Update On Susy Field Theory And Invariants Of Smooth Four-Manifolds

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StringMath@ICTP,
June 9, 2024

1 Motivation

2 Topologically Twisted d=4 N=2 Field Theory

3 Review Of The Donaldson-Witten Paradigm

4 d=5: ``K-theoretic Donaldson invariants''

5 Family Donaldson invariants

Glorious History Of
4d Field Theory & Four-Manifold Topology

Instantons (BPST) (1975)

Donaldson invariants (1982)

TQFT (1988)

Seiberg-Witten Invariants (1994)

Revolution of 1995

But not all questions are answered…

X: $\mathrm{d} = 4$, Smooth, compact, oriented, $\partial X = \emptyset$.

For **simplicity**: Connected & $\pi_1(X) = 0$

Essential in Donaldson & Seiberg-Witten theory:
X admits an almost complex structure

$\Longleftrightarrow$ $b_2^+(X)$ is odd

Misses ``half'' the world of four-manifolds!

We'll relax that condition in the final part of the talk.

Generalizations Of The Donaldson-Witten Paradigm

Other 4d N=2 theories

5d theories

6d theories

Coupling to background supergravity…

These *<u>might</u>* lead to truly new invariants that are independent of the Donaldson/SW invariants …

…. or not …

Today’s talk: Some of these generalizations lead to interesting issues in QFT, analytic number theory, and topology.

So it is worth thinking about.

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Witten's Homomorphism

$$\varphi_W : SO(4) \to \frac{Spin(4) \times SU(2)^R}{\langle\, (-1,-1,-1\,)\rangle}$$

$\langle(-1,-1,-1)\rangle \cong \mu_2$ acts trivially on vectormultiplets

Twisted SYM based on φ_W
gives function $Z_W : H_*(X) \to \mathbb{C}$
Z_W only depends on:

1. oriented diffeomorphism type of X
2. A choice of 't Hooft flux

For other $d = 4, N = 2$ theories one

<u>must</u>

choose other tangential structures to define the twisted theory.

$N = 2^*$ one <u>must</u> choose a spin-c structure

(Labastida-Marino; Manschot-Moore)

$Z: H_*(X) \to \mathbb{C}$ only depends on

1. oriented diffeomorphism type of X
2. A choice of 't Hooft flux

<u>AND</u> 3. an ``ultraviolet'' spin-c structure

What is the general tangential structure?

WIP with Vivek Saxena and Ranveer Singh gives generalization of top twisting to arbitrary renormalizable Lagrangian

d=4

N=2 theories

Goal: Further generalization to arbitrary class S theories.

Basic Data For A Renormalizable Lagrangian d=4 N=2 Theory

$\tilde{G}$: ss cpt Lie group, $\pi_1 = 0$

$\rho : \tilde{G} \to End(\mathcal{R})$: quaternionic

$$\Rightarrow \quad G^f := (\rho(\tilde{G}))' \subset O(\rho)$$

$$\tilde{G}^{cover} := \tilde{G} \times Spin(4) \times SU(2)^R \times G^f$$

Structure Group Of The Physical Theory

$$\tilde{G}^{cover} := \tilde{G} \times Spin(4) \times SU(2)^R \times G^f$$

Choice of subgroup $C^{phys} \subset Z(\tilde{G}^{cover})$
(acting trivially on $\mathcal{R}$)

Fields of the physical theory are based on a bundle P^{phys} with connection ∇^{phys} for structure group

$$G^{phys} := \tilde{G}^{cover} / C^{phys}$$

$\mathcal{F}^{phys}$: (Sheaf) of all fields

1. $\left(P^{phys}, \nabla^{phys}\right)$

2. Flat gerbe (1-form symmetry field)

3. Masses, couplings, hypermultiplet fields

Transfer Of Structure Group

$\varphi: G_1 \to G_2$ Homomorphism of groups

$\Rightarrow$ Can map a principal G_1 bundle $P_1 \to M$
to a principal G_2 bundle $\varphi_* P_1 \to M$

Transition functions $g_{\alpha\beta}: U_{\alpha\beta} \to G_1$ map to new
transition functions $\varphi\left(g_{\alpha\beta}\right): U_{\alpha\beta} \to G_2$

$$\varphi_*(P_1) := P_1 \times_{G_1} G_2$$

$$B\varphi: \; BG_1 \to BG_2$$

φ_* extends to category of bundles with connection:

∇_1 on P_1

$$\Rightarrow \quad \nabla_2 = \varphi_*(\nabla_1)\ on\ \varphi_*(P_1)$$

$$Hol(\nabla_2, \gamma) = \varphi\big(Hol(\nabla_1, \gamma)\big)$$

$$
\begin{tikzcd}
 & \left(\tilde{G} \times Spin(4) \times SU(2)^R \times G^f\right) \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "p_3"] \arrow[drr, "p_4"] & & \\
\tilde{G} & Spin(4) & SU(2)^R & G^f
\end{tikzcd}
$$

$p_{1,*}\left(P^{phys}, \nabla^{phys}\right) \coloneqq (P_1, A)$ Dynamical gauge field bundle

$$(p_2)_*\left(P^{phys}, \nabla^{phys}\right) = \left(Fr(X), \nabla^{LC}\right)$$

$p_{3,*}\left(P^{phys}, \nabla^{phys}\right) \coloneqq$ R-symmetry bundle & connection

$p_{4,*}\left(P^{phys}, \nabla^{phys}\right) \coloneqq$ Flavor bundle & connection

The Untwisted Theory Does Not Need A Spin Structure

Only need a principal bundle with structure group:

$$G^{phys} = (\tilde{G} \times Spin(4) \times SU(2)^R \times G^f)/C^{phys}$$

Choose $(p_2 \times p_3)(C^{phys}) = \langle(-1,-1,-1)\rangle$

$$\subset Z(\,Spin(4) \times SU(2)^R)$$

Such bundles can very well exist on nonspin manifolds

Role of the gerbe b is to allow introduction of 't Hooft flux

Best done via slab/sandwich/quiche/SymTFT picture

(see e.g. Dan Freed's StringMath2022 talk)

b has char. class $\mu(b) \in H^2\left(X, C^{grb}\right)$

$$C^{grb} := p_1\left(C^{phys}\right)$$

Existence of P^{phys} Puts A Condition On `` 1-Form Symmetry Fields''

$$\rho \cong \bigoplus_u \rho_u$$

Generic hm masses $\Rightarrow G^f \cong \mathbb{T}^f = \prod_u U(1)$

$$\left(p_{4,u}\right)_*\left(P^{phys}\right) = \mathcal{L}_u \quad \Rightarrow \quad c(u) = c_1(\mathcal{L}_u)$$

$$\mu(b) + w_2(X) + c(u) = 0 \in H^2\left(X, \mathbb{Z}_{n_u}\right)$$

Generalizes several special cases in the literature:

[Moore-Witten, Labastida-Marino,Manschot-Moore, Aspman-Furrer-Manschot]

$$
\begin{tikzcd}
\mathcal{F}^{phys} \arrow[r, "e^{-S}"] \arrow[d, "\pi"'] & \mathbb{C} \\
\mathcal{F}^{bck} \arrow[r, "Z = \pi_*:"] & \mathbb{C}
\end{tikzcd}
$$

Dynamical fields are the fiber of π

In general the fibration is nontrivial

Topological Twisting Via Transfer Of Structure Group

Upgrade φ_W to $\varphi^{tw}: \ G^{tw} \ \rightarrow \ G^{phys}$

Define $\left(P^{phys}, \nabla^{phys}\right) := (\varphi^{tw})_*(P^{tw}, \nabla^{tw})$

Such that $Z\left(\nabla^{tw,bck}\right)$ is invariant under continuous deformations of $\nabla^{tw,bck}$

because for such backgrounds

$\exists \ Q$ such that $Q^2 = 0$

$$
\begin{tikzcd}
\mathcal{F}^{tw} \arrow[r, "\varphi_*^{tw}"] \arrow[d, "\pi"'] & \mathcal{F}^{phys} \arrow[r, "e^{-S}"] \arrow[d, "\pi"'] & \mathbb{C} \\
\mathcal{F}^{tw,bck} \arrow[r] & \mathcal{F}^{bck} \arrow[r, "Z = \pi_*:"] & \mathbb{C}
\end{tikzcd}
$$

We construct explicit:

$$
G^{tw} = \left(\tilde{G} \times Spin(4) \times \mathbb{T}^f\right)/C^{tw} \xrightarrow{\varphi^{tw}} G^{phys}
$$

Z^{tw} only depends on

1. Diffeomorphism type of X

2. ‘t Hooft flux $\cong \mu(b)$

3. ``*Generalized Spin-c structure*'':

Principal bundle P^c for

$$(Spin(4) \times \mathbb{T}^f)/C^{tw,bck}$$

$$p_{1,*}P^c = Fr(X)$$

Just like generalized symmetries & their ‘t Hooft anomalies …

Topological twisting data should be an RG invariant, so it should be possible to extract $\mathcal{F}^{tw}$ from the LEET SW theory

will allow us to find twisting data for general class S theories.

