From
Rational Conformal Field Theories
To Modular Tensor Categories
To Nonabelions

Gregory Moore, Rutgers University

Perspectives In Theoretical Physics
Series on History Of Physics,
September 26, 2024

Preliminaries

1. This talk is intended to provide primary source material for a hypothetical historian of science interested in RCFT, MTC, and nonabelions.

2. It will be focused (mostly) on *my* work and *my* role in these developments 1986-1989. The account will be biased.

3. Memory is notoriously unreliable.

4. We all have a tendency to inflate our importance, particularly in recollection of past achievements

Preliminaries

5. History is fragile: Written documents get lost or destroyed: Floods, fires, mold. Emails get lost or deleted. Magnetic tape decays….

6. But total loss of memory is a loss of identity, as in Alzheimer's patients. The same is true for the scientific community. I commend the organizers for creating this series of talks.

7. I thought it might be valuable to take a look backward and reflect on the road traveled.
Thank you for providing this opportunity.

1 **Prehistory: A. 2dCFT and B. String Theory**

2 Three Roads To RCFT: 1986-1987

3 Princeton: Fall 1987- January 1988

4 Braiding & Fusion & S & T: Moore & Seiberg Spring 1988

5 Chern-Simons Theory: July 1988- July 1989

6 MTC & Anyons: August 1989 – November 1989

Prehistory A: 2dCFT 1980-1986

1. Friedan's thesis
2. Minimal Models: BPZ: 1983
3. WZW model
4. Modular invariance

ANNALS OF PHYSICS 163, 318–419 (1985)

Nonlinear Models in $2+\varepsilon$ Dimensions*

DANIEL HARRY FRIEDAN†

Lawrence Berkeley Laboratory, University of California, Berkeley, California 9472

Received February 10, 1984

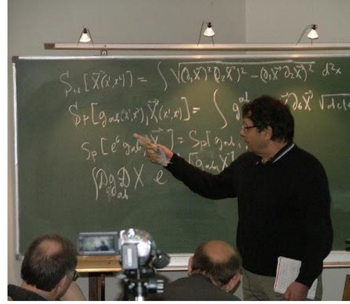
$$\Lambda^{-1}\frac{\partial}{\partial \Lambda^{-1}} g_{ij} = -\beta_{ij}(g)$$

$$\beta_{ij}(T^{-1}g) = -\varepsilon T^{-1} g_{ij} + R_{ij} + \tfrac{1}{2} T R_{ipqr} R_{jpqr} + O(T^2).$$

Conformal models related to solutions of Einstein's equations

Leads to the idea that the space of 2d field theories is some amazing generalization of the space of manifolds

Nuclear Physics B241 (1984) 333–380



INFINITE CONFORMAL SYMMETRY IN TWO-DIMENSIONAL QUANTUM FIELD THEORY

A A BELAVIN, A M POLYAKOV and A B ZAMOLODCHIKOV

L D Landau Institute for Theoretical Physics, Academy of Sciences, Kosygina 2, 117334 Moscow, USSR

Received 22 November 1983

We present an investigation of the massless, two-dimensional, interacting field theories Their basic property is their invariance under an infinite-dimensional group of conformal (analytic) transformations It is shown that the local fields forming the operator algebra can be classified according to the irreducible representations of Virasoro algebra, and that the correlation functions are built up of the "conformal blocks" which are completely determined by the conformal invariance Exactly solvable conformal theories associated with the degenerate representations are analyzed In these theories the anomalous dimensions are known exactly and the correlation functions satisfy the systems of linear differential equations

Virasoro symmetry: $vir(c)_L \oplus vir(c)_R$

$$G_{nm}^{lk}(x,\bar{x}) = \langle k|\phi_l(1,1)\phi_n(x,\bar{x})|m\rangle .$$

$$G_{nm}^{lk}(x,\bar{x}) = \sum_p C_{nm}^{p} C_{klp} A_{nm}^{lk}(p|x,\bar{x}),$$

$$A_{nm}^{lk}(p|x,\bar{x}) = \mathfrak{F}_{nm}^{lk}(p|x)\bar{\mathfrak{F}}_{nm}^{lk}(p|\bar{x}),$$

$$\mathfrak{F}_{nm}^{lk}(p|x) = x^{\Delta_p-\Delta_n-\Delta_m} \sum_{\{k\}} \beta_{nm}^{p\{k\}} x^{\Sigma k_i} \frac{\langle k|\phi_l(1,1)L_{-k_1}\ldots L_{-k_N}|p\rangle}{\langle k|\phi_l(1,1)|p\rangle}$$

of the conformal algebra. The complete set of the fields $\{A_j\}$ consists of some number (which can be infinite) of the conformal families

$$\{A_j\} = \bigoplus_n [\phi_n]. \tag{3.11}$$

(vii) If the parameter c satisfies the equation

$$\frac{\sqrt{25-c}-\sqrt{1-c}}{\sqrt{25-c}+\sqrt{1-c}}=\frac{p}{q},$$

where p and q are positive integers, the "minimal" conformal quantum field theory can be constructed so that it be exactly solvable in the following sense (i) A finite number of conformal families $[\phi_n]$ is involved in the operator algebra, each of them being degenerate, (ii) all anomalous dimensions Δ_n are known exactly, (iii) all correlation functions of the theory can be computed as solutions of special systems of linear partial differential equations There are infinitely many conformal quantum

VOLUME 52, NUMBER 18 PHYSICAL REVIEW LETTERS 30 APRIL 1984

Conformal Invariance, Unitarity, and Critical Exponents in Two Dimensions

Daniel Friedan, Zongan Qiu, and Stephen Shenker
Enrico Fermi and James Franck Institutes and Department of Physics, University of Chicago, Chicago, Illinois 60637
(Received 31 January 1984)

Conformal invariance and unitarity severely limit the possible values of critical exponents in two-dimensional systems.

