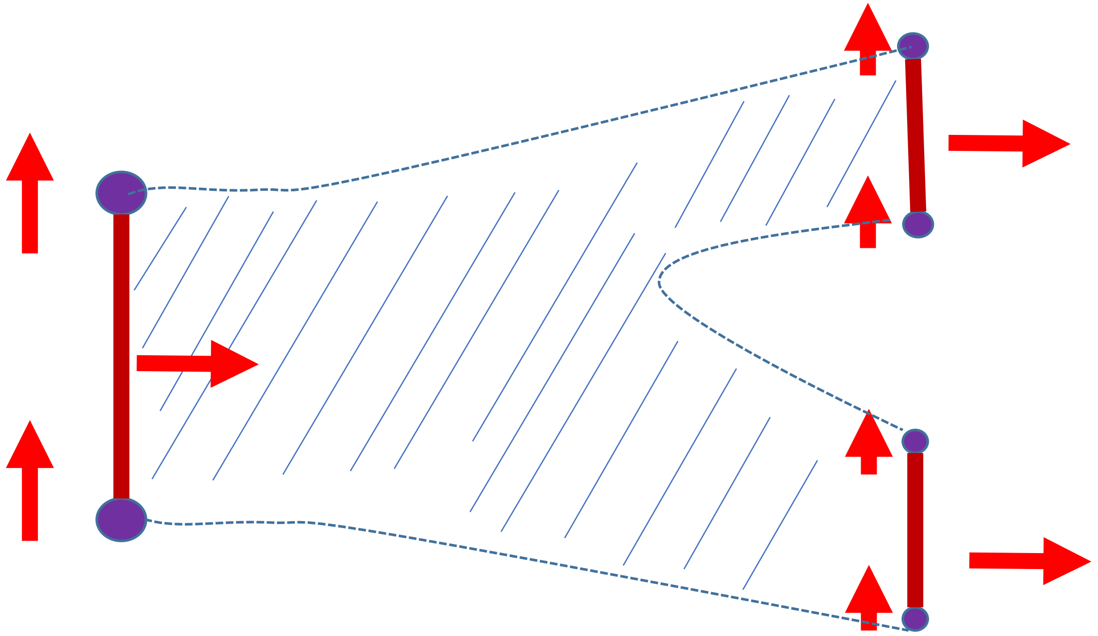
KITP, March 13, 2023

1. Preliminary Remarks
2. Fields Without Fields
3. Finite Homotopy Theories
4. Defects & Domain Walls
5. Symmetry Action Via Quiche
6. Composition Of Defects
7. Open Problems

The n-Category $Bord_n$

Definition: An n-category is a category C whose morphism spaces are n-1 categories.

Bord_n: Objects (0-morphisms) = 0-dimensional manifolds ;
1-Morphisms = 1d bordisms between 0-folds;
2-Morphisms = 2d bordisms between 1d bordisms; …



Field Theory Without Fields

Generalize functorial picture of field theory from Atiyah, Segal, ….

An n-dimensional ``field theory'' is a monoidal functor $F: Bord_n \to \mathcal{C}$

Result of ``doing the path integral.''

M: n-dimensional, compact $\Rightarrow F(M) \in \mathbb{C}$

N: (n-1)-dimensional, compact $\Rightarrow F(N) \in Obj(VECT)$

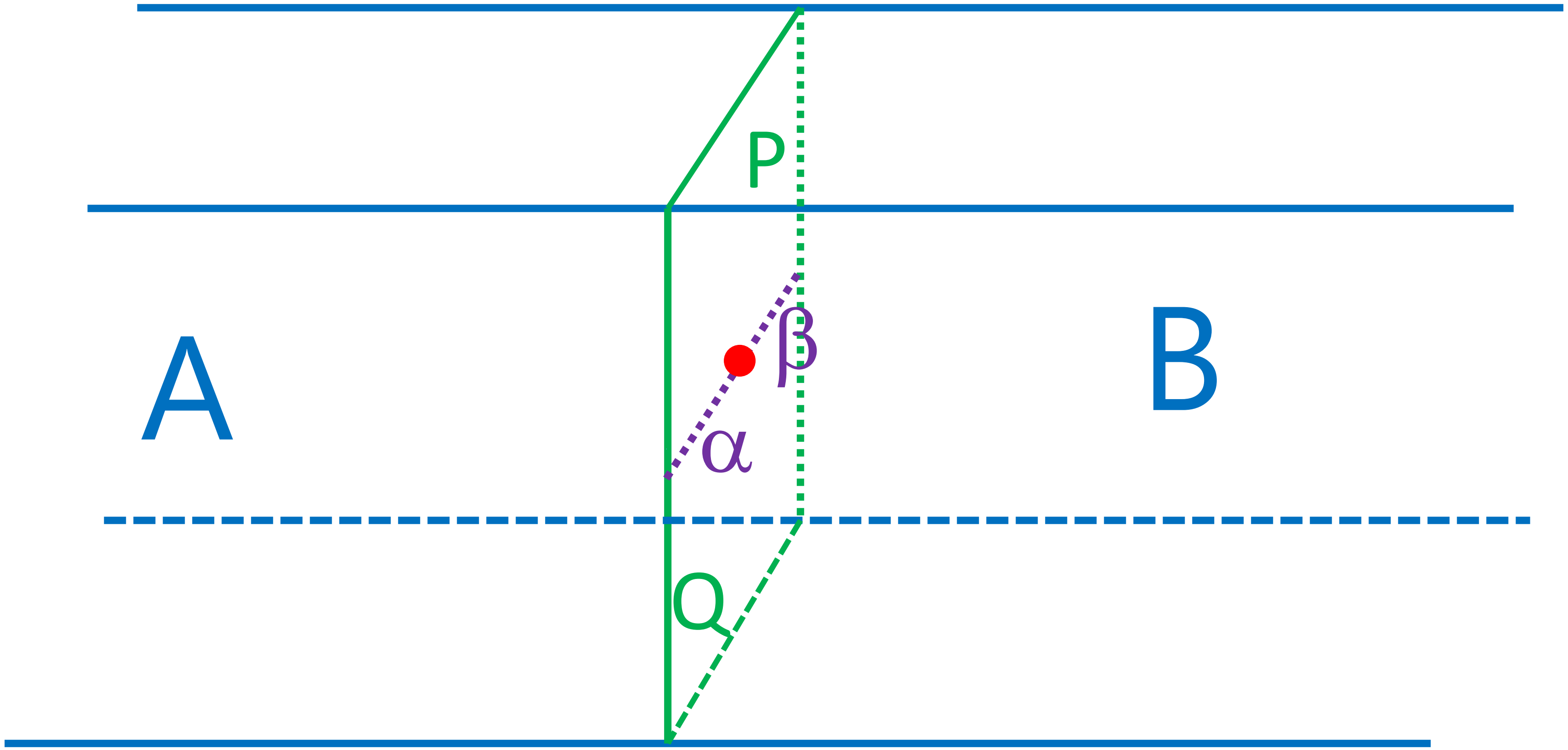
– not necessarily linear (e.g. $\mathcal{C} = HolSym$, Moore-Tachikawa 2011)

R: (n-2)-dimensional, compact $\Rightarrow F(R)$: more complicated mathematical object, e.g. object in a ``higher category.''

Defects Within Defects

Baez & Dolan, …., Lurie, …

Kapustin, arXiv:1004:2307.



Adding Fields: Background Fields

Fields should be locally defined on n-manifolds,
pull back under local diffeomorphisms, satisfy a sheaf property

Orientation, (s)pin structure, G-bundle with connection, Riemannian metric, differential cochain, foliation, …

Freed & Hopkins: [1301.5959, sec. 3]

Def: A $field$ $\mathcal{F}$ is a sheaf on Man_n^{op} valued in simplical sets Set_Δ

$$F: Bord_n(\mathcal{F}) \to \mathcal{C}$$

1. Preliminary Remarks
2. Fields Without Fields
3. **Finite Homotopy Theories**
4. Defects & Domain Walls
5. Symmetry Action Via Quiche
6. Composition Of Defects
7. Open Problems

Important and rich class of theories, underlies almost all our examples.

Kontsevich, Quinn, Freed, Turaev, …

TOPOLOGICAL QUANTUM FIELD THEORIES FROM COMPACT LIE GROUPS

DANIEL S. FREED, MICHAEL J. HOPKINS, JACOB LURIE, AND CONSTANTIN TELEMAN

Let G be a compact Lie group and BG a classifying space for G. Then a class in $H^{n+1}(BG;\mathbb{Z})$ leads to an n-dimensional topological quantum field theory (TQFT), at least for $n = 1, 2, 3$. The theory for $n = 1$ is trivial, but we include it for completeness. The theory for $n = 2$ has some infinities if G is not a finite group; it is a topological limit of 2-dimensional Yang-Mills theory. The most direct analog for $n = 3$ is an L^2 version of the topological quantum field theory based on the classical Chern-Simons invariant, which is only partially defined. The TQFT constructed by Witten and Reshetikhin-Turaev which goes by the name 'Chern-Simons theory' (sometimes '*holomorphic* Chern-Simons theory' to distinguish it from the L^2 theory) is completely finite.