Conjecture:

Any $d = 4, N = 2$ theory (with generic masses) admits a topological twisting defining a function $Z : H_*(X) \to \mathbb{C}$ that is *only* a function of

1. Oriented diffeomorphism type of X

2. Iso class of background gerbes for B^2C

3. ``*Generalized Spin-c structure*''

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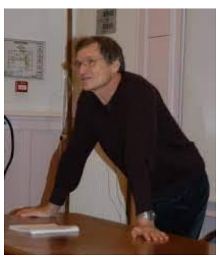
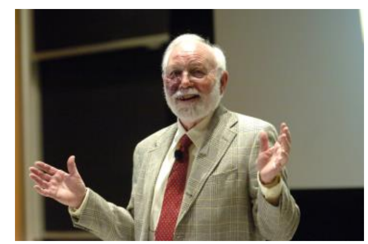
$$Q\ A_\mu = \psi_\mu \qquad Q\ \psi_\mu = -D_\mu \phi \qquad Q\ \phi = 0$$

$$A \in \mathcal{A}(P_1) \qquad \phi \in \Omega^0(X, ad\ P_1 \otimes \mathbb{C})$$

An important viewpoint in the section on families below



Baulieu & Singer, 1988



$\mathcal{G}$ −equivariant cohomology of $\mathcal{A}(P_1)$

$\mathcal{G} := Aut(P_1)$ Group of gauge transformations

Q −closed observables: $\mathcal{O}(pt) = Tr\left(\phi^2(pt)\right)$

$$\{K, Q\} = d \Rightarrow \quad \mathcal{O}_j := K^j \mathcal{O}$$

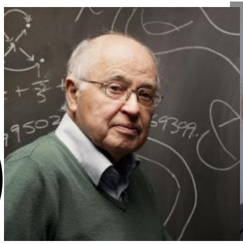
$\mathcal{O}(\Sigma_j) := \int_{\Sigma_j} \mathcal{O}_j$ only depends on $[\Sigma_j] \in H_j(X)$

Function on $H_*(X)$: $Z_W(\Sigma) = \langle e^{\mathcal{O}(\Sigma)} \rangle_{tw}$

Witten (1988):

Z_W independent of metric $g_{\mu\nu}$ on X

Witten (1988) & Atiyah& Jeffrey(1990)