A propos the fragility of history -

When I moved from Chicago to Rutgers in 1989, I brought along a couple of backup tapes of my computer files. They wouldn't have contained notebooks; I was still writing on paper. But the tapes did contain the computer programs I wrote to explore my conjectures about the Virasoro unitarity constraints. Around that time there was a famous particle detector at SLAC called the Mark II detector. I gave a Fermi Institute seminar (in front of the hex people among others) on the unitarity results from the 2nd version of my program. I called the talk "Results from the Mark II Ghost Detector". Many years after the move to Rutgers, I got around getting the material on the tapes transferred to hard disks. It was too late. The magnetic tapes were unreadable. - Daniel Friedan, Sept. 2024

BPZ had a notion of fusion rules as selection rules, but not as an algebra.

$\phi_\alpha(z)$ be the primary field with the dimension (5.22) The result of the above calculation can be represented by the following symbolic formulae

$$\psi_{(1,2)}\phi_{(\alpha)} = \left[\phi_{(\alpha-\alpha_+)}\right] + \left[\phi_{(\alpha+\alpha_+)}\right],$$

$$\psi_{(2,1)}\phi_{(\alpha)} = \left[\phi_{(\alpha-\alpha_-)}\right] + \left[\phi_{(\alpha+\alpha_-)}\right] \tag{5 24}$$

Here the square brackets denote the contributions of the corresponding conformal families to the operator product expansion of $\psi(z)\phi_{(\alpha)}(z_1)$. In (5 24) overall factors, standing in front of these contributions are omitted These factors cannot certainly be determined by simple calculations like the one performed above* As we shall see in the next section, some of these coefficients could vanish

It can be shown that the "fusion rule" (5.24) is generalized to the cases of arbitrary degenerate fields $\psi_{(n\ m)}$ as follows.

$$\psi_{(n,m)}\phi_\alpha = \sum_{l=1-m}^{1+m} \sum_{k=1-n}^{1+n} \left[\phi_{(\alpha+l\alpha_-+k\alpha_+)}\right], \tag{5 25}$$

WZW Model

Non-Abelian Bosonization in Two Dimensions

Edward Witten*
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

Abstract. A non-abelian generalization of the usual formulas for bosonization of fermions in 1+1 dimensions is presented. Any fermi theory in 1+1 dimensions is equivalent to a local bose theory which manifestly possesses all the symmetries of the fermi theory.

Received September 29, 1983

Argued for conformal invariance when WZ term is added.

CURRENT ALGEBRA AND WESS-ZUMINO MODEL IN TWO DIMENSIONS

V G KNIZHNIK and A B ZAMOLODCHIKOV
Landau Institute of Theoretical Physics, Moscow, USSR

Received 6 June 1984

We investigate quantum field theory in two dimensions invariant with respect to conformal (Virasoro) and non-abelian current (Kac-Moody) algebras The Wess-Zumino model is related to the special case of the representations of these algebras, the conformal generators being quadratically expressed in terms of currents The anomalous dimensions of the Wess-Zumino fields are found exactly, and the multipoint correlation functions are shown to satisfy linear differential equations In particular, Witten's non-abelean bosonisation rules are proven

Generalized BPZ to models with Lie group symmetry

STRING THEORY ON GROUP MANIFOLDS

Doron GEPNER and Edward WITTEN

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544, USA

Received 26 May 1986

A number of issues concerning affine Lie algebras and string propagation on group manifolds are addressed. We show that a 1 + 1 dimensional quantum field theory which gives a realization of current algebra (for any non-abelian Lie group G) will always give rise to an "integrable" representation. It is known that string propagation on the group manifold can give rise to a realization of current algebra for any G and any k, but precisely which representations occur for given k has not been determined previously. We do this here by studying modular invariance and by making a semiclassical study for large k. These results permit a complete description of the operator product algebra. Some examples based on SO(3) and SU(3)/Z_3 are worked out in detail.

WZW Model

$$\mathcal{H} \cong \bigoplus_{0 \le \lambda \cdot \rho \le k} \quad V_\lambda \otimes \bar{V}_\lambda$$

N.B. A *finite* sum

Compare Peter-Weyl theorem

$$L^2(G) \cong \bigoplus_\lambda \; V_\lambda \otimes \bar{V}_\lambda$$

$$``\mathcal{H} \cong L^2\left(LG^{(k)}\right)"$$

It's all about group theory, and becomes finite dimensional group theory in the semiclassical limit $k \to \infty$

Modular Invariance

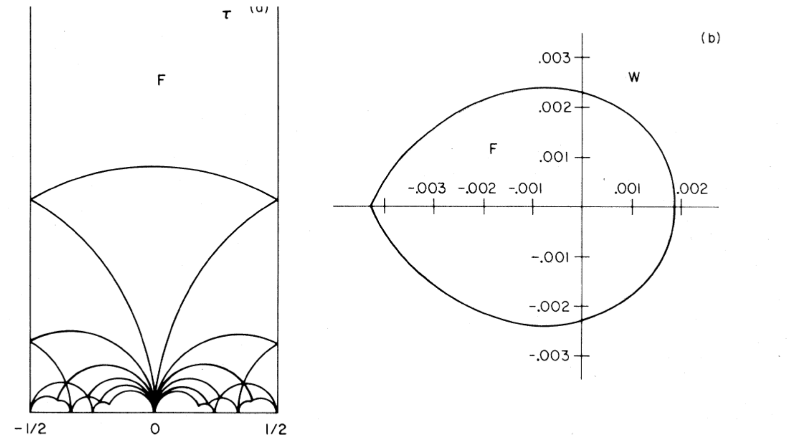
CAL REVIEW D VOLUME 5, NUMBER 8

Loop Graph in the Dual-Tube Model*

Joel A. Shapiro
Center for Theoretical Physics, Department of Physics and Astrono
University of Maryland, College Park, Maryland 20742
and Physics Department,† Rutgers University, New Brunswick, New Jer

(Received 6 December 1971)

The one-loop graph in the dual-tube model is constructed. The conditions fc gences or new singularities are exactly those found by Lovelace for factorizat "Pomeranchukon" in the strip model. Loops correspond to electrostatics on ı only if spurious particles are permitted to circulate in the loops.