The theories we construct here are extended, or multi-tiered, TQFTs which go all the way down to points. For the $n = 3$ Chern-Simons theory, which we term a '0-1-2-3 theory' to emphasize the extension down to points, we only treat the cases where G is finite or G is a torus, the latter being one of the main novelties in this paper. In other words, for toral theories we provide an answer to the longstanding question: What does Chern-Simons theory attach to a point? The answer is a bit subtle as Chern-Simons is an *anomalous* field theory of oriented manifolds.[1] This framing anomaly was already flagged in Witten's seminal paper [Wi]. Here we interpret the anomaly as an *invertible* 4-dimensional topological field theory $\mathscr{A}$, defined on oriented manifolds. The Chern-Simons theory is a "truncated morphism" $Z\colon 1 \to \mathscr{A}$ from the trivial theory to the anomaly theory. For example, on a closed oriented 3-manifold X the anomaly theory produces a complex line $\mathscr{A}(X)$ and the Chern-Simons invariant $Z(X)$ is a (possibly zero) element of that line. This is the standard vision

π −finite space $\mathcal{X}$: (Homotopy type of) a topological space with finitely many components, finitely many nonzero homotopy groups, all of which are finite groups.

$$
\begin{tikzcd}
 & \vdots & \\
K(\pi_3, q_3) \arrow[r] & \mathcal{X}^{(3)} \arrow[d] & \cdots \\
K(\pi_2, q_2) \arrow[r] & \mathcal{X}^{(2)} \arrow[d] \arrow[r] & BK(\pi_3, q_3) = K(\pi_3, q_3 + 1) \\
 & \mathcal{X}^{(1)} = K(\pi_1, q_1) \arrow[r] & BK(\pi_2, q_2) = K(\pi_2, q_2 + 1)
\end{tikzcd}
$$

$$\mathcal{X} = K(G,1) = BG \qquad G - \text{gauge theory}$$

$$\mathcal{X} = K(A, q+1) = B^{q+1}A$$

Will be used to describe ``q-form symmetry for group A “

$$\begin{array}{ccc} K(A,2) & \longrightarrow & \mathcal{X} \\ & & \downarrow \\ & & BG \end{array}$$

Classifying space of a ``2 −group”

Will be used to describe ``2-group symmetry“

π −finite spaces $\mathcal{X}$ also known as ``higher groups”

Want: an m-dimensional TFT $\sigma_{\mathcal{X}}^{(m)}$ where the (dynamical!) ``fields'' are, notionally, maps to $\mathcal{X}$, considered up to homotopy.

But we need to <u>*specify*</u> the codomain $\mathcal{C}$

TFT <u>*should*</u> really be denoted $\sigma_{\mathcal{X},\mathcal{C}}^{(m)}$ but in the paper it is written $\sigma_{\mathcal{X}}^{(m)}$

Monoidal m-category: $\otimes$ and $1_{\mathcal{C}}$

VECT: 1-category of fin. dim. complex vector spaces. $1_{\mathcal{C}} = \mathbb{C}$

ALG(VECT): Algebras, bimodules, bimodule maps

$$1_{\mathcal{C}} = \mathbb{C}$$

CAT: Categories, Functors, Natural transformations

With suitable $\otimes$, tensor unit $1_{CAT} = VECT$

$$\Omega\mathcal{C} := Hom_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$$

A monoidal $(m-1)$-category. We ALWAYS take:

$$\Omega^{m-1}\mathcal{C} = VECT \qquad \Omega^{m}\mathcal{C} = \mathbb{C}$$

In our paper *different* choices of 2-categories $\Omega^{m-2}\mathcal{C}$ are used in *different* examples…

$\Omega^{m-2}\mathcal{C}$ = CAT or $\Omega^{m-2}\mathcal{C}$ = ALG(VECT)

Latter choice leads to language of modules over an algebra

Example:

For 2d gauge theory for finite group G we get:

$\sigma^{(2)}_{BG}(pt) = REP(G)$ OR $\sigma^{(2)}_{BG}(pt) = \mathbb{C}[G]$

Depending on the choice of $\mathcal{C}$

$$\mathcal{C}' = ALG(\mathcal{C})$$

Objects (``0-morphisms’’) are
<u>*algebra objects*</u> in $\mathcal{C}$.

$\mathcal{C}$ is an m-category

$\mathcal{C}'$ is an $(m+1)$-category

ALG(VECT): 2-category of Algebras, bimodules, bimodule maps

ALG(CAT)=TENSCAT: 3-category of tensor categories:

Tensor categories, Bimodule categories, Bimodule functors, Natural transformations

For a compact k $-$fold M_k without boundary

$$\sigma_{\chi}^{(m)}(M_k) \in Obj(\Omega^k\, \mathcal{C}) \ ,0 \leq k \leq m$$

We'll now say something

about the values of $\sigma_{\chi}^{(m)}(M_k)$

for $k = m, m-1, m-2, m-3$

$\sigma_{\mathcal{X}}^{(m)}$ for $k = m - 1$

Notation: For any manifold M of any dimension $\mathcal{X}^M := Map(M, \mathcal{X})$

$\sigma_{\mathcal{X}}^{(m)}(M_{m-1})$ for M_{m-1} Compact $(m-1)$ –fold, without boundary

will be an *object* in $\Omega^{m-1}\mathcal{C} = VECT$

$\sigma_{\mathcal{X}}^{(m)}(M_{m-1})$: ``Space of states'' on the spatial slice M_{m-1}

N.B. Vectors determined by a bordism $\emptyset \rightarrow M_{m-1}$ might very well be zero, hence are not ``states'' in the sense of quantum theory.

Standard field theory: $\mathcal{X}^M = Map(M, \mathcal{X})$ is just the space of (scalar) fields in a sigma model with target $\mathcal{X}$

In standard scalar field theory we would have a Riemannian metric on M and the states would be described by normalizable wavefunctionals of the field configurations: $\Psi[\phi(x)]$ with $\phi \in \mathcal{X}^M$.

Hilbert space of states: $L^2(\mathcal{X}^M)$

Here we just work up to homotopy equivalence.

So we just want the vector space of *locally constant* functions on $\mathcal{X}^{M_{m-1}}$

$$\sigma_{\mathcal{X}}^{(m)}(M_{m-1}) := \mathrm{Fun}\left(\pi_0\left(\mathcal{X}^{M_{m-1}}\right)\right)$$

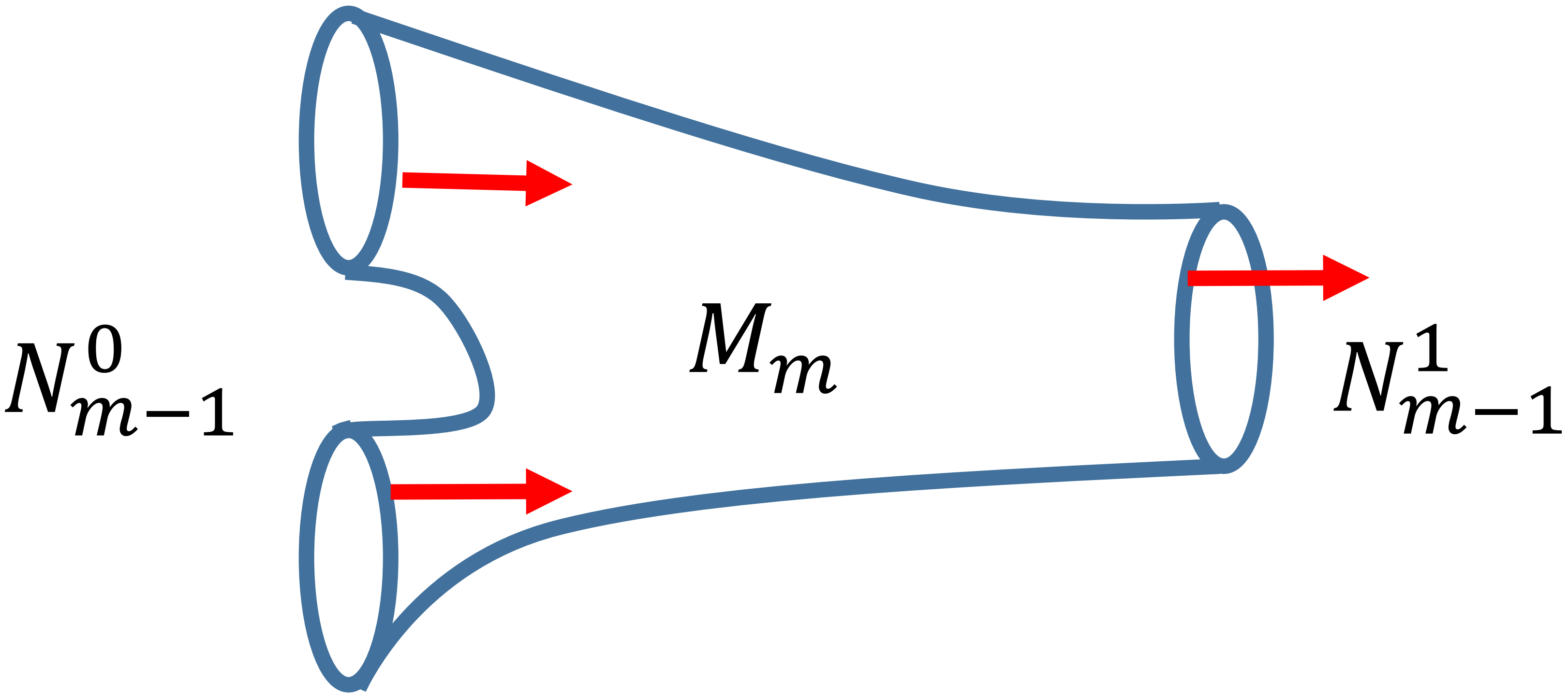
$$\sigma_{\mathcal{X}}^{(m)}(M_{m-1}) \coloneqq \mathrm{Fun}\big(\pi_0(\mathcal{X}^{M_{m-1}})\big)$$

Example: If $\mathcal{X} = K(A,q)$ then $\pi_0(\mathcal{X}^{M_{m-1}}) = H^q(M_{m-1}, A)$

If $\mathcal{X} = K(G,1)$ then $\pi_0(\mathcal{X}^{M_{m-1}}) = \{$ isom. Classes of principal G −bundles over M_{m-1} $\}$ so our ``statespace'' is functions of G-bundles over the spatial manifold.