$Z_W(\Sigma)$ path integral localizes to an integral over

$$\mathcal{M} \subset \mathcal{A}(P_1)/\mathcal{G}$$

$$\mathcal{M} := \{A \in \mathcal{A}(P_1): F(A)^+ = 0\}/\mathcal{G}$$

$$F^+ := \frac{1}{2}(F + * F)$$

Donaldson Invariants

Donaldson: $\mu: H_*(X) \to H^*(\mathcal{M})$

$$Z_D: H_*(X) \to \mathbb{Q}$$

$$Z_D(\Sigma) = \int_{\mathcal{M}} e^{\mu(\Sigma)}$$

$\mathcal{M}$ depends on $g_{\mu\nu}$, but Z_D does not

Main Statement

$$Z_W = Z_D \qquad =: Z_{DW}$$

Z_{DW} only depends on

1. oriented diffeomorphism type of X
2. A choice of ‘t Hooft flux

Evaluation Of $Z_{DW}(\Sigma)$

Z_{DW} independent of $g_{\mu\nu}$ on X

Consider metric $L^2 g_{\mu\nu}$ in the limit $L \to \infty$

$\Rightarrow$ Use Seiberg-Witten LEET

$$Z_{DW} = Z_{Coul}^{J} + Z_{SW}^{J}$$

Witten 94
Moore-Witten 97

$b_2^+ = 1$: J: oriented line $H^{2,+}(X) \subset H^2(X)$

One can deduce Z_{SW}^{J} from $\mathrm{Z}_{\mathrm{Coul}}^{J}$

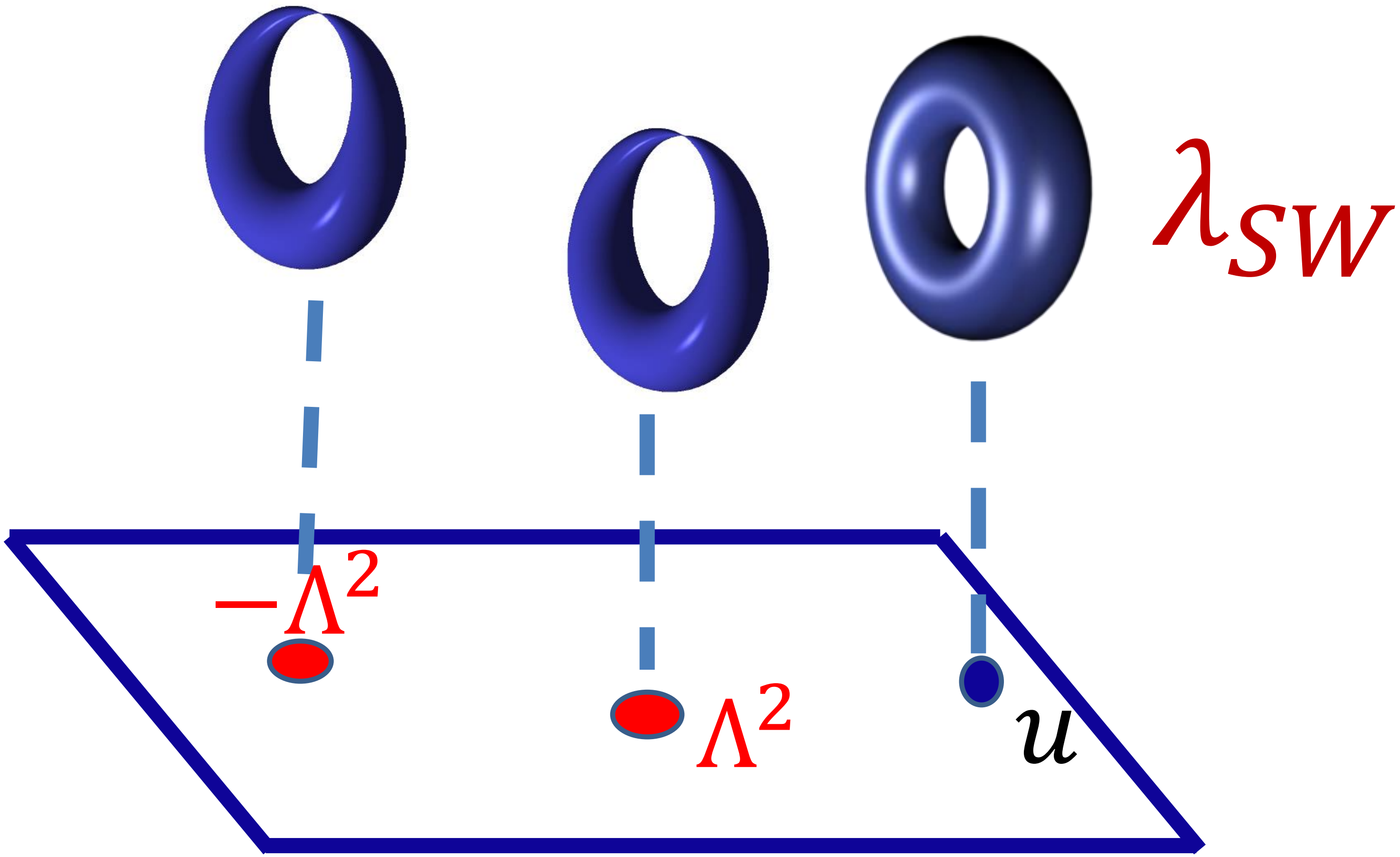
Moore-Witten, 1997

So start with $\mathrm{Z}_{\mathrm{Coul}}^{J}$

Z^{J}_{Coul} Integral over the Coulomb branch of vacua on $\mathbb{R}^4$,

$G = SU(2)$: parametrized by $u = \langle Tr\phi^2 \rangle$,

Measure is computed using SW LEET for U(1) VM



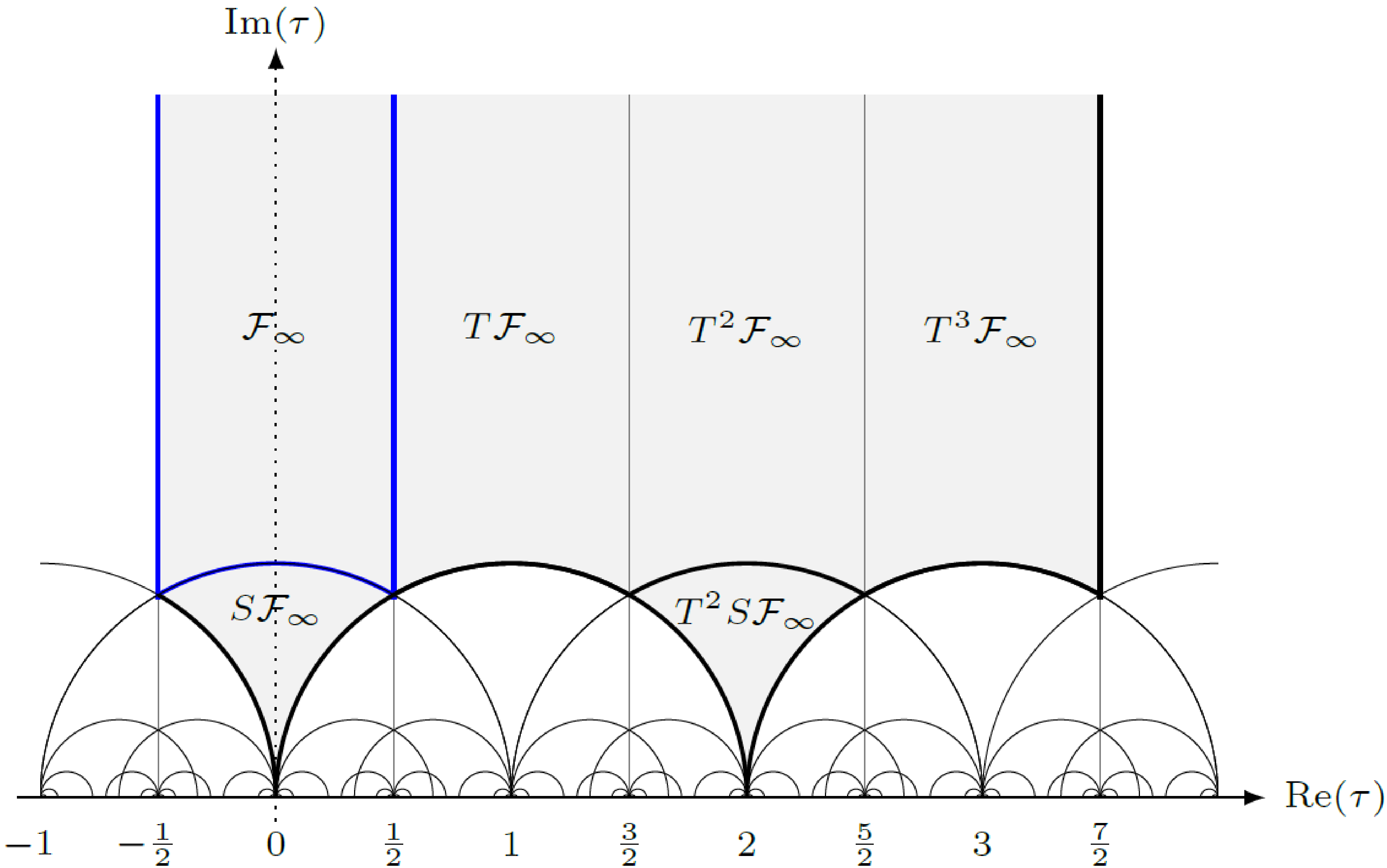
u-Plane Integral

SW94: Coulomb branch has a modular parametrization:

$$u(\tau) = \frac{\vartheta_2^4 + \vartheta_3^4}{2\,\vartheta_2^2\vartheta_3^2} = \frac{1}{8}q^{-\frac{1}{4}} + \frac{5}{2}q^{\frac{1}{4}} + \cdots$$

$$q = e^{2\pi i\tau}$$

Coulomb branch $\cong UHP/\Gamma^0(4)$



$$\Gamma^0(4) = \{\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} mod 4\} \subset SL(2,\mathbb{Z})$$

$$Z^J_{Coul}(\Sigma) = \int_{\mathcal{F}(\Gamma^0(4))} d\tau d\bar{\tau}\ \mathcal{H}(\tau) \frac{\partial}{\partial\bar{\tau}}\ \hat{G}^J(\tau,\bar{\tau},\Sigma)$$

$\hat{G}^J(\tau,\bar{\tau},\Sigma)$ Comes from the photon path integral

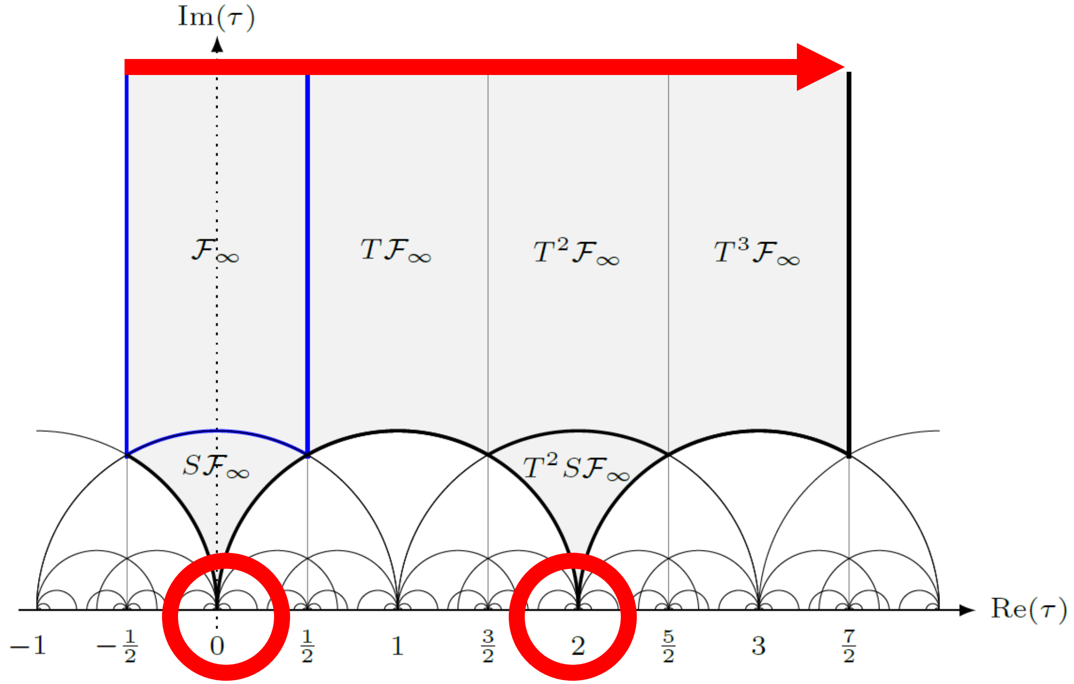
Not holomorphic in τ $(or\ u)$

Continuously metric dependent.

$\hat{G}^J(\tau,\bar{\tau},\Sigma)$: Is a mock Jacobi form

G. Korpas, J. Manschot, G. Moore, I. Nidaiev (2019)

$$Z^{J}_{Coul}(\Sigma) \sim \sum_{cusps} [\mathcal{H}(\tau) G^{J}(\tau,\Sigma)]_{q^0}$$



Cusps = $\infty \cup \;\; \tau = 0 \; [Monopole] \; \cup \;\; \tau = 2 \;\; [Dyon\,]$

Singularities at $u = \pm\Lambda^2$ spoil topological invariance.