(a) Fundamental regions of the modular group. The region marked F is the one I choose to integrate over. (b) The region F in the w plane $[w = \exp(2\pi i \tau)]$.

VOLUME 54, NUMBER 6 PHYSICAL REVIEW LETTERS 11 FEBRUARY 1985

Heterotic String

David J. Gross, Jeffrey A. Harvey, Emil Martinec, and Ryan Rohm
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
(Received 21 November 1984)

A new type of superstring theory is constructed as a chiral combination of the closed $D=26$ bosonic and $D=10$ fermionic strings. The theory is supersymmetric, Lorentz invariant, and free of tachyons. Consistency requires the gauge group to be $\mathrm{Spin}(32)/Z_2$ or $E_8 \times E_8$.

OPERATOR CONTENT OF TWO-DIMENSIONAL CONFORMALLY INVARIANT THEORIES

John L. CARDY

Department of Physics, University of California, Santa Barbara CA 93106, USA

Received 22 November 1985
(Revised 3 January 1986)

A Classification of Open String Models

W. Nahm

Physikalisches Institut der Universität Bonn, Nussallee 12, D-5300 Bonn 1, Federal Republic of Germany

Prehistory B: String Theory 1984-1986

1. Anomalies – cool thing to work on
   @ Harvard 1982-1985

(Anomalies have a complex history)

2. Green-Schwarz anomaly cancellation

3. Heterotic String Theory & Calabi-Yau

4. Narain; Narain, Sarmadi, and Witten

Nuclear Physics B239 (1984) 477–507

CHIRAL ANOMALIES, HIGHER DIMENSIONS, AND DIFFERENTIAL GEOMETRY

Bruno ZUMINO

Lawrence Berkeley Laboratory and Department of Physics, University of California, Berkeley, CA 94720, USA

WU Yong-Shi* and A. ZEE

Department of Physics, University of Washington, Seattle, WA 98195, USA

Received 6 June 1983
(Final version received 6 December 1983)

We determine the abelian and non-abelian chiral anomalies in $2n$-dimensional spacetime by a differential geometric method which enables us to obtain the anomalies without having to calculate Feynman diagrams, as has been done by Frampton and Kephart. The advantage of this method is that the construction automatically satisfies the Wess-Zumino consistency condition, a condition which has direct physical interpretation. We hope that our analysis sheds light on the mathematical structure associated with chiral anomalies. The mathematical analysis is self-contained and a brief review of differential forms and other mathematical tools is included.

Proc. Natl. Acad. Sci. USA
Vol. 81, pp. 2597–2600, April 1984
Mathematics

Dirac operators coupled to vector potentials

(elliptic operators/index theory/characteristic classes/anomalies/gauge fields)

M. F. Atiyah[†] and I. M. Singer[‡]

[†]Mathematical Institute, University of Oxford, Oxford, England; and [‡]Department of Mathematics, University of California, Berkeley, CA 94720

Contributed by I. M. Singer, January 6, 1984

ABSTRACT **Characteristic classes for the index of the Dirac family $\not{\partial}_A$ are computed in terms of differential forms on the orbit space of vector potentials under gauge transformations. They represent obstructions to the existence of a covariant Dirac propagator. The first obstruction is related to a chiral anomaly.**

The analytic family indexed by $\mathfrak{A}/\mathcal{G}$ can be defined directly in terms of the Hilbert bundles $\mathcal{H}^{\pm} = \mathfrak{A}\times_{\mathcal{G}} L_2(S^{\pm} \otimes E)$ over $\mathfrak{A}/\mathcal{G}$. Covariance means $\{\not{\partial}_{\phi A}\}_{\phi\in\mathcal{G}}$ gives an elliptic operator $\not{\partial}_{\mathcal{G}\cdot A}$ mapping the fiber $\mathcal{H}^{+}_{\mathcal{G}\cdot A}$ to $\mathcal{H}^{-}_{\mathcal{G}\cdot A}$. The analytic index of this family is Ind $\not{\partial}$ above.

When $M = S^4$ and $\mathcal{G}$ is the group of gauge transformations leaving the north pole fixed, the index for the Dirac family

Physics B234 (1983) 269–330

GRAVITATIONAL ANOMALIES

Luis ALVAREZ-GAUMÉ[1]

Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Edward WITTEN[2]

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

Received 7 October 1983

It is shown that in certain parity-violating theories in $4k+2$ dimensions, general covariance is spoiled by anomalies at the one-loop level. This occurs when Weyl fermions of spin-$\frac{1}{2}$ or -$\frac{3}{2}$ or self-dual antisymmetric tensor fields are coupled to gravity. (For Dirac fermions there is no trouble.) The conditions for anomaly cancellation between fields of different spin is investigated. In six dimensions this occurs in certain theories with a fairly elaborate field content. In ten dimensions there is a unique theory with anomaly cancellation between fields of different spin. It is the chiral $n=2$ supergravity theory, which is the low-energy limit of one of the superstring theories. Beyond ten dimensions there is no way to cancel anomalies between fields of different spin.

ANOMALY CANCELLATIONS IN SUPERSYMMETRIC $D = 10$ GAUGE THEORY AND SUPERSTRING THEORY ☆

Michael B. GREEN
Queen Mary College, University of London, London E1 4NS, UK
and California Institute of Technology, Pasadena, CA 91125, USA

and

John H. SCHWARZ
California Institute of Technology, Pasadena, CA 91125, USA

Received 10 September 1984

Supersymmetric ten-dimensional Yang–Mills theory coupled to $N = 1$, $D = 10$ supergravity has gauge and gravitational anomalies that can be partially cancelled by the addition of suitable local interactions The remaining pieces of all the anomalies cancel if the gauge group is SO(32) or $E_8 \times E_8$. These cancellations are automatically incorporated in the type I superstring theory based on SO(32). A superstring theory for $E_8 \times E_8$ has not yet been constructed.

Walking into the office of Alvarez-Gaume & Ginsparg and hearing about the news from Aspen.

SUPERSTRINGS FROM 26 DIMENSIONS?