``Quantization of the mapping space $\mathcal{X}^{M_{m-1}}$ ''

Amplitudes



$$M_m\!: \;\; N^0_{m-1} \to N^1_{m-1}$$

Correspondence Course

$$\begin{tikzcd} & S \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ R_1 & & R_2 \end{tikzcd}$$

Generalizes notion of functions from R_1 to R_2

We would like to compose correspondences:

$$\begin{tikzcd} & & & S_{13} \arrow[dl, "g_1"'] \arrow[dr, "g_3"] & & & \\ & & S_{12} \arrow[dl, "f_1"'] \arrow[dr, "f_{12}"] & & S_{23} \arrow[dl, "f_{23}"'] \arrow[dr, "f_3"] & & \\ & R_1 & & R_2 & & R_2 & \end{tikzcd}$$

Homotopy Fiber Product

$$\begin{tikzcd} & S_1 \times_h S_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ S_1 \arrow[dr, "f_1"'] & & S_2 \arrow[dl, "f_2"] \\ & R & \end{tikzcd}$$

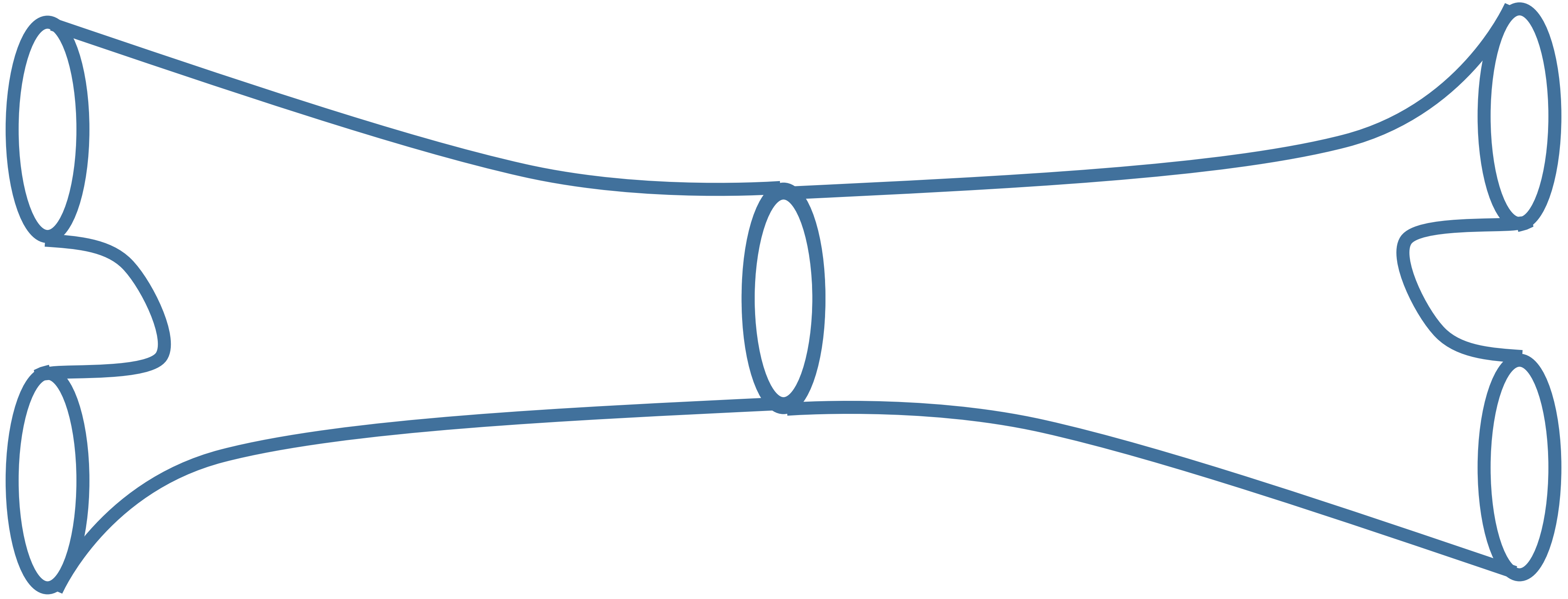
$$S_1 \times_h S_2 := \{\, (s_1, s_2, \gamma) : \ \gamma : f_1(s_1) \to f_2(s_2) \,\}$$

Amplitudes $M_m\colon\ N^0_{m-1} \to N^1_{m-1}$

$$
\begin{tikzcd}
 & \chi^{M_m} \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
\chi^{N^0_{m-1}} & & \chi^{N^1_{m-1}}
\end{tikzcd}
$$

$$p_{1,*} \circ p_0^* \ : \sigma_\chi^{(m)}(N^0_{m-1}) \to \sigma_\chi^{(m)}(N^1_{m-1})$$

$$p_{1,*}(\Psi)(h) := \sum_{[\phi]\in\pi_0\left(p_1^{-1}(h)\right)} \left(\prod_{i=1}^{\infty} \left|\pi_i(p_1^{-1}(h),\phi)\right|^{(-1)^i}\right) \Psi(\phi)$$



The fact that amplitudes compose properly follows naturally from properties of homotopy fiber products.

Partition function $(k = m)$ just take $N^0_{m-1} = N^1_{m-1} = \emptyset$

$$\sigma_{\mathcal{X}}^{(m)} \text{ for } k = m-2$$

$\Omega^{m-2}\mathcal{C} = CAT \quad \Rightarrow \quad \sigma_{\mathcal{X}}^{(m)}(M_{m-2})$ must be a category.

Should be some kind of locally ``constant vector spaces'' over $\mathcal{X}^{M_{m-2}}$

$$\sigma_{\mathcal{X}}^{m}(M_{m-2}) := VECT(\pi_{\leq 1}(\mathcal{X}^{M_{m-2}}))$$

``Quantization of the mapping space $\mathcal{X}^{M_{m-2}}$ ''

$\mathcal{X} = BG \Rightarrow \pi_{\leq 1}(\mathcal{X}^{M_{m-2}}) =$
Groupoid of principal G −bundles over M_{m-2}

$$\sigma_{\mathcal{X}}^{(m)} \text{ for } k = m - 3$$

$$\mathcal{C} = ALG(CAT) \ \& \ m = 3$$

$\mathcal{X} = BG \ : \ \sigma^{(3)}(pt) = VECT[G]$ as a tensor category:

$$(V_1 * V_2)_g = \bigoplus_{g_1 g_2 = g} (V_1)_{g_1} \otimes (V_2)_{g_2}$$

At each categorical level there is some ``quantization'' of a suitable correspondence of mapping spaces.

Not quantization in terms of symplectic geometry, but in the above homotopical sense.

Precise general formulation of ``quantization'' in this setting is given (to some extent) in FHLT sec. 8.4

In addition to the choice of $\mathcal{X}$ and $\mathcal{C}$ one can also consider a ``twisted'' construction based on a choice of cocycle $\lambda \in Z^m(\mathcal{X})$

For $\mathcal{X} = BG$ these would be Dijkgraaf-Witten theories.

e.g. $\sigma_{\mathcal{X},\lambda}^{(m)}(M_{m-1})$: Vector space of locally-constant sections of a line bundle $L_\lambda \rightarrow \mathcal{X}^{M_{m-1}}$

1 Preliminary Remarks

2 Fields Without Fields

3 Finite Homotopy Theories

4 Defects & Domain Walls

5 Symmetry Action Via Quiche

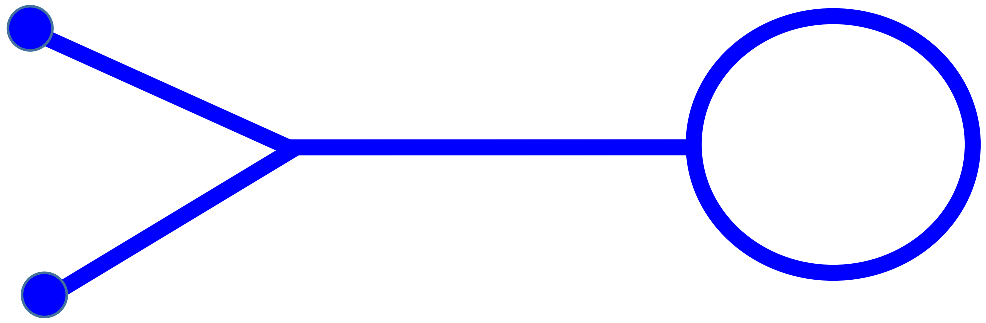
6 Composition Of Defects

7 Open Problems

We can extend FHLT to a general theory of defects in theories $\sigma_{X,\lambda}^{(m)}$

Suggests a general framework for defects in general TQFT's.