Restore it with integral over ``Higgs branch vacua''

$$Z_{SW}^{J}(\Sigma) = \sum_{c \in Spin^c(X)} SW^J(c)\ f_c(\Sigma)$$

$SW^J(c) \in \mathbb{Z}$ Seiberg-Witten invariants

$f_c(\Sigma)$: Trigonometric function of Σ

computed from $Z_{Coul}^{J}(\Sigma)$

What Do The Other (Lagrangian) Theories Compute?

The path integral for topologically twisted Lagrangian theories localizes to intersection theory on $\mathcal{M}$

$\mathcal{M}$: moduli stack of the Nonabelian Seiberg-Witten equations

[Labastida-Marino 1997; Losev-Shatashvili-Nekrasov1997]

(instanton moduli space is a special case)

Nonabelian Monopole Equations

aka Nonabelian Seiberg-Witten Equations

$$G^{tw} = \left(\tilde{G} \times Spin(4) \times \mathbb{T}^f\right)/C^{tw}$$

$p_{2,*}(P^{tw}, \nabla^{tw}) = (Fr(X), \nabla^{LC})$ is <u>fixed</u>

$\mathcal{R} = V \oplus V^*$, where V is a complex representation of $\tilde{G}$

$V \otimes S^+$: Representation of G^{tw}

$\mathcal{V}$: Associated bundle: $M \in \Gamma(\mathcal{V})$

$$F^+ = q(M, \bar{M}) \qquad \gamma \cdot D\, M = 0$$

! Need an orientation on $\mathcal{M}$

Discussions with D. Freed and M. Hopkins seek to describe the 5d invertible theory whose Hilbert space is the orientation line of $\mathcal{M}$.

Freed-Hopkins construct a KO class T on $\mathcal{A}(P_1)/\mathcal{G}$ that restricts to the tangent bundle on the instanton moduli stack,

using elaborate topological methods compute $w_i(T)$ for $i = 1,2$

!!! turns out to be cohomological in $w_i(X), w_i(P), \mathrm{i} = 1, \dots, 4$

a surprise since $w_i(T)$ mod-two indices in KO theory.

And is consistent with a mod-two index computation I worked out with E. Witten (January 2024).

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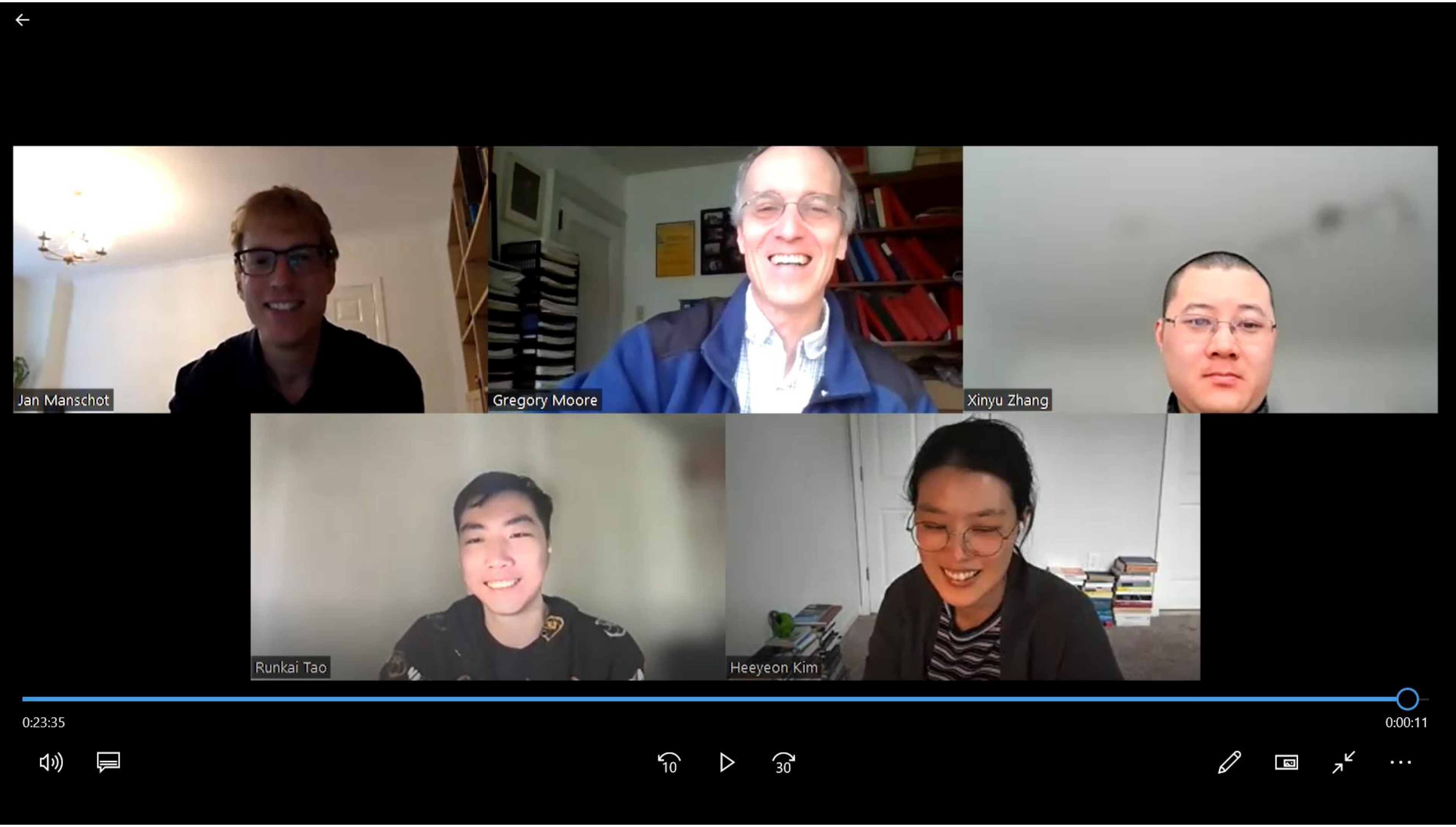
``K-Theoretic Donaldson Invariants''



$\mathcal{N} = 1$ 5D SYM

$$G^{phys} = (SU(2)^g \times Spin(4) \times SU(2)^R \times U(1)^{(I)})/\mathbb{Z}_2$$

Vectormultiplet: $V = (A, \sigma, \lambda_\alpha^A)$

$$J^{(I)} = tr\, F^2 \;\Rightarrow\; V^{(I)} = \left(A^{(I)}, \sigma^{(I)}, \dots\right)$$ Seiberg 1995

$A^{(I)}$ a connection on $P^{(I)}$:

a principal $\mathrm{U}(1) - \mathrm{bundle\ over}$ X_5

Electric Coupling Of Instanton Particle

$$\int_{X_5} A^{(I)} \wedge J^{(I)} = \int_{X_5} A^{(I)} \wedge tr\, F^2$$

Supersymmetrization gives entire 5d SYM action coupled to background $V^{(I)}$

$$S[V; V^{(I)}] = \int_{X_5} A^{(I)} \wedge tr\, F^2 \; + \; \int \sigma^{(I)}\, tr\, F * F \; + \cdots$$

$$\sigma^{(I)} \sim g_{5d,SYM}^{-2}$$

$$
\begin{aligned}
S = &\ \frac{\kappa}{8\pi^2}\int_{X_5} iA^{(I)} \wedge \mathrm{tr}(F \wedge F) \\
&- \frac{\kappa}{8\pi^2}\int_{X_5} d^5x\,\sqrt{g}\,\mathrm{tr}\bigg[\frac{1}{2}{\lambda^{(I)}}^A\Gamma^{mn}\lambda_A F_{mn} - \frac{1}{4}\lambda^A\Gamma^{mn}\lambda_A F^{(I)}_{mn} + \frac{i}{2}\lambda^A\lambda^B D^{(I)}_{AB} + i{\lambda^{(I)}}^A\lambda^B D_{AB} \\
&+ \sigma^{(I)}\left(\frac{1}{2}F^{mn}F_{mn} + D^m\sigma D_m\sigma + \frac{1}{2}D^{AB}D_{AB} + i\lambda_A\Gamma^m D_m\lambda^A + i\lambda_A[\sigma, \lambda^A]\right) \\
&+ \sigma {F^{(I)}}_{mn}F^{mn} + 2\sigma D^m\sigma^{(I)}D_m\sigma + \sigma {D^{(I)}}^{AB}D_{AB} + i\sigma {\lambda^{(I)}}_A\Gamma^m D_m\lambda^A \\
&+ i\sigma\lambda_A\Gamma^m D_m{\lambda^{(I)}}^A\bigg].
\end{aligned}
$$

Is $\int_{X_5} A^{(I)} \wedge trF^2$ globally well-defined?