Peter G.O. FREUND[1]
Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA

Received 13 November 1984

The finite type I superstring theories of Green and Schwarz (SO(32) and (?)$E_8 \times E_8$) in ten dimensions are viewed as special dimensional reductions on 16-tori from the non-supersymmetric Veneziano–Nambu–Goto strings in 26 dimensions. Fermions appear as solitons of the two-dimensional string field theory. Various problems of such an approach are pointed out and possible solutions outlined.

VOLUME 54, NUMBER 6 PHYSICAL REVIEW LETTERS 11 FEBRUARY 1985

Heterotic String

David J. Gross, Jeffrey A. Harvey, Emil Martinec, and Ryan Rohm
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
(Received 21 November 1984)

A new type of superstring theory is constructed as a chiral combination of the closed $D=26$ bosonic and $D=10$ fermionic strings. The theory is supersymmetric, Lorentz invariant, and free of tachyons. Consistency requires the gauge group to be $\mathrm{Spin}(32)/Z_2$ or $E_8 \times E_8$.

Right-moving CFT does not have to be the same as the Left-moving CFT

VACUUM CONFIGURATIONS FOR SUPERSTRINGS

P. CANDELAS

Center for Theoretical Physics, University of Texas, Austin, Texas 78712
and Institute for Theoretical Physics, Santa Barbara, California 93106, USA

Gary T. HOROWITZ

Physics Department, University of California, Santa Barbara, California 93106, USA

Andrew STROMINGER

The Institute for Advanced Study, Princeton, New Je

Edward WITTEN

Joseph Henry Laboratories, Princeton University, Prince

Received 2 January 1985

We study candidate vacuum configurations in ten-dimens ity and superstring theory that have unbroken $N=1$ supers condition permits only a few possibilities, all of which have van $E_8 \times E_8$ case, one of these possibilities leads to a model that i group with four standard generations of fermions.

CONFORMAL INVARIANCE, SUPERSYMMETRY AND STRING THEORY

Daniel FRIEDAN[1]

Enrico Fermi and James Franck Institutes and Department of Physics, University of Chicago, Chicago, IL 60637, USA

Emil MARTINEC[2]

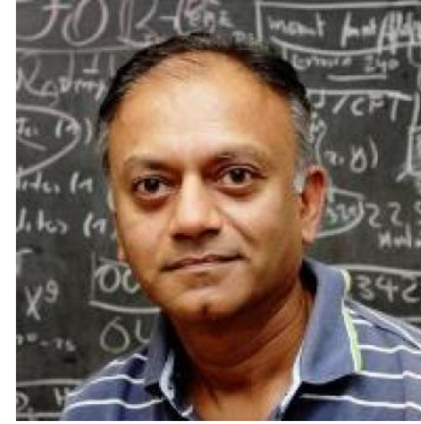
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

Stephen SHENKER[1]

Enrico Fermi and James Franck Institutes and Department of Physics, University of Chicago, Chicago, IL 60637, USA

Received 16 December 1985

Covariant quantization of string theories is developed in the context of conformal field theory and the BRST quantization procedure. The BRST method is used to covariantly quantize superstrings, and in particular to construct the vertex operators for string emission as well as the supersymmetry charge. The calculation of string loop diagrams is sketched. We discuss how conformal methods can be used to study string compactification and dynamics.



Volume 169B, number 1 PHYSICS LETTERS 20 March 1986

NEW HETEROTIC STRING THEORIES IN UNCOMPACTIFIED DIMENSIONS < 10

K.S. NARAIN

Rutherford Appleton Laboratory, Chilton, Didcot, OX11 0QX, UK

Received 6 December 1985

It is shown that infinitely many heterotic string theories exist in uncompactified dimensions less than 10, that are one-loop finite (for massless external legs). Tachyons are removed by compactifying into tori $(10-d)$ and $(26-d)$ dimensions of the right-moving superstring and left-moving bosonic string sectors, respectively. The condition for modular invariance is shown to be equivalent to self-duality condition on even lorentzian lattices with $(10-d)$ and $(26-d)$ timelike and spacelike directions, respectively. The construction results in a $(10-d)(26-d)$ parameter family of one-loop finite string theories. The zero mass sector of these theories for $d=4$ and 6 correspond to $N=4$ and 2 supergravity coupled to super Yang–Mills with many possible groups, some of which cannot be obtained by compactifying $d=10$ heterotic string theory.

Nuclear Physics B279 (1987) 369–379
North-Holland, Amsterdam

A NOTE ON TOROIDAL COMPACTIFICATION OF HETEROTIC STRING THEORY

K.S. NARAIN and M.H. SARMADI

Rutherford Appleton Laboratory, Chilton, Didcot, Oxon OX11 OQX, UK

E. WITTEN

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

Received 9 June 1986
(Revised 4 July 1986)

The connection of recently constructed lower dimensional heterotic strings with conventional toroidal compactification is clarified.

NSW: It's just a toroidal compactification of heterotic string with flat metric, B-field, and gauge field

$$Z(\tau, \bar{\tau}) = (\eta\bar{\eta})^{-d} \sum_{p \in \Gamma} q^{\frac{1}{2}p_L^2} \, \bar{q}^{\frac{1}{2}p_R^2}$$

$$q = e^{2\pi i \tau}$$

Infinite sum: Γ is an even unimodular lattice embedded into $\mathbb{R}^{d,d+8s}$

Siegel-Narain theta function

Simplest case: Target space S^1_R

$$\Theta(\tau,\bar\tau) = \sum_{n,w\in\mathbb{Z}} q^{\frac{1}{2}p_L^2}\,\bar q^{\frac{1}{2}\mathrm{p}_\mathrm{R}^2}$$

$$p_L = \frac{1}{\sqrt{2}}\left(\frac{n}{R} + w\,R\right) \qquad p_R = \frac{1}{\sqrt{2}}\left(\frac{n}{R} - w\,R\right)$$

When $R^2 = \frac{a}{b} \in \mathbb{Q}$ $\quad \Theta = \sum_{s,s'\in\frac{\mathbb{Z}}{N\mathbb{Z}}} d_{s,s'}\,\Theta_s\,\overline{\Theta}_{s'} \qquad N = 2ab$

Noted c. 1987 by Friedan & Shenker; Moore;
Dijkgraaf, Verlinde, Verlinde (DVV) ; perhaps many others.