Defects are associated to subsets $Z \subset M_m$ where Z need not be smooth…



Questions To Answer:

What data are necessary to specify a defect?

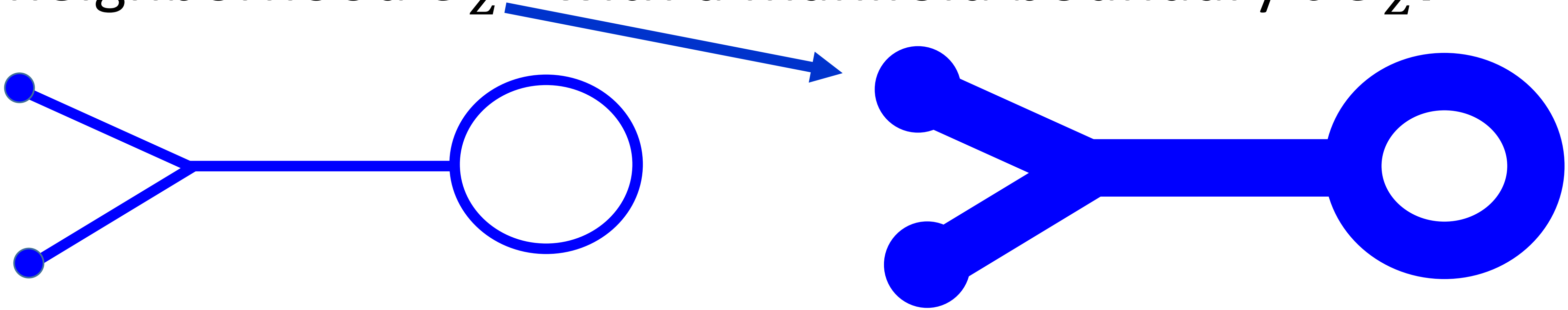
i.e. what are the ``labels'' carried by a defect?

Classical labels, semiclassical labels, quantum labels, local labels, global labels.

How does the presence of such defects affect the quantum values $\sigma_{\chi,\lambda}^{(m)}(M_k, \mathcal{D}(Z))$

Is there a product law on defects?
How do the labels compose?

Global labels: Surround Z by a ``small'' neighborhood U_Z with a manifold boundary ∂U_Z.



∂U_Z will be of codimension 1 so there is an associated statespace $\sigma(\partial U_Z)$

$$\delta_{\mathcal{D}(Z)} \in \sigma(\partial U_Z) = state\ space \in Obj(VECT)$$

i.e. $\delta_{\mathcal{D}(Z)}$ is a vector in the complex vector space $\sigma(\partial U_Z)$

Local Labels: When Z is a smooth submanifold we can hope to characterize the defect by examining the neighborhood of a point $p \in Z$.

Basic idea: Try to implement KK reduction along the linking sphere $S^{\ell-1}$ of $p \in Z$ where $\ell := cod(Z \subset M)$

Local Label $\delta_{\mathcal{D}(p)} \in Obj\left(\underbrace{Hom\left(1_{\Omega^{\ell-1}\mathcal{C}}, \sigma(S^{\ell-1}) \right)}_{(m-(\ell-1))-1 \,=\, (m-\ell)\,-\text{category}} \right)$

Sanity check: $\ell = m$. Local label = global label.

$$\Omega^{m-1}\mathcal{C} = VECT \qquad 1_{\Omega^{\ell-1}\mathcal{C}} = \mathbb{C}$$

$\sigma(S^{m-1}) =$ Vector space of ``states'' on S^{m-1}

$\delta_{\mathcal{D}(p)}$ is a vector in statespace on S^{m-1}:

State/operator correspondence.

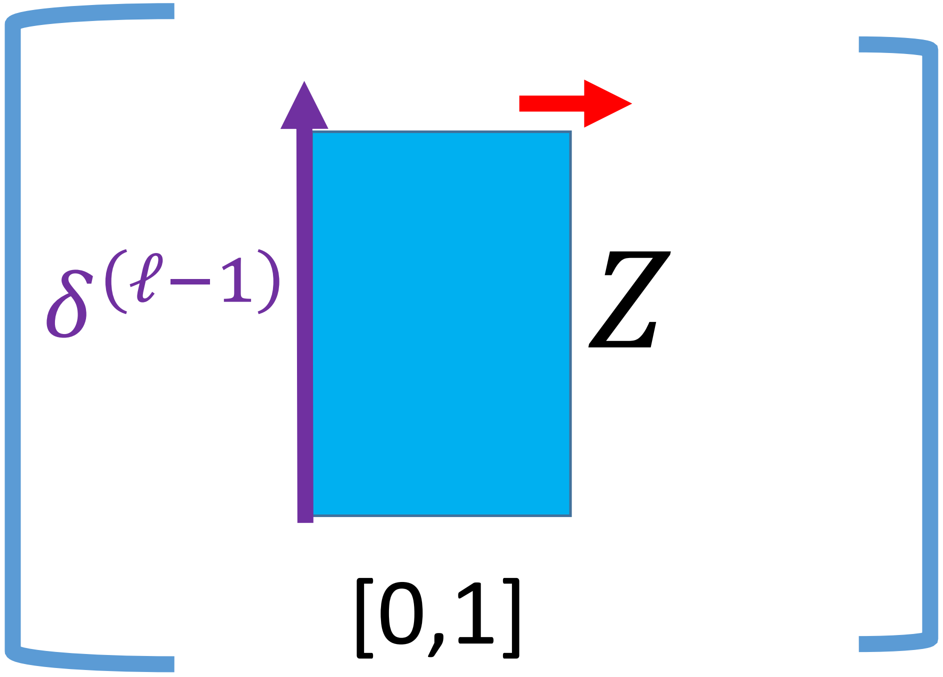
Already for line defects the choice of $\Omega^{m-2}\mathcal{C}$ makes a difference.

Claim: Z smooth with trivialized normal bundle then local labels can be integrated to global labels:

``KK Reduction": $\sigma^{(\ell-1)}(N) := \sigma\left(N \times S^{\ell-1}\right)$

Data of local defect defines a left boundary theory $\delta^{(\ell-1)}$ for $(m-\ell+1)$ − dimensional theory $\sigma^{(\ell-1)}$