Yes!

$\left(P^{(I)}, A^{(I)}\right)$ corresponds to an element of $\breve{H}^2(X_5)$

$CS(A)$ corresponds to an element of $\breve{H}^4(X_5)$

$$\breve{H}^2(X_5) \times \breve{H}^4(X_5) \to \breve{H}^1(pt) \cong \mathbb{R}/\mathbb{Z}$$

Is $\int_{X_5} A^{(I)} \wedge trF^2$ globally well-defined?

Well…. almost …

When $w_2(P_1) \neq 0$ instanton particles have fractional charge so the exponentiated electric coupling can have an anomaly

Expect: Cancelled by anomaly in the fermion determinant.
(Discussion with Freed & Hopkins)

Now take $X_5 = X_4 \times S^1$

$\theta \sim \oint_{S^1} A^{(I)}$ const. on X_4

$P^{(I)}$ & $F^{(I)}$ pulled back from X_4

We have a *partial topological twist* based on transfer of structure group

$$\varphi\colon\ \mathbb{Z}_2 \times SO(4) \to (Spin(5) \times SU(2)^R)/\mathbb{Z}_2$$

Background fields: $\varphi_*(\nabla^{LC})$

Topological on X_4 but a nontopological, spin, theory on S^1

$$Q^2 = \partial_t$$

SQM With Target $\mathcal{M}$

Topological on $X_4 \Rightarrow$

Can shrink $X_4 \Rightarrow$

Describe the twisted theory in terms of

SQM with target space

the moduli stack of instantons

[Nekrasov, 1996]

Potential Global Anomalies

5D: $Pfaff\left(\gamma \cdot D_{X_4 \times S^1}\right)$ not well-defined on $\mathcal{A}/\mathcal{G}$

1D: $Pfaff(\gamma \cdot D_{S^1})$ not well-defined on $L\mathcal{M}$

Anomaly if $\mathcal{M}(X_4)$ is not spin

[Witten ’85; Atiyah ‘85]

All controlled by ``the same’’

6d mod-two index in KO theory

A computation with E. Witten gives a useful formula for it.

Special case of our result:
X admits an ACS and $G = PSU(2N)$:

$\mathcal{M}$ is spin iff $w_2(P_1) \cdot w_2(TX_4) = 0$

Note: Independent of instanton number k .

Compatible with more general formula from Freed-Hopkins for $w_i(T)$.

Line Bundle On $\mathcal{M}$

$$\int_{X_4 \times S^1} A^{(I)} \wedge J^{(I)} = \int_{X_4 \times S^1} A^{(I)} \wedge tr\, F^2$$

$\Rightarrow$ SQM($\mathcal{M}$) couples to a
``line bundle with connection'' $\mathcal{L}^{(I)} \to \mathcal{M}$

$$\check{H}^2(X_4) \times \check{H}^4(X_4 \times \mathcal{A}(P_1)/\mathcal{G}) \rightarrow \check{H}^2(\mathcal{A}(P_1)/\mathcal{G})$$

w/ Kim, Manschot, Tau, Zhang:

$\mathbb{Z}_2$: Global anomaly : $w_2(P_1) \cdot n^{(I)} \neq 0$

$$n^{(I)} := c_1\left(P^{(I)}\right) \in H^2(X_4, \mathbb{Z})$$

(Conjectural) Anomaly Cancellation

$$w_2(P^g) \cdot \left(w_2(X_4) + n^{(I)}\right) = 0$$

$$Pfaff(\gamma \cdot D_{S^1}) \cdot e^{i \oint \mathcal{A}^{(I)}}$$

Conjecture $\mathcal{A}^{(I)}$:
``U(1) gauge field'' of a Spin-c structure on $\mathcal{M}$

One can show
X_4 admits an ACS $\Rightarrow$ $\mathcal{M}$ is spin-c

When global anomalies cancel, the partition function is a function of

1. Oriented diffeomorphism type of X_4

2. ‘t Hooft flux

3. $n^{(I)} := c_1\left(P^{(I)}\right)$

4. $\mathcal{R} = R\,\Lambda_{4d}$

$$\mathcal{R}^4 = \exp\left[-8\,\pi^2 \frac{R}{g_{5d,YM}^2} + i\,\theta\right]$$

$$Z(\mathcal{R}, n^{(I)}) =$$

$$\sum_{k=0}^{\infty} \mathcal{R}^{\frac{d_k}{2}} \; Tr_{\mathcal{H}_k}\{(-1)^F \exp(-\mathcal{R}\, D^2_{\mathcal{L}^{(I)}} \;)\}$$

$$d_k = \dim_{\mathbb{R}} \mathcal{M}_k \; = 4h^{\vee} k \; - \dim G \, \frac{\chi + \sigma}{2}$$

In good cases, the Witten index is the L^2 index of the Dirac operator $D_{\mathcal{L}^{(I)}}$

$\Rightarrow$ ``K-theoretic Donaldson invariants''

[Nekrasov, 1996; Losev, Nekrasov, Shatashvili, 1997]

2006:

K-THEORETIC DONALDSON INVARIANTS VIA INSTANTON COUNTING

LOTHAR GÖTTSCHE, HIRAKU NAKAJIMA, AND KŌTA YOSHIOKA

To Friedrich Hirzebruch on the occasion of his eightieth birthday

Abstract. In this paper we study the holomorphic Euler characteristics of determinant line bundles on moduli spaces of rank 2 semistable sheaves on an algebraic surface X, which can be viewed as K-theoretic versions of the Donaldson invariants. In particular if X is a smooth projective toric surface, we determine these invariants and their wall-crossing in terms of the K-theoretic version of the Nekrasov partition function (called 5-dimensional supersymmetric Yang-Mills theory compactified on a circle in the physics literature). Using the results of [43] we give an explicit generating function for the wall-crossing of these invariants in terms of elliptic functions and modular forms.

2019:

VERLINDE FORMULAE ON COMPLEX SURFACES I: K-THEORETIC INVARIANTS

L. GÖTTSCHE, M. KOOL, AND R. A. WILLIAMS

Abstract. We conjecture a Verlinde type formula for the moduli space of Higgs sheaves on a surface with a holomorphic 2-form. The conjecture specializes to a Verlinde formula for the moduli space of sheaves. Our formula interpolates between K-theoretic Donaldson invariants studied by the first named author and Nakajima-Yoshioka and K-theoretic Vafa-Witten invariants introduced by Thomas and also studied by the first and second named authors. We verify our conjectures in many examples (e.g. on K3 surfaces).

What We Add

We study $Z\left(\mathcal{R}, n^{(I)}\right)$ using both the Coulomb branch integral

and,

for X_4 a toric Kahler manifold, toric localization.

Total partition function is a sum of two terms

$$Z^J(\mathcal{R}, n^{(I)}) = \mathrm{Z}^J_{\mathrm{Coul}}(\mathcal{R}, n^{(I)}) + Z^J_{SW}(\mathcal{R}, n^{(I)})$$

$\mathrm{Z}^J_{\mathrm{Coul}}(\mathcal{R}, n^{(I)})$: Coulomb branch integral of the 4d theory from reduction

$$X_5 = X_4 \times S^1 \to X_4$$

One can deduce Z^J_{SW} from $\mathrm{Z}^J_{\mathrm{Coul}}$

Coulomb Branch For $X_5 = X_4 \times S^1$

For 5d SYM gauge group of rank 1:

Coulomb branch = $\mathbb{C}$

parametrized by :

$$U = \langle\, Tr_F P exp \oint_{S^1} (\sigma dx^5 + i\, A) \,\rangle$$

``1-form symmetry'' $\wp : U \to -U$

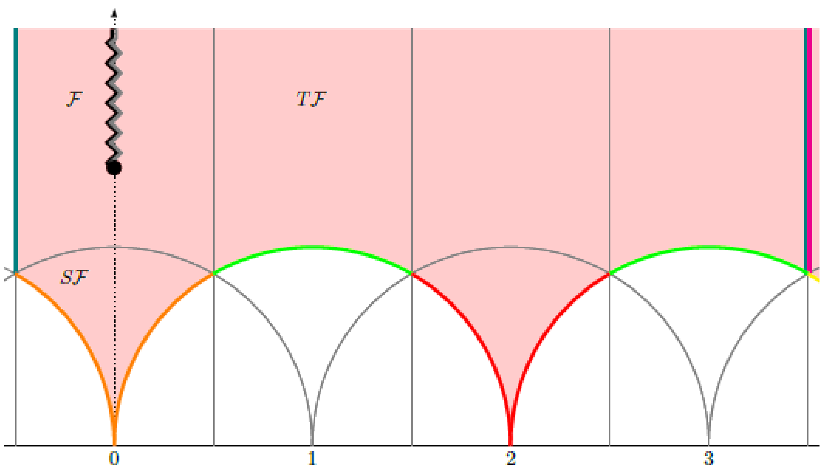
Modular Parametrization Of U −plane

The Coulomb branch is a branched double cover of the modular curve for $\Gamma^0(4)$

$$U^2 = 4\,(\mathcal{R}^4 - 2\,\mathcal{R}^2\,u(\tau) + 1)$$

$$u(\tau) = \frac{\vartheta_2(\tau)^2}{\vartheta_3(\tau)^2} + \frac{\vartheta_3(\tau)^2}{\vartheta_2(\tau)^2}$$

Hauptmodul for $\Gamma^0(4)$



Similar to Aspman, Furrer & Manschot

$$U^2 = 4\,(\mathcal{R}^4 - 2\,\mathcal{R}^2\,u(\tau) + 1)$$

$\Rightarrow$ $\wp$ is also the deck transformation of the double cover

$$\mathrm{Z}^J_{\mathrm{Coul}}(\mathcal{R}, n) = \int_{\mathbb{C}_U} \Omega_{Coul}$$

$$\wp^* \Omega_{Coul} = (-1)^{w_2(P^g)\cdot\left(w_2(X)+n^{(I)}\right)}\,\Omega_{Coul}$$

$\Rightarrow$ $Z^J_{Coul} = 0$ when there is a global anomaly

$$Z^{J}_{\mathrm{Coul}}(\mathcal{R}, n^{(I)}) = 2\int_{\mathcal{F}(\Gamma^0(4))} d\tau d\bar{\tau} \frac{\nu}{U} \, C^{n^2} \;\; \Psi^{J}\left(\tau, \frac{n^{(I)}}{2}\zeta\right)$$

$$\nu(\tau) = \frac{\vartheta_4^{13-b_2}}{\eta^9}$$

$C(\tau, \mathcal{R})$ Suitably modular invariant and holomorphic ``contact term''

$\Psi^{J}\left(\tau, \frac{n^{(I)}}{2}\zeta\right)$ Photon partition function

$$\Psi^J(\tau, z) = \sum_{k \in H^2(X,\mathbb{Z})} \left(\frac{\partial}{\partial \bar{\tau}} \, E_k^J \right) q^{-\frac{k^2}{2}} e^{-2\pi i \, k \cdot z} \, (-1)^{k \cdot K}$$

$$E_k^J = Erf\left(\sqrt{Im \tau} \, \left(k + \frac{Im \, z}{Im \, \tau} \right) \cdot J \right)$$

Not holomorphic.

Continuously metric dependent

$$z \to \frac{n^{(I)}}{2} \, \zeta(\tau, \mathcal{R}) \qquad \zeta(\tau, \mathcal{R}) \sim \frac{\partial^2 \mathcal{F}}{\partial a \, \partial m_{inst}}$$

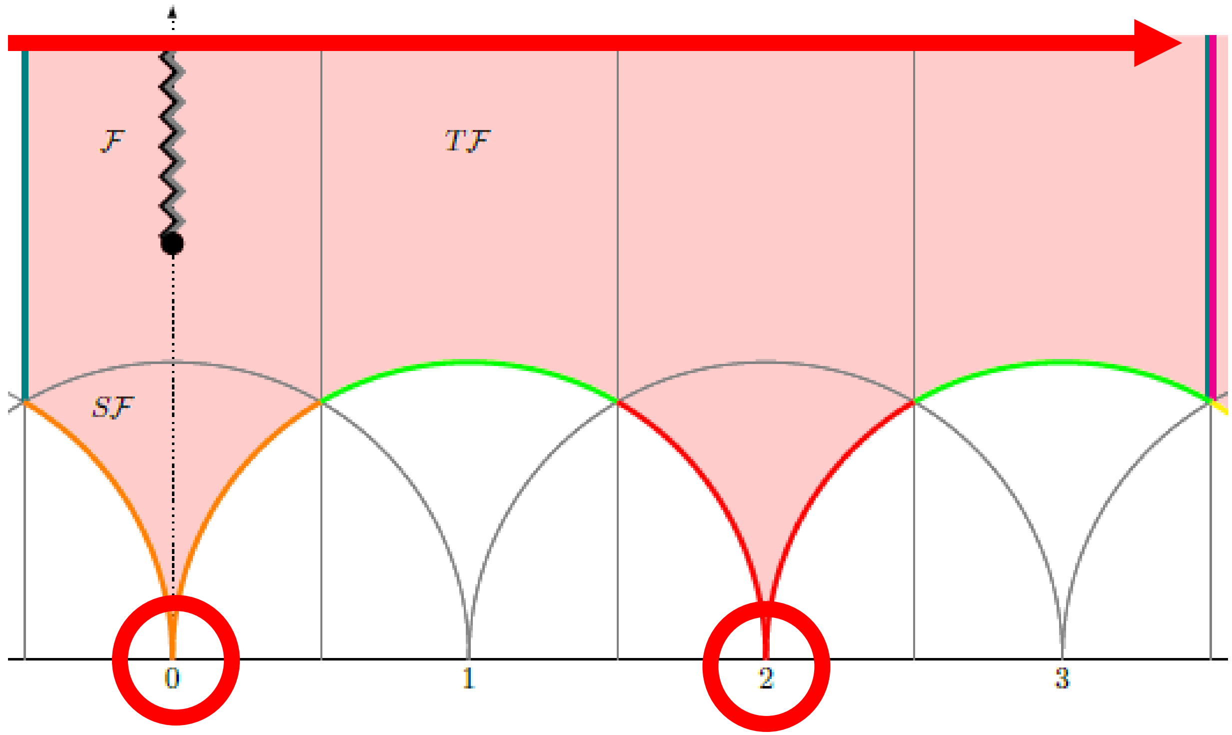
Measure As A Total Derivative

$$Z^{J}_{\mathrm{Coul}}(\mathcal{R}, n^{(I)}) = \int_{\mathcal{F}} d\tau d\bar{\tau}\, \mathcal{H}(\tau) \;\; \Psi^{J}\left(\tau, \frac{n^{(I)}}{2}\,\zeta\right)$$

$\exists$ suitably modular invariant and nonsingular $\hat{G}^{J}(\tau,\bar{\tau})$

$$\frac{\partial}{\partial\bar{\tau}}\hat{G}^{J} = \Psi^{\mathrm{J}}$$

(It can be hard to find explicit formulae for $\hat{G}^{J}$ one needs the theory of mock modular forms....)



$$Z^{J}_{\mathrm{Coul}}(n,\mathcal{R}) = \sum_{i=1,2,3} \oint d\tau_i \mathcal{H}(\tau_i) G^{J}(\tau_i)$$

?????????