Harvard 1985-1986

Despite opposition from the senior faculty, kids run the show.

L. Alvarez-Gaume, S. & V. Della Pietra, J. . Distler, P. Ginsparg, G. Moore, P. Nelson, J. Polchinski, C. Vafa

Trying to learn about the great developments from Princeton & Chicago. Lots of work on 2dCFT and string theory from the worldsheet perspective.

Complex geometry and the theory of quantum strings

A. A. Belavin and V. G. Knizhnik

L. D. Landau Institute for Theoretical Physics of the USSR Academy of Sciences
(Submitted 18 February 1986)
Zh. Eksp. Teor. Fiz. **91**, 364–390 (August 1986)

A summation over closed orientable surfaces of genus $p \geqslant 2$ (p-loop vacuum amplitudes in the theory of bosonic strings) in the critical dimension $\mathscr{D} = 26$ reduces to an integration over the moduli space M_p of complex structures of Riemann surfaces of genus p. The analytic properties of the integration measure are studied as a function of the complex coordinates on M_p. It is shown that the measure multiplied by $(\det \operatorname{Im} \hat{\tau})^{13}$ (where $\hat{\tau}$ is the period matrix of the Riemann surface) is the absolute square of a function holomorphic and nowhere vanishing on M_p. This function has a second-order pole at the infinity $D = \overline{M}_p / M_p$ of the compactified moduli space M_p. By these properties the measure is determined uniquely, up to an arbitrary constant factor, fact which allows one to construct explicit formulas in terms of theta functions for surfaces of genus $p = 2, 3$. The theory contains power and logarithmic divergences, related respectively to the renormalization of the tachyon wave function and of the slope. The relation of these results to Mumford's theorem is discussed. The quantum geometry of critical strings turns out to be a complex geometry.

$$Z_g = \int_{\mathcal{M}_g} |\psi(m)|^2 \, (\det \operatorname{Im} \tau)^{-13}$$

$$= \langle \psi, \psi \rangle$$

I went to David Mumford's office to explain some of what was going on at the physics department: Theta functions, modular forms, integration on moduli space of curves.

His *Tata Lectures On Theta* were enormously helpful to the work of several of us at Harvard at the time.

Mumford isomorphism $K(\mathcal{M}_g) \otimes \lambda^{\otimes -13} \cong \mathcal{O}$

Full of excitement I called Phil Nelson, at the time he was visiting Joe Polchinski in UT Austin.

Too late.

There was competition between Harvard and Moscow – but it was not acrimonious.

1. Prehistory: A. 2dCFT and B. String Theory

2. Three Roads To RCFT: 1986-1987

3. Princeton: Fall 1987- January 1988

4. Braiding & Fusion & S & T: Moore & Seiberg Spring 1988

5. Chern-Simons Theory: July 1988- July 1989

6. MTC & Anyons: August 1989 – November 1989

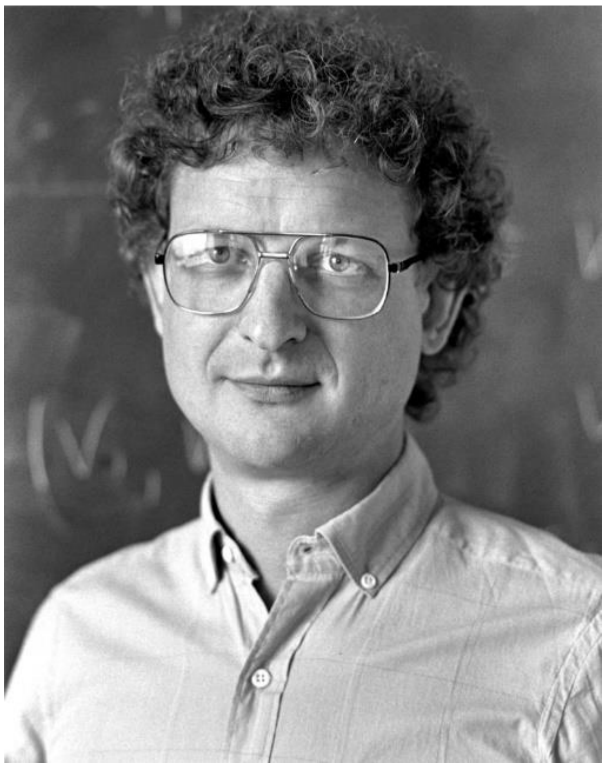
Three Roads To RCFT

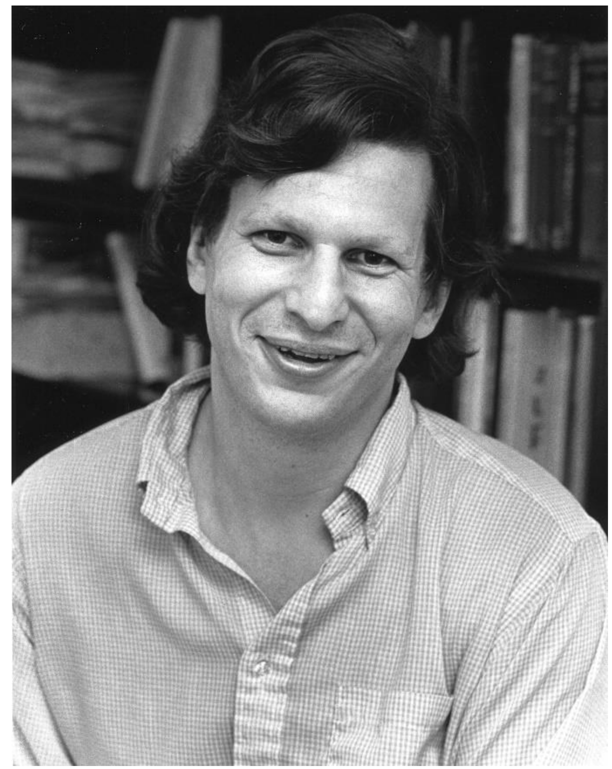
a.) Friedan & Shenker: Modular Geometry

b.) Moore: Atkin-Lehner Symmetry

c.) Dijkgraaf-Verlinde-Verlinde (DVV):
c=1 classification and beyond

Friedan & Shenker 1986-1987





Nuclear Physics B281 (1987) 509–545
North-Holland, Amsterdam

THE ANALYTIC GEOMETRY OF TWO-DIMENSIONAL CONFORMAL FIELD THEORY*

Daniel FRIEDAN and Stephen SHENKER

Enrico Fermi and James Franck Institutes and Department of Physics, University of Chicago, Chicago, Illinois 60637, USA