$\sigma^{(\ell-1)}$



$\in Hom\left(\sigma^{(\ell-1)}(\emptyset), \sigma^{(\ell-1)}(Z)\right)$

Gives vector $\delta_{\mathcal{D}(Z)}$ in vector space

$$\sigma^{(\ell-1)}(Z) = \sigma\left(Z \times S^{(\ell-1)}\right)$$

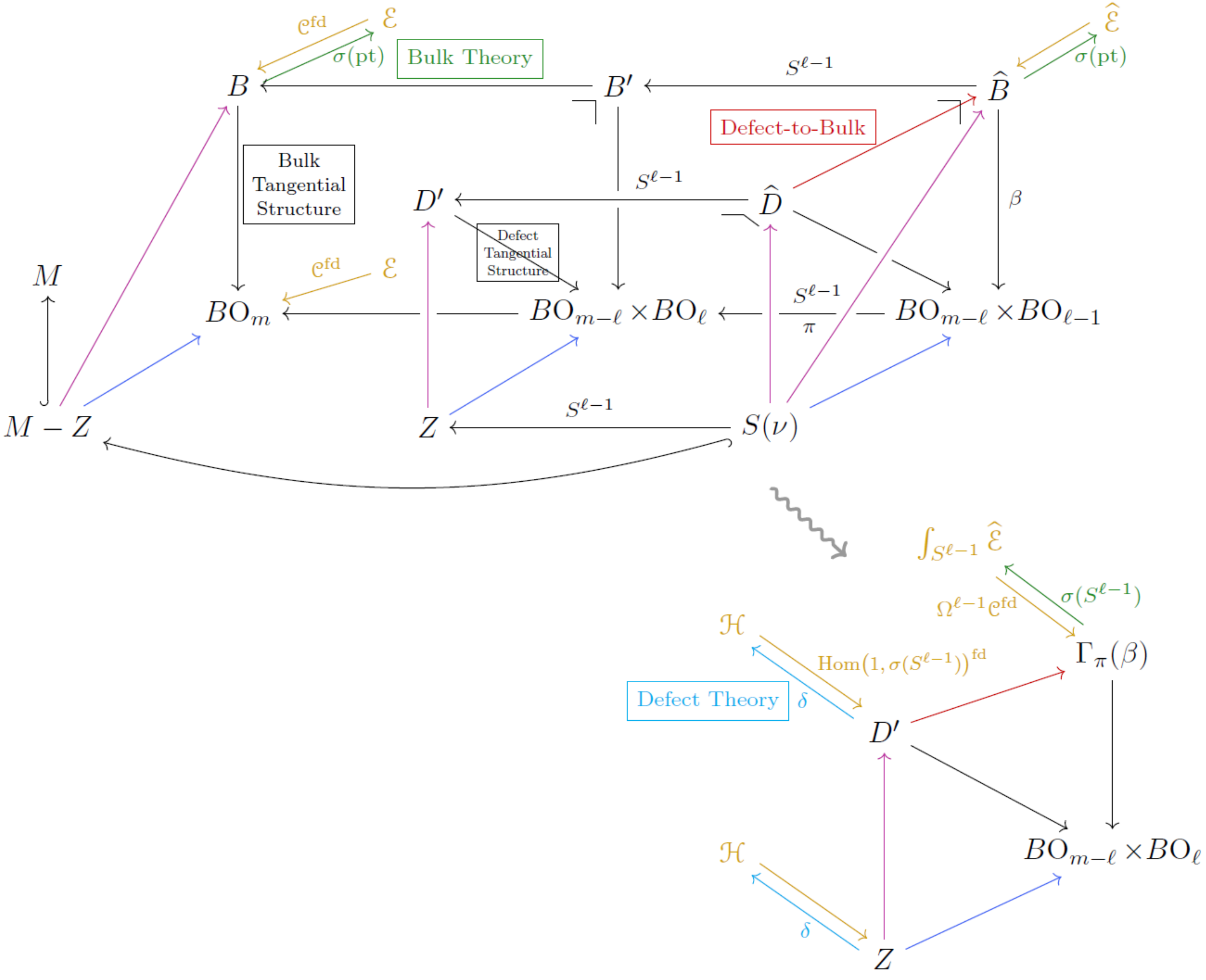


Figure 8. Local defect data, including tangential structures

When Z is not smooth we treat it as a stratified space and consider the links starting with the lowest codimension and then move up.

Semiclassical Data

For $\sigma_{\mathcal{X}}^{(m)}$ we can derive the local and global labels from `semiclassical data’’ (thought of as dynamical fields for the defect)

DEF: Semiclassical local defect data: $\psi: \mathcal{Y} \to \mathcal{X}^{S^{(\ell-1)}}$

Apply ``quantization procedure’’ of FHLT to the correspondence:

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl] \arrow[dr, "\psi"] & \\ * & & \mathcal{X}^{S^{(\ell-1)}} \end{tikzcd}$$

Simplest example: $\ell = m$: Point defect

Local label $\in Hom\left(\mathbb{C}, \sigma_{\mathcal{X}}^{(m)}(S^{m-1})\right)$ i.e. is a vector in

$$\sigma_{\mathcal{X}}^{(m)}(S^{m-1}) = Fun\left(\pi_0\left(\mathcal{X}^{S^{m-1}}\right)\right)$$

We *compute* this vector to be the pushforward of the function $\Psi = 1$ on $\mathcal{Y}$:

$$h \in \mathcal{X}^{S^{m-1}} \qquad \Psi(h) = \sum_{\phi \in \pi_0(\psi^{-1}(h))} \prod_{i=1}^{\infty} |\pi_i(\psi^{-1}(h), \phi)|^{(-1)^{i-1}}$$

Semiclassical Approach to Global Labels

For Defects In $\sigma_{\mathcal{X}}^{(m)}$

Mapping space $\mathcal{M}$ is space of pairs $(\phi_{blk}, \phi_{dfct})$

$\phi_{blk}: M \to \mathcal{X}$ $\qquad$ $\phi_{dfct}: Z \to \mathcal{Y}$

$$\begin{tikzcd} & \mathcal{Y} \arrow[dd, "\psi"] \\ Z \arrow[ur, "\phi_{dfct}"] \arrow[dr, "\phi_{blk,Z}"'] & \\ & \mathcal{X}^{S^{(\ell-1)}} \end{tikzcd}$$

``Quantization'' of $\mathcal{M}$ gives partition functions, ``statespaces'', amplitudes, etc. in the presence of the defect with local sc label ψ.

Domain Walls & Boundary Theories

Specialize to $\ell = 1$: $\mathcal{X}^{S^0} = \mathcal{X} \coprod \mathcal{X}$

and then generalize to give semiclassical data for a domain wall between FHT's:

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & \\ \mathcal{X}_1 & & \mathcal{X}_2 \end{tikzcd}$$

Boundary theories: $\mathcal{X}_1 = \emptyset$ OR $\mathcal{X}_2 = \emptyset$

``Dirichlet'': $\mathcal{Y} = pt.$ & choose component of $\mathcal{X}$

``Neumann'': $\mathcal{Y} = \mathcal{X}$ & $\psi = Identity.$

Names arise from the case of G –gauge theory with $\mathcal{X} = BG$

But lots of other boundary theories are possible….

Example: $\mathcal{X} = BG$ & $\lambda \in Z^m(BG, \mathbb{C}^*)$

$\sigma^{(m)}_{\mathcal{X},\lambda}$: m-dimensional Dijkgraaf-Witten theory.

A general set of semiclassical boundary conditions:

$f: H \to G$ & $\mu \in C^{m-1}(BH, \mathbb{C}^*)$: $\delta\mu = (Bf)^*(\lambda)$

$$\mathcal{Y} = BH \xrightarrow{\psi = Bf} BG$$

If $\partial M_m = N_{m-1}$ then the relevant mapping space is

$$\mathcal{M} = \left\{ \left(\phi_{blk}, \phi_{bdy}\right) : \begin{tikzcd} & BH \arrow[dd, "Bf"] \\ N \arrow[ur, "\phi_{bdy}"] \arrow[dr, "\phi_{blk}\big|_N"'] & \\ & BG \end{tikzcd} \right\}$$

Reduction of structure group on the boundary from G to H

Adding a (homotopical) sigma model $N \to G/\mathrm{f}(H)$, as expected when we break G −symmetry to H −symmetry on the boundary.

$$\mathcal{C} = ALG(CAT) = TENSCAT \ \ \& \ \ m = 3$$

$\sigma^{(3)}_{\chi,\lambda}\Big($ 

$\Big) \in Hom\left(1_{\mathcal{C}}\, , \sigma^{(3)}_{\chi,\lambda}(pt)\,\right)$

Will be a module category for the tensor category

$\sigma^{(3)}_{BG}(pt) = VECT[G]$. Will be $VECT[G/H\,]$

$$(V * W)_{gH} \ := \bigoplus\nolimits_{g', g''H} \ L_{g', g''H} \otimes V_{g'} \otimes W_{g''H}$$

$g'(g''H) = gH$ $\quad L_{g', g''H}$: Constructed from the cocycle λ

Defects Within Defects

One could go on to develop this formalism to describe defects within defects

Used in the paper to discuss composition of N/D and D/N boundary conditions, and duality domain walls.

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl] \arrow[dr] & \\ \mathcal{Y}_1 \arrow[dr] & & \mathcal{Y}_2 \arrow[dl] \\ & \mathcal{X} & \end{tikzcd}$$

Nontrivial Topological Effects

Classical labels: $\pi_0\left(\mathcal{X}^{S^{\ell-1}}\right)$ They are inadequate. Section 4.4.

$$m = 3, B^2A \to \mathcal{X} \to BG, \mathcal{C} = TENSCAT$$

$\sigma_{\mathcal{X}}^{(3)}(pt) = VECT[A^{\vee} \times G\,]$: Vector bundles over G with coeff's in $VECT[A^{\vee}]$

$$(W_1 * W_2)_g = \oplus_{g_1 g_2 = g}\, K_{g_1,g_2} \otimes W_{g_1} \otimes W_{g_2}$$

$K_{g_1,g_2} \to A^{\vee}$: A line bundle computed from Postnikov map $k\colon BG \to B^3A$

For a line in a D boundary theory the classical labels are $g \in G$

Quantum Labels: Object in $VECT[G \times A^{\vee}]$ with above composition.

1. Preliminary Remarks
2. Fields Without Fields
3. Finite Homotopy Theories
4. Defects & Domain Walls
5. Symmetry Action Via Quiche
6. Composition Of Defects
7. Open Problems

Generalized, categorical, noninvertible,… ``symmetries’’

We describe a framework for understanding these terms using the sandwich or quiche picture

Motivation 1:

TFT $\sim$ Algebra

$$\sigma_{BG}^{(2)}(pt) = \mathbb{C}[G]$$ Algebra

$$\sigma_{BG}^{(3)}(pt) = VECT[G]$$ $\otimes$ $-$category

(algebra object in CAT)

Boundary theory $\sim$ module for the algebra

$\Rightarrow$ Important notions from algebra: Regular representation,....

It is good to separate the notion of abstract group (algebra)from it’s action on a module.