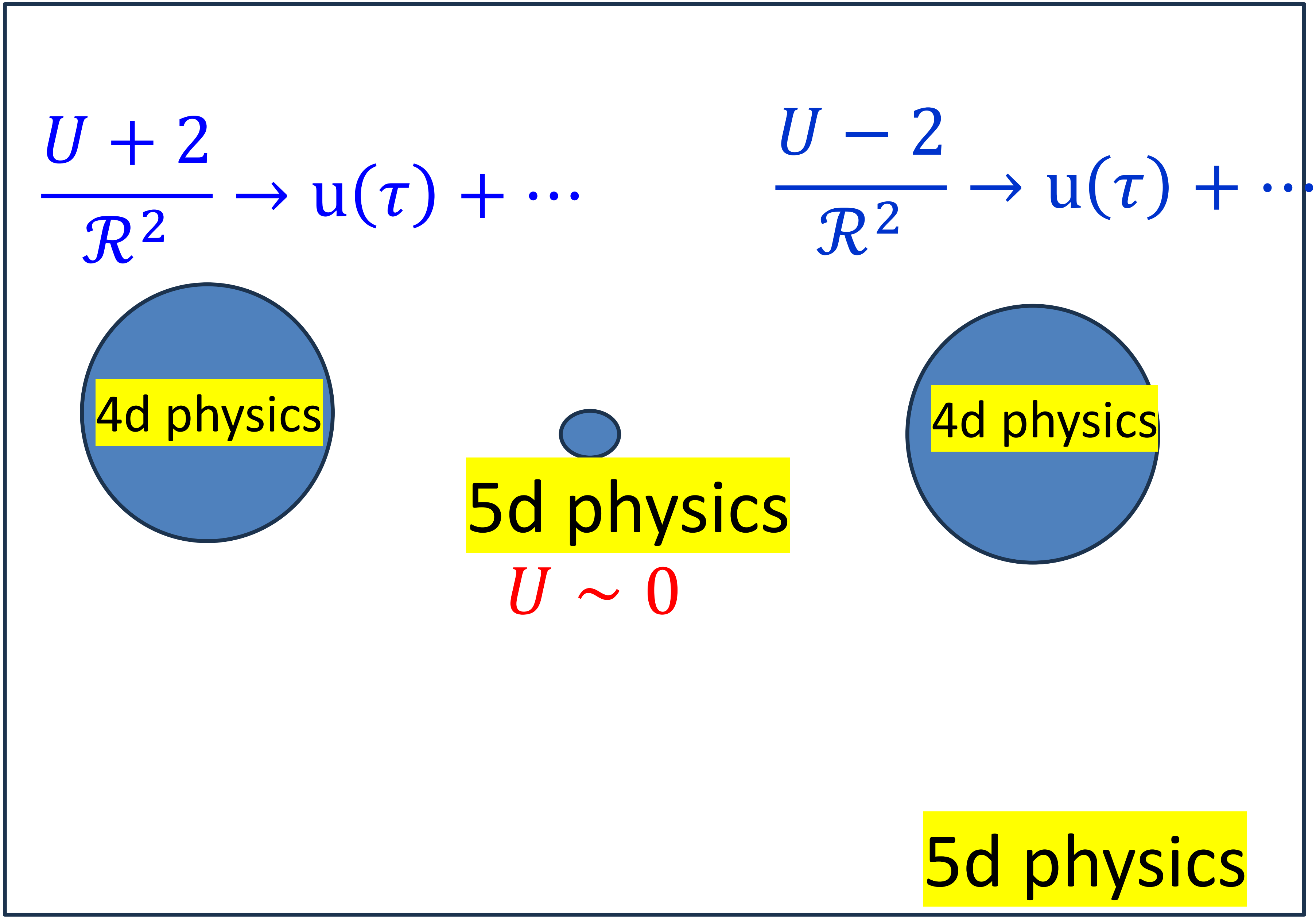
$U, C, \hat{G}^J$ are functions of τ and of $\mathcal{R}$

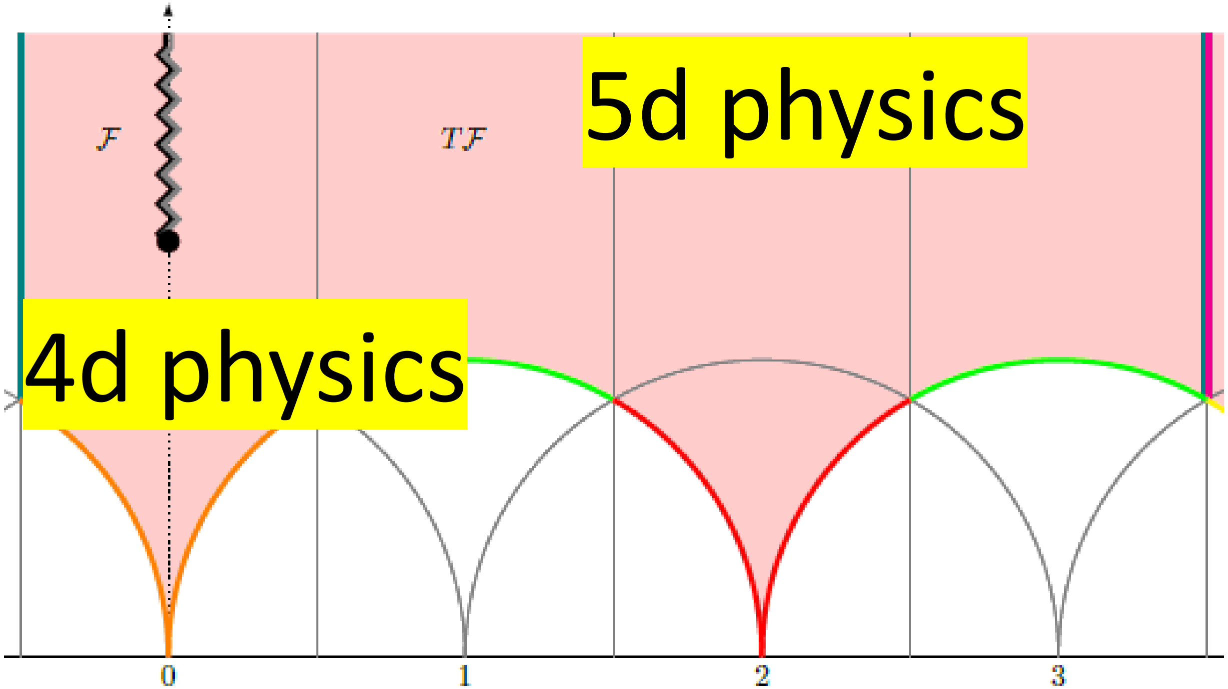
Subtle order of limits: $\mathcal{R} \to 0$ vs. $\mathrm{Im}\,\tau \to \infty$

Example: $u(\tau) \sim \frac{1}{8} q^{-\frac{1}{4}} + \frac{5}{2} q^{\frac{1}{4}} - \frac{31}{4} q^{\frac{3}{4}} + \mathcal{O}\left(q^{\frac{5}{4}}\right)$

$$U^2 = 4\left(\mathcal{R}^4 - 2\,\mathcal{R}^2\, u(\tau) + 1\right)$$

$U \to \pm 2$ vs $U \to \infty$


$\frac{U+2}{\mathcal{R}^2} \to \mathrm{u}(\tau) + \cdots$
$\frac{U-2}{\mathcal{R}^2} \to \mathrm{u}(\tau) + \cdots$
4d physics
5d physics
$U \sim 0$
4d physics
5d physics



$$\frac{\pi \tau_{bp}}{4} = i \log(1/\mathcal{R}) + \cdots$$

$$
\mathrm{Z}_{\text{Coul}}\left(\mathcal{R}, n^{(I)}\right) = 2 \sum_{i} \left[\, \nu(\tau, \mathcal{R}) \;\; C(\tau, \mathcal{R})^{n^2} \;\; G(\tau, \mathcal{R}) \right]_{q_i^0}
$$

If we *first* expand the expressions in [....] in $\mathcal{R}$ around $\mathcal{R} = 0$ *then* take the constant q^0 term at each order in $\mathcal{R}$ we find remarkable and nontrivial agreement with GNY.

Examples Of Explicit Results

$$X = \mathbb{CP}^2$$

$$Z_{Coul}(n, \mathcal{R}) = \left[\nu(\tau, \mathcal{R}) \; C(\tau, \mathcal{R})^{n^2} \; G(\tau, \mathcal{R}) \right]_{q^0}$$

$$G(\tau, \mathcal{R}) = -\frac{e^{i\,\pi n \frac{\zeta(\tau,\mathcal{R})}{2}}}{\vartheta_4(\tau)} \sum_{\ell \in \mathbb{Z}} (-1)^\ell \; \frac{q^{\frac{\ell^2}{2} - \frac{1}{8}}}{1 - e^{i\,\pi n\,\zeta(\tau,\mathcal{R})} q^{\ell - \frac{1}{2}}}$$

$$\zeta(\tau, \mathcal{R}) = \left(\vartheta_2(\tau)\vartheta_3(\tau)\right)^{-1} \int_0^{\mathcal{R}} \frac{dx}{\sqrt{1 - 2u(\tau)x^2 + x^4}}$$

Examples Of Explicit Results

Wall Crossing Formula:

$$Z^{J}_{\mathrm{Coul}} - Z^{J'}_{\mathrm{Coul}} = \left[\nu \, C^{n^2} \Theta^{J,J'}(\tau, \mathcal{R}) \right]_{q^0}$$

$$\Theta^{J,J'} = \sum_{k \in H^2(X,\mathbb{Z})} \left(s_k^J - s_k^{J'} \right) q^{-\frac{k^2}{2}} e^{-2\pi i \, k \cdot n \, \zeta(\tau,\mathcal{R})} \, (-1)^{k \cdot K}$$

$$s_k^J := sign\left(\sqrt{Im \tau} \, \left(k + \frac{Im \, \zeta(\tau, \mathcal{R})}{Im \, \tau} \right) \cdot J \right)$$

Using toric localization and the 5d instanton partition function we derived exactly the same formula for wall-crossing @ ∞

This would be another entire seminar….