Received 16 June 1986

Two-dimensional conformal field theory is formulated as analytic geometry on the universal moduli space of Riemann surfaces.

The analytic geometry construction was a synthesis of BPZ and Cardy. - Daniel Friedan, Sept. 2024

Conformal blocks are sections of projectively flat (infinite-dimensional) vector bundles over the moduli of curves.

Correlation functions for fixed moduli $(m, \bar{m})$ are pairings with a metric

$$Z(\bar{m}, m) = h(\bar{\psi}, \psi) = \overline{\psi^a(m)} h_{\bar{a}b} \psi^b(m), \tag{43}$$

Moduli include positions of punctures.

a, b label Virasoro conformal primaries: Generically sum is infinite-dimensional (but finite dimensional in the BPZ case)

The simplest tractable nontrivial examples in higher genus are gaussian models, the nonlinear models whose target spaces are multidimensional tori. The generalized characters are theta functions. The Ising model can also be represented explicitly as a gauge system. The Ising partition function in genus g is written in terms of pfaffians or square roots of determinants of chiral Dirac operators:

$$Z = 2^{-g} \sum_{s} |\text{pfaffian } \partial\!\!\!/_s|^2 . \tag{44}$$

The sum is over the $2^{2g-1} + 2^{g-1}$ spin structures in genus g which generically have no zero modes, i.e. for which the Z_2 index vanishes. For $c < 1$, the rank of the vector bundle V_g is finite, and grows exponentially in g. For $c \geq 1$, the rank of V_g is infinite in general, because an infinite number of highest weights appear in the theory. In exceptional cases the metric of the infinitely many generalized characters is highly degenerate, so the rank of W_g is much smaller than the number of highest weights in the theory. In chiral theories this collapse is taken to the extreme; the rank of W_g is 1, for all g.

Friedan & Shenker, unpublished, 1986-1987 speculating:

Space of CFTS is an arithmetic variety. The degenerate points are like the ``rational points'' of this variety.

I remembered that Peter Freund was thinking about p-adic string theory in 1986-87. His August 1987 paper cites private communication with Shenker and me on the subject. This was the idea of reducing a rational conformal field theory at each of the prime numbers to build a p-adic version. - Daniel Friedan, Sept. 2024

Most likely Daniel coined the term ``rational conformal field theory'' in the 1986-1987 school year.

NON-ARCHIMEDEAN STRINGS ☆

Peter G.O. FREUND and Mark OLSON
Enrico Fermi Institute and Department of Physics, The University of Chicago, Chicago, IL 60637, USA

Received 13 August 1987

A full set of factorized, dual, crossing-symmetric tree-level N-point amplitudes is constructed for non-archimedean closed strings. Momentum components and space–time coordinates are still valued in the field of real numbers, quantum amplitudes in that of complex numbers. It is the world-sheet parameters, which one integrates over, that become p-adic. For the closed string the parameters are valued in quadratic extensions of the fields Q_p of p-adic numbers (p=prime).

(in the adelic sense) in a larger structure meant to shed light on the behavior of the physical string. The adelic point of view has been advocated, though in a different context, also by Witten [2] and Friedan and Shenker [8].

[8] D. Friedan and S. Shenker, private communication.

Atkin-Lehner Symmetry 1986-1987

At Harvard/IAS in 1986-1987 I was interested in what string theory could say about the cosmological constant problem.

At the time, because $\Lambda \leq 10^{-120}$ *in ``natural units''* it was ``obvious'' to most people that it is identically zero.

Crossing the street at Harvard Square with Steve Weinberg

(Measured value $\cong 10^{-122}$ in Planck units $\cong$ $(0.008\, eV)^4$)

In string theory the (perturbative approx. to the) cosmological constant is the partition function summed on genus g surfaces.

Heterotic string vacuum amplitude (Belavin & Knizhnik; Friedan & Shenker):

$$Z = \langle \psi_L , \psi_R \rangle \qquad \psi_{L,R} \in \Gamma(\mathcal{V} \rightarrow \mathcal{M}_g)$$

Maybe $Z = \langle \psi_L , \psi_R \rangle = 0$ because of a selection rule due to a symmetry?

Discussion with Pierre Deligne at IAS tea time about symmetries of moduli spaces: Atkin-Lehner symmetry.