Relations between algebra elements will universally be true in all modules.

Field theory: Compute relations among defects by computations within the TFT

Motivation 2:

4d Yang-Mills for compact group $G = SU(N)$

From Lagrangian we can't tell if the gauge group is
G or $G^{adj} = PSU(N)$ or G/A with $A \subset Z(G)$

F: $4d$ G gauge theory: partition function/Hilbert space:
Sum over all $G -$ bundles: Just need $c_2(P)$

For $PSU(N)$ gauge theory: To compute the partition function/Hilbert space:

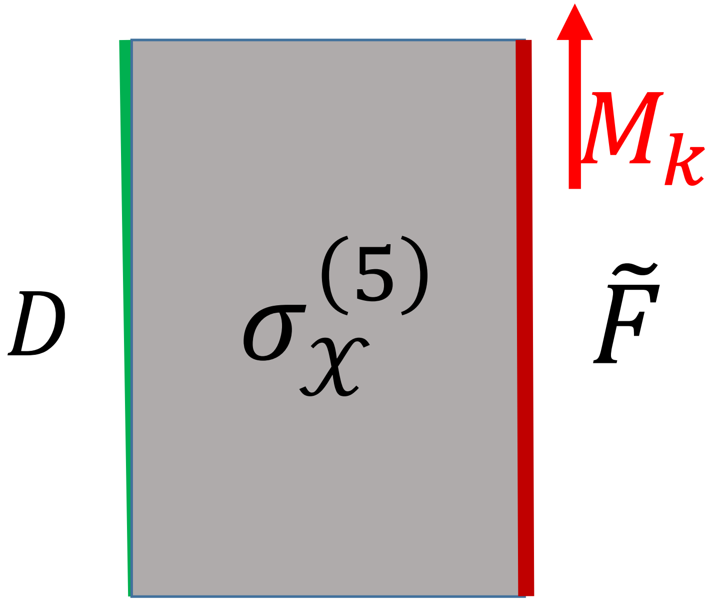
Sum over all G^{adj} − bundles: Need $c_2(P)$ …

AND $w_2(P) \in H^2(M; Z(SU(N)))$

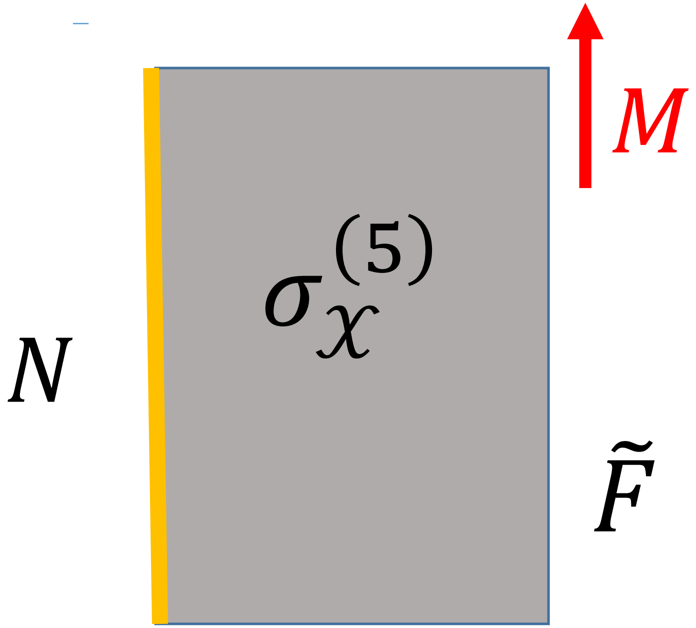
Obstruction to lifting the $PSU(N)$ bundle to an $SU(N)$ bundle is a $\mathbb{Z}_N$ − gerbe on the 4d spacetime.

PSU(N) gauge theory as a boundary theory for $\sigma_{\mathcal{X}}^{(5)}$ with $\mathcal{X} = B^2\mathbb{Z}_N$:

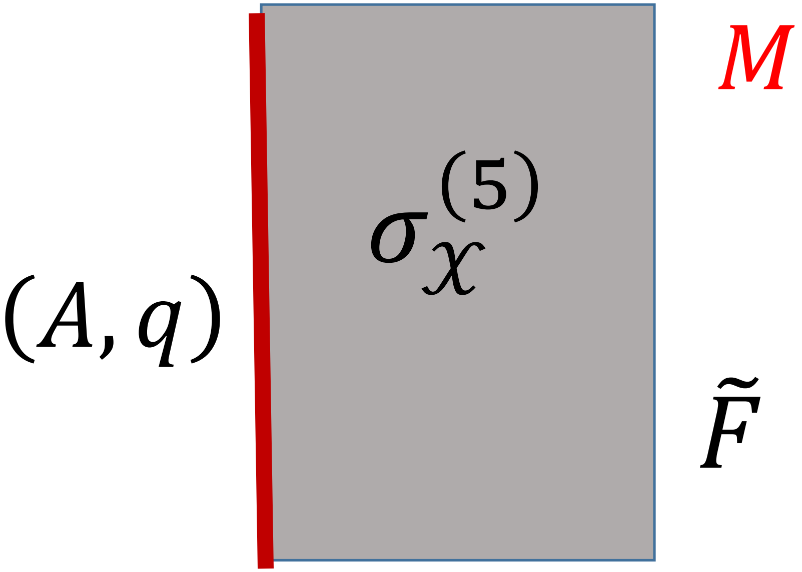
New boundary field: Isomorphism of restriction of $\mathbb{Z}_N$ gerbe from 4d with $\mathbb{Z}_N$ gerbe describing obstruction to lifting PSU(N) bundle to SU(N) bundle. So $\tilde{F}$ is not exactly $PSU(N)$ gauge theory.



This is just F := SU(N) gauge theory because the Dirichlet bc trivializes the A –gerbe, forcing us to couple YM only to *SU(N)*-bundles



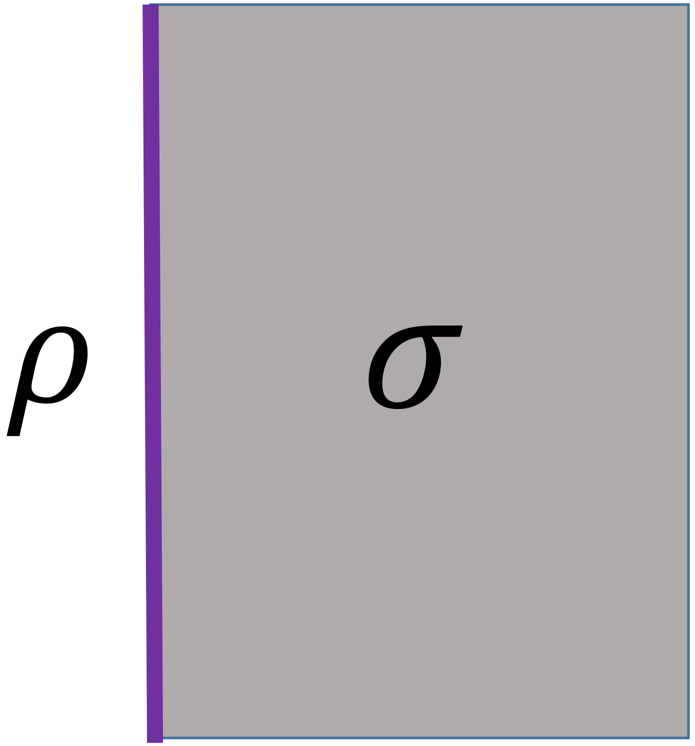
This PSU(N) gauge-theory



This is $SU(N)/A$ gauge-theory for $A \subset Z(SU(N))$ with topological coupling determined by $\mathcal{P}_q(w_2(P))$

Definition 1: An n-dimensional ***quiche*** is a pair (ρ, σ) with

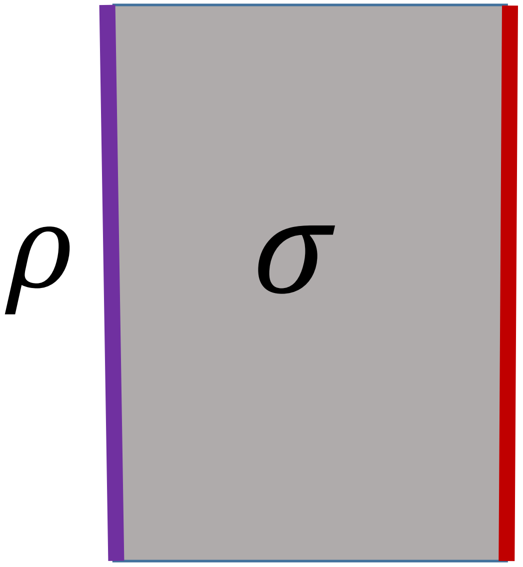
σ: $(n + 1)$ –dimensional TFT



ρ: n –dimensional ***topological*** boundary theory

(=``right module for σ''
$\Rightarrow$ ``right ∂ –theory'')

Definition 2: An action by the quiche (ρ, σ) on an n-dimensional field theory F, (not necessarily topological), is a ``left'' boundary theory (``left module for σ'') $\tilde{F}$ (not necessarily topological) and an isomorphism:



$$\tilde{F} \xrightarrow[\cong]{\theta} F$$

Note: Different θ's for same $(\rho, \sigma, \tilde{F})$ differ by elements of $Aut(F)$:

Partially justifies the viewpoint that this is a ``symmetry.''