Using the wall-crossing behavior of $Z^{J}_{\mathrm{Coul}}(\mathcal{R}, n)$
at the strong coupling cusps allows
one to ***derive*** Z^{J}_{SW} $\Rightarrow$ partition function for $b_2^+ > 1$

$$S(\mathcal{R}, n) = \frac{2^{2\chi+3\sigma-\chi_h}}{(1-\mathcal{R}^2)^{\frac{1}{2}n^2+\chi_h}} \sum_c SW(c) \left(\frac{1+\mathcal{R}}{1-\mathcal{R}}\right)^{c\cdot\frac{n}{2}}$$

$Z(\mathcal{R}, n)$ = Terms in the power
series with $\mathcal{R}^d$ with $d = \frac{\chi+\sigma}{4} \ mod\ 4$

Agrees with, and generalizes, GKW Conjecture 1.1

E_1 Theory

$$\mathcal{R}^4 = \exp\left[-8\,\pi^2 \frac{R}{g^2_{5d,YM}} + i\,\theta \right]$$

$g^2_{5d,sym} = \infty$ corresponds to the E_1 5d superconformal theory [Seiberg 1995]

And our formulae above indeed have special properties at $\mathcal{R}^4 = 1$.

Moving Up To Six Dimensions

All this should generalize to (anomaly-free) 6d SYM theories on $X_4 \times \mathbb{E}$

$$Index\left(D_{\mathcal{L}^{(I)}}\right) \to Ell(\sigma(\mathcal{M}))$$

Anomaly free $\Rightarrow$ Moduli space of nonabelian Seiberg-Witten equations

Relation to tmf[2d (0,1) theory]
Gukov, Pei, Putrov, and Vafa?

1 Motivation

2 Topologically Twisted d=4 N=2 Field Theory

3 Review Of The Donaldson-Witten Paradigm

4 d=5: ``K-theoretic Donaldson invariants''

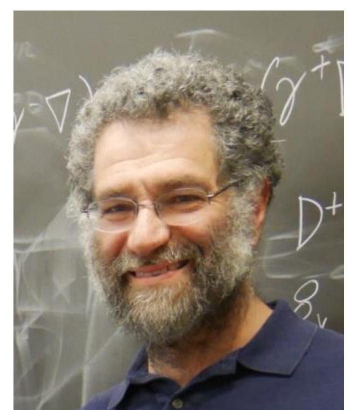
5 Family Donaldson invariants

Family Donaldson Invariants

There is an interesting generalization to invariants for families of four-manifolds.

Mentioned by Donaldson long ago.

A modest amount of work has been done in the math literature .



Families Of Metrics

Couple twisted theory to a family of metrics: $g_{\mu\nu}(x; s)$

$s \in \mathcal{P}$: Parameters of the family.

$Z_{DW}(g_{\mu\nu}(x; s))$ is independent of s.

A suitable coupling to background supergravity gives a partition function which is a closed differential form $\mathrm{Z}_{\mathrm{i}_1 \dots i_p} \mathrm{ds}^{i_1} \wedge \cdots ds^{i_p}$.

Periods of $\mathrm{Z}_{\mathrm{i}_1\ldots i_p}\mathrm{ds}^{i_1}\wedge\cdots ds^{i_p}$ are the family Donaldson invariants.

No restriction on b_2^+. No assumption of ACS.

Does it see the other half of the world of four-manifolds?

Universal Family

$$\mathcal{P} = Met(X)/Diff^+(X)$$

$$\pi_j\left(\frac{Met(X)}{Diff^+(X)}\right) \cong \pi_{j-1}\big(Diff^+(X)\big)$$

$$\pi_1\left(\frac{Met(X)}{Diff^+(X)}\right) \cong \pi_0\big(Diff^+(X)\big)$$

$\pi_0\big(Diff^+(X)\big)$: 4d mapping class group

Donaldson-Witten a la Baulieu-Singer

$$P \to X \qquad\qquad \mathcal{G} := Aut(P_1)$$

$\mathcal{G}$ −equivariant cohomology of $\mathcal{A}(P_1)$

$$\left(\Omega^*(\mathcal{A}(P_1)) \otimes S^*(Lie\mathcal{G}) \right)^{\mathcal{G}}$$

$$Q\, A_\mu = \psi_\mu \qquad Q\, \psi_\mu = -D_\mu \phi \qquad Q\, \phi = 0$$

Atiyah & Jeffrey + Baulieu & Singer

Z_{DW} : Pushforward in $\mathcal{G}$ −equivariant cohomology.

$$\mathcal{G}_d := Diff^+(X)$$

$\mathcal{G}_d$ −equivariant cohomology of $MET(X)$

$$Q\ g_{\mu\nu} = \Psi_{\mu\nu} \qquad Q\ \Psi_{\mu\nu} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu \qquad Q\Phi^\mu = 0$$

These arise from truncated twisted N=2 superconformal gravity

Φ^μ : Ghost field

Action e^{-S} is a closed equivariant class in the $\mathcal{G} \rtimes \mathcal{G}_d$ − equivariant cohomology of $MET(\mathbb{X}) \times \mathcal{A}(P)$

Push-forward in $\mathcal{G}$ −equivariant cohomology is a $\mathcal{G}_d$ −equivariant class on $MET(\mathbb{X})$

Thanks to heroic computations by JC and VS we have explicit actions e^{-S} obtained by coupling to truncated & twisted $N = 2$ conformal supergravity

Coupling To Twisted Truncated Background Supergravity

$$S[g,\Psi,\Phi] = S_{DW} + \int \sqrt{g}\,(\,\Psi^{\mu\nu}\Lambda_{\mu\nu} + \Phi^{\mu}Z_{\mu} + \Psi^{\mu\sigma}\Psi^{\nu}{}_{\sigma}\ \Upsilon_{\mu\nu}\ )$$

$$T_{\mu\nu}^{SYM} = \{Q, \Lambda_{\mu\nu}\} \qquad \Lambda_{\mu\nu} = Im\,\tau_{IJ}\left(\,F_{\rho\mu}^{-,I}\,\chi_{\nu}^{\rho,J}\right) + \cdots$$

$$Z_{\mu} = \mathcal{F}_{IJK}\,\psi_{\mu}^{I}\ F_{\rho\sigma}^{+,J}\ \chi^{\rho\sigma\,K} + \cdots$$

$$\Upsilon_{\mu\nu} = Im\,\tau_{IJ}\ \chi_{\mu\rho}^{I}\,\chi_{\nu}^{\rho,J} + \cdots$$

$\gamma \subset \frac{Met(X)}{Diff^{+}(X)}$ nontrivial 1-cycle from some nontrivial element of $\pi_0\left(Diff^{+}(X)\right)$

$$\oint_{\gamma} ds \int_{X} \mathrm{vol}(g) \frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}s} \langle \Lambda^{\mu\nu} \rangle$$

$$Q\left(\Lambda_{\mu\nu}\right) = T_{\mu\nu}^{SYM} + \cdots$$

This raises several questions:

$\Lambda^{\mu\nu}$ is NOT Q-closed!!!

Does our period integral localize to moduli spaces of instantons?

Does tree-level exactness (of LEET) persist?

Questions & Future Directions

Topological data for twisting the general d=4 N=2 theory?

Invertible theory governing orientation of nonabelian SW moduli

Global anomalies of 5D SYM …

Generalization to elliptic invariants from 6d theories on $X \times E$

Puzzles concerning the family generalization of Donaldson invariants

Other puzzles and directions I did not have time to mention ….


NOT
That's all Folks!