Our first reference complaint:

Subject: sandwiches
From: Jeff Harvey <jaharvey@ >
Date: 9/16/2022, 11:58 AM
To: Gregory Moore <gwmoore@ >

An open-faced sandwich is not a quiche, it is a tartine.

What is wrong with you?

Example: G-Symmetry In Quantum Mechanics

F: n=1 dimensional field theory

$F(pt) = \mathcal{H}$ Hilbert space

$F([0,t]) = U(t) = e^{-t\,H} \in Hom(\mathcal{H},\mathcal{H})$

Actually: $F\big(germ(pt)\big) = (\mathcal{H}, H)$ Kontsevich & Segal

Suppose $\rho: G \to U(\mathcal{H})$ has image commuting with H

G need not be Abelian (need not be finite!)

Won't be sensitive to higher homotopy so take $\sigma \to \sigma_{BG}^{(2)}$

Need to define the left σ $-$module $\tilde{F}$

$\sigma($ 

$)$ $\in Hom_{ALG(VECT)}\big(\sigma(pt), \sigma(\emptyset)\big)$

$= Hom_{ALG(VECT)}(\mathbb{C}[G], \mathbb{C})$

$= \{\, \mathbb{C}[G] - \mathbb{C} \ \ bimodules \,\}$

$\sigma($ 

$)$ $:= \mathcal{H}$ as a <u>*left*</u> $\mathbb{C}[G]$ $-$module

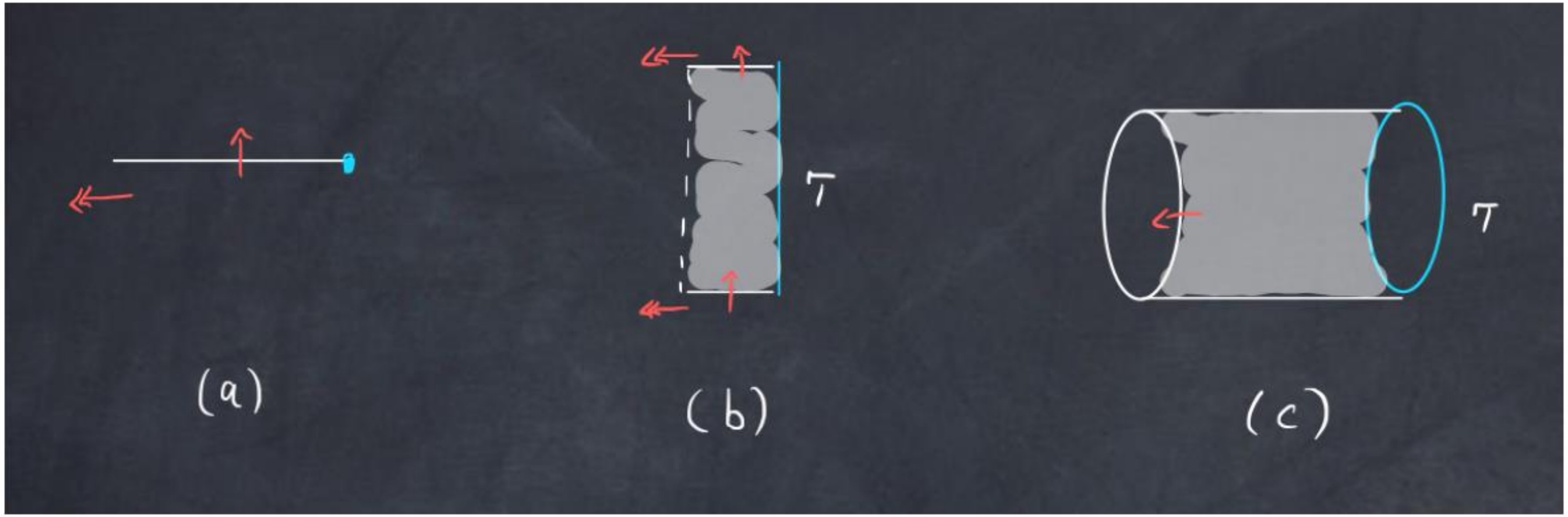


FIGURE 13. Three bordisms evaluated in (3.9) in the theory $(\sigma, \widetilde{F})$

(a) the left module ${}_{\mathbb{C}[G]}\mathcal{H}$

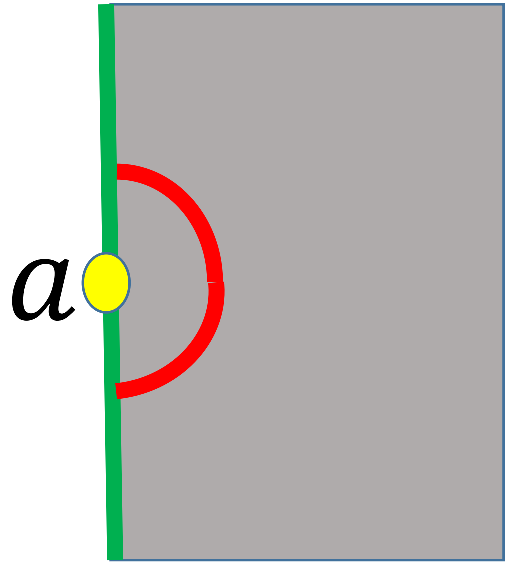
(b) $e^{-\tau H/\hbar}\colon {}_{\mathbb{C}[G]}\mathcal{H} \longrightarrow {}_{\mathbb{C}[G]}\mathcal{H}$

(c) the central function $g \longmapsto \mathrm{Tr}_{\mathcal{H}}\bigl(S(g)e^{-\tau H/\hbar}\bigr)$ on G

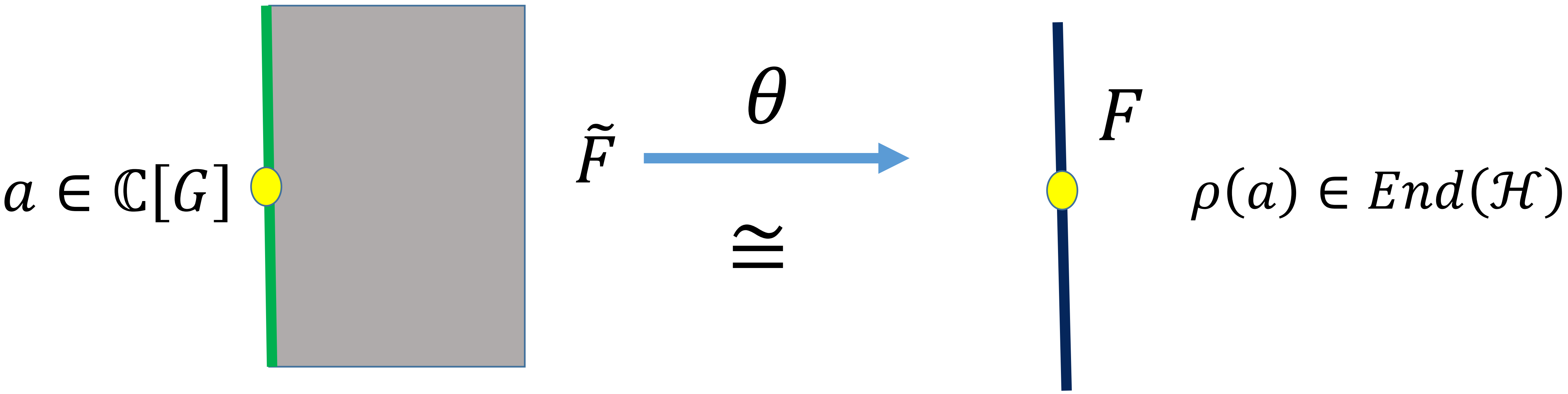
Quiche: $\left(\rho, \sigma_{BG}^{(2)}\right)$ with ρ = Dirichlet

$\sigma_{BG}^{(2)}($  $)$ $= \mathbb{C}[G]$ as a $\mathbb{C} - \mathbb{C}$ bimodule

Quantization of G –bundles on [0,1] trivialized at both ends: {Trivialized bundles} $= G$, so quantization gives functions on G.



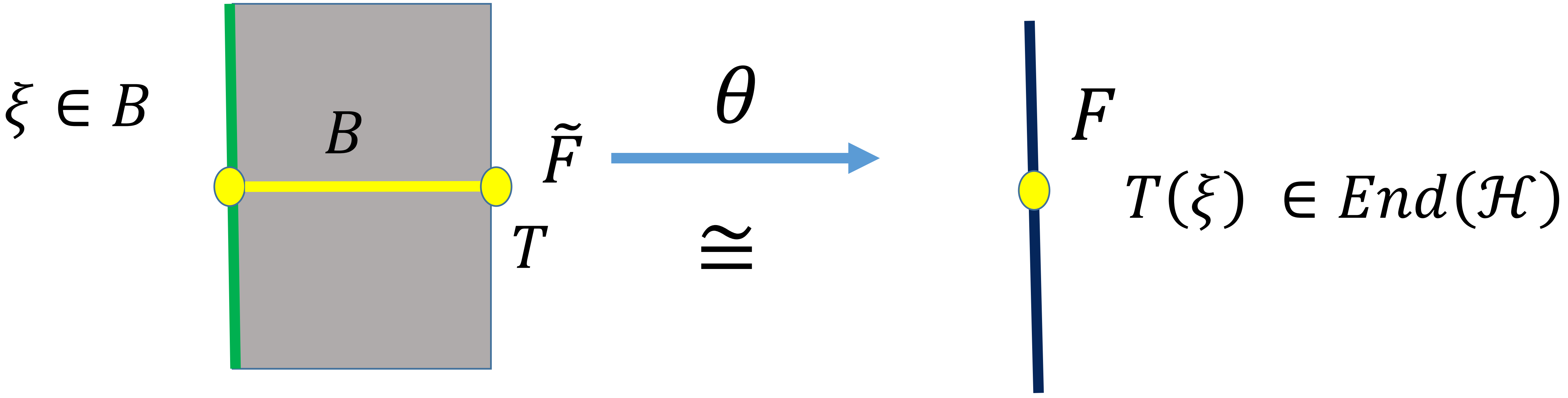
Topological ρ –defects in the Dirichlet boundary are labeled by $a \in \mathbb{C}[G]$



Insertion on topological boundary $\Rightarrow$

$\rho(a)$ commutes with $U(t) \Rightarrow$

$\rho(a)$ commutes with H



$$T: B \to End(\mathcal{H})$$

Not topological: Gives general operator on $\mathcal{H}$

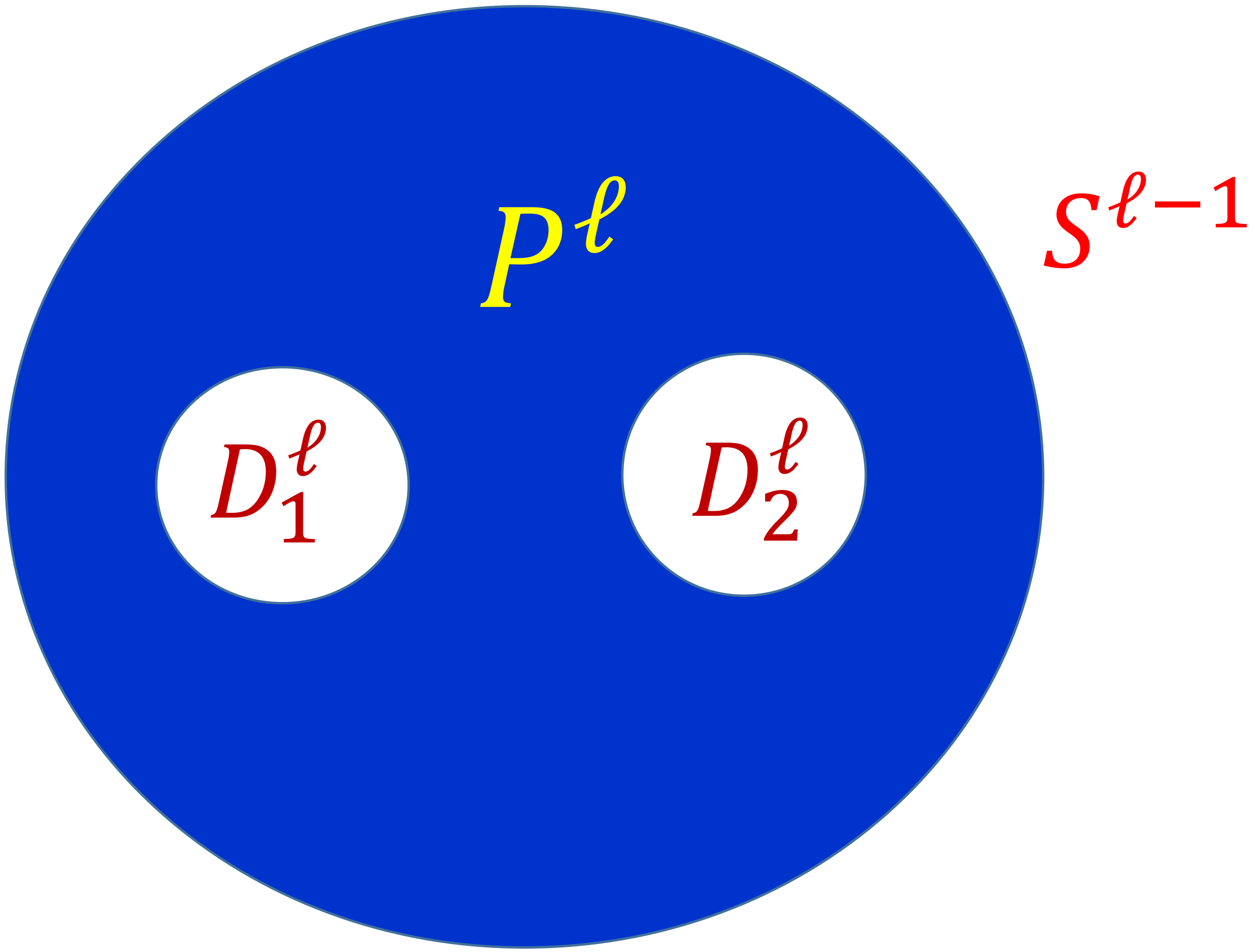
In general,….

all manipulations, e.g. OPE's of defects, etc. done within the TFT σ give universal relations independent of the field theory F on which the symmetry acts.

Some ``generalized topological symmetry'' operators on F might be very hard to describe within F but easy to describe in a quiche.

Example 4.4: Slice knot defects in 3d field theory that do not bound a disk.

Given defects $(\mathcal{D}_1, Z_1)$ & $(\mathcal{D}_2, Z_2)$ with Z_1, Z_2 codimension ℓ , parallel, trivialized normal bundles:



N.B. The product of cod ℓ defects is expressed in terms of cod ℓ defects.

In FHT, if the local defects are described by semiclassical data as above, this translates to the equation:

$$\begin{tikzcd}[column sep=small]
& (\mathcal{Y},\ \pi_1^*\mu_1 + \pi_2^*\mu_2 + \mu) \arrow[dl, dashed] \arrow[d, dashed, "g"] \arrow[ddr, "\psi = r_1 \circ g"] & \\
(\mathcal{Y}_1 \times \mathcal{Y}_2,\ \pi_1^*\mu_1 + \pi_2^*\mu_2) \arrow[d, "\psi_1 \times \psi_2"'] & (\mathcal{X}^P,\ \mu) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\
(\mathcal{X}^{S^{\ell-1}} \times \mathcal{X}^{S^{\ell-1}},\ \pi_1^*(\tau^{\ell-1}\lambda) + \pi_2^*(\tau^{\ell-1}\lambda)) & & (\mathcal{X}^{S^{\ell-1}},\ \tau^{\ell-1}\lambda)
\end{tikzcd}$$

$\mathcal{Y}$: homotopy fiber product of $\psi_1 \times \psi_2$ and r_0

Example: Domain walls in finite gauge theory

$$\mathcal{D}\left(\begin{tikzcd} & H_{12} \arrow[dl, "f_1"'] \arrow[dr, "f_{12}"] & \\ G_1 & & G_2 \end{tikzcd}\right) * \mathcal{D}\left(\begin{tikzcd} & H_{23} \arrow[dl, "f_{23}"'] \arrow[dr, "f_3"] & \\ G_2 & & G_3 \end{tikzcd}\right)$$

$$= \sum_{[g]} \mathcal{D}\left(\begin{tikzcd} & Z_{12}(g) \arrow[dl, "f_1\pi_1"'] \arrow[dr, "f_3\pi_3"] & \\ G_1 & & G_3 \end{tikzcd}\right)$$

$$[g] \in f_{12}(H_{12})\backslash G_2/f_{23}(H_{23})$$

$$Z_{(12)}(g) = \{\, (h_{12}, h_{23}) |\, f_{12}(h_{12})\; g\; f_{23}(h_{23})^{-1} = g \,\}$$

1. Preliminary Remarks
2. Fields Without Fields
3. Finite Homotopy Theories
4. Defects & Domain Walls
5. Symmetry Action Via Quiche
6. Composition Of Defects
7. Open Problems

Some Future Directions

Several examples in the paper show topological subtleties in labeling and composition laws of defects. Physical consequences?

Some applications are described in the paper: Duality defects, modular invariant combinations of left & rightmovers in 2d CFT, … It would be nice to see more.

Given $(\mathcal{X}, \lambda)$ can we find a ``traditional'' field theory (gauge fields, fermion fields, p-form fields, …) on which $\left(\rho, \sigma_{\mathcal{X}}^{(m)}\right)$ acts?

Some Future Directions

Extension to families of QFT's.
e.g. higher Berry curvatures?

Spacetime symmetries.
(Start with P,T-invariance)

Continuous symmetries?

Thanks for your attention!