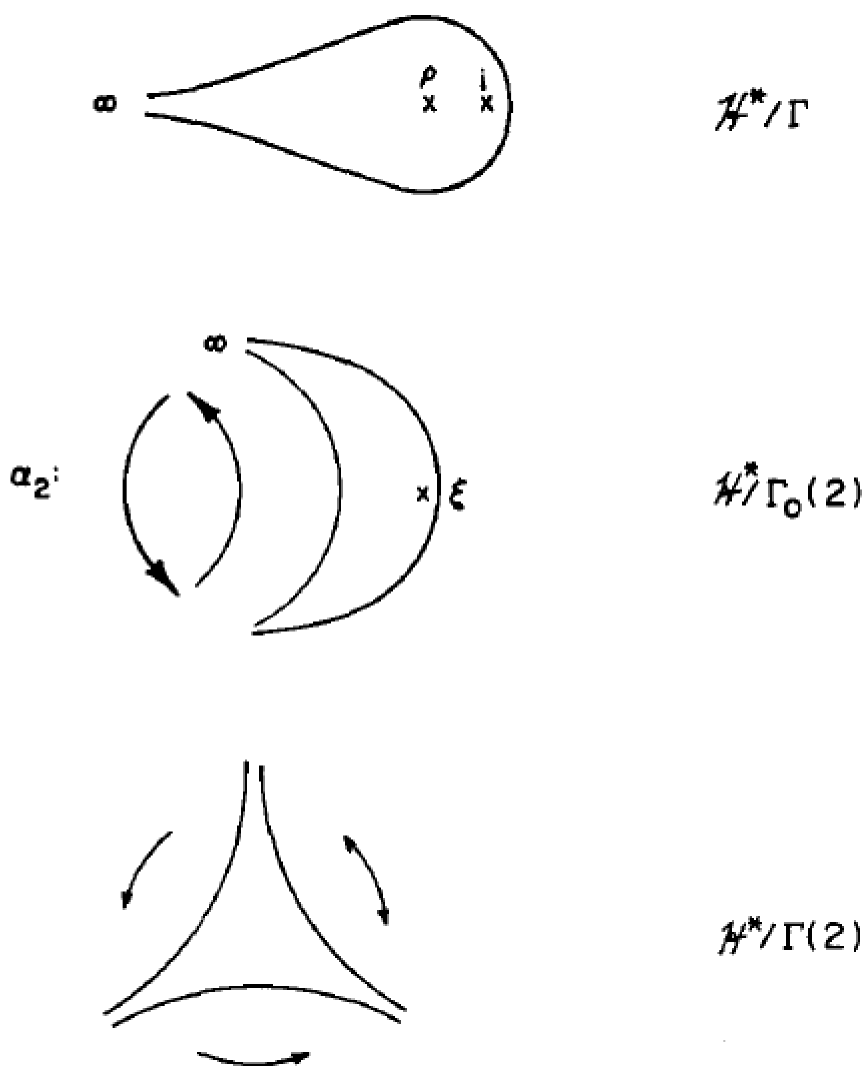


Fig. 3. Some modular surfaces and their symmetries.

$$Z_{Vac,1} = \int_{\mathcal{F}(\Gamma')} \langle \psi_L, \psi_R \rangle (\tau, \bar{\tau})$$

$$PSL(2, \mathbb{Z}) = \coprod_i \Gamma' \cdot \gamma_i$$

Letting χ_i be a basis for the γ_i −transforms of ψ_L and ψ_R

$$Z = \int_{\mathcal{M}_1} Z(\tau, \bar{\tau})$$

$$Z(\tau, \bar{\tau}) = \sum_{i=1}^{N} N_{ij}\, \chi_i(\tau)\, \overline{\chi_j}(\bar{\tau})$$

Use asymmetric orbifolds of Narain compactifications.

First example: $2d$ compactification of heterotic string.

Amazing cancellation:

$$Z = \int_{\mathcal{M}_1} \frac{d^2\tau}{(Im\,\tau)^2} \frac{\Theta_{Leech}(\tau)}{\eta^{24}} = 0$$

(Later pointed out to be an easy consequence of integration by parts because $\partial_{\bar{\tau}}\left(\widehat{E_2}\right) \sim (Im\,\tau)^{-2}$.)

n	a_n	I_n	$a_n I_n$	Σ_n
-1	1	-11.596	-11.569	-11.569
0	24	$\pi/3$	25.132	13.5367
1	196884	$-.79482 \times 10^{-4}$	-15.6487	-2.112
2	21493760	$.10669 \times 10^{-6}$	2.2932	0.1812
3	864299970	$-.22243 \times 10^{-9}$	-0.1922	-.011

Since $|I_n| < \frac{1}{\pi n} e^{-\sqrt{3}\pi n}$ the series converges rapidly. Graphing the data in the above table we have:

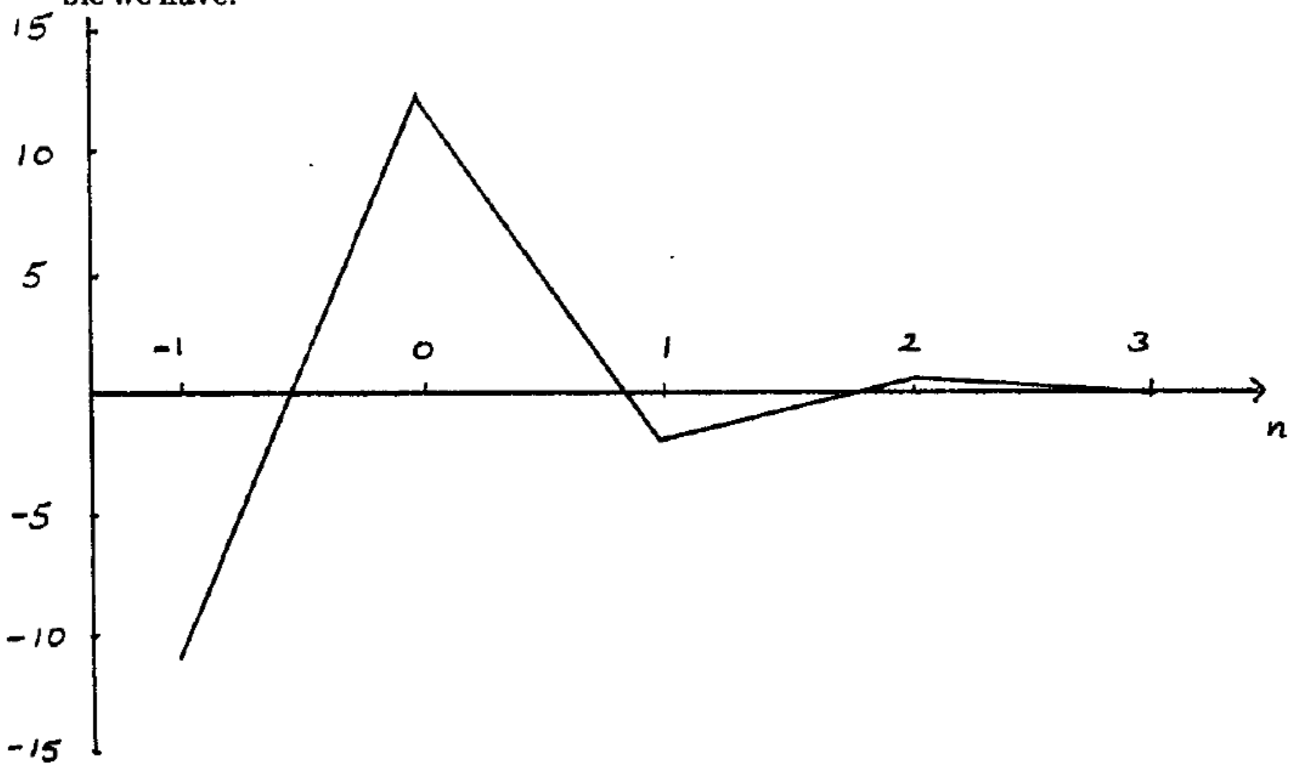


fig.5: Graph of the cumulative cosmological constant against level number

I spent a lot of time, with very limited success, trying to make models.

I decided to focus on what kind of CFT's give partition functions which are finite sums of holomorphic times antiholomorphic functions of moduli

Nuclear Physics B293 (1987) 139–188
North-Holland, Amsterdam

ATKIN-LEHNER SYMMETRY

Gregory MOORE

School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA
and Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Received 24 March 1987

The vanishing of the one-loop string cosmological constant in nontrivial nonsupersymmetric backgrounds can be understood by viewing the path integral as an inner product of orthogonal wave functions. For special backgrounds the string theory has an extra symmetry, expressed as a transformation on moduli space. When left- and right-moving wave functions transform in different representations of this symmetry the cosmological constant must vanish. Specific examples of the mechanism are given at one loop for theories in two and four dimensions. Various suggestions are made for the higher loop extension of this idea.

function which is holomorphically factorized. For these backgrounds the one loop partition function should be a finite sum of Petersson inner products for congruence subgroups:

$$\sum_i c_i \langle f_i | g_i \rangle . \tag{2.24}$$

We will see in some examples that the coefficients c_i needn't be positive. In this picture, the inclusion $\Gamma_0(n) \subset \Gamma_0(m)$ whenever m divides n, suggests that we should think of the subgroups of the modular group as a vast net spreading out from Γ and reconverging, amongst the groups with infinite index, on $\langle T \rangle$. As we have mentioned the index provides a measure of the complexity of the background. Note that this way of measuring distances between backgrounds is rather different from the naive notion of distance between backgrounds, and is somewhat reminiscent of p-adic distance.

Recently, Friedan and Shenker have independently investigated backgrounds of this type, and have developed a notion of rational conformal field theories, in which the background is specified by rational data [32]. In their picture, the metric on the infinite dimensional bundle of primary fields becomes highly degenerate for certain backgrounds, leading to a partition function of the form (2.24).

Probably the first mention of ``rational conformal field theories'' in print.

The Road from Utrecht

July 14-30, 1987

Nonperturbative Quantum Field Theory

Edited by

G. 't Hooft
A. Jaffe
G. Mack
P. K. Mitter and
R. Stora

NATO ASI Series

As you remember, when we first met at Cargese 1987, I was working with Herman and Robbert on c=1 CFT's. This is also where I learned about the isolated c=1 models of Paul Ginsparg.
Other subjects that were covered at that school were orbifolds etc. I was very much influenced by the work of Dan Friedan and Steve Shenker (especially their paper on the analytic geometry of CFT), but also by all the developments related to the classification of modular invariants for minimal models and WZW models. People were also already thinking about extensions of the Virasoro algebra, such as W-algebras (e.g. by Kareljan Schoutens who was also at Utrecht). In the c=1 case we knew that rational radii were special and have an enlarged algebra .I think the relationship with theta functions and bosonization of fermions also made clear that there was something special about rational models.
In my recollection the concept of RCFT was natural consequence of the work of Dan and Steve, and I think Dan may have been the first to use this term. - Erik Verlinde, September 2024

CONFORMAL FIELD THEORY AT C=1

Robbert Dijkgraaf, Erik Verlinde and Herman Verlinde

Institute for Theoretical Physics
Princetonplein 5, P.O. Box 80.006
3508 TA Utrecht, The Netherlands

1. C=1 MODELS

1.1. Introduction

Conformal field theory [1] is a subject of great interest to various disciplines in physics. Although we are still far from a complete understanding, partial results seem to indicate that a beautiful, deep mathematical structure lies at its roots. The situation for central charge $c<1$ is by now very well understood [2]. The case $c=1$ however forms in many aspects a natural boundary. Here we meet the new features of an infinity of primary fields and the existence of marginal operators and deformations. Furthermore, the $c=1$ models allow a natural interpretation as a string compactification. As such they can serve as an instructive example of what is to be expected at higher c values.

FUSION RULES AND MODULAR TRANSFORMATIONS IN 2D CONFORMAL FIELD THEORY

Erik VERLINDE

Institute for Theoretical Physics, University of Utrecht, P. O. Box 80.006, 3508 TA Utrecht, The Netherlands

Received 15 March 1988

We study conformal field theories with a finite number of primary fields with respect to some chiral algebra. It is shown that the fusion rules are completely determined by the behavior of the characters under the modular group. We illustrate with some examples that conversely the modular properties of the characters can be derived from the fusion rules. We propose how these results can be used to find restrictions on the values of the central charge and conformal dimensions.

Erik’s definition: Close to the modern one

Extended chiral algebra (holomorphic VOA) with finitely many (highest weight) representations.

The germ of the idea is already in the DVV Cargese paper:

1.4. Rational gaussian models

A different approach towards classification uses the concept of rational conformal field theories. A characteristic property of these models is that they contain extra chiral primary operators, which can be used to construct an extension of the Virasoro algebra. The theory is called rational if, in addition, its operator content falls into a finite set of irreducible representations of this extended algebra. An alternative characterization of rationality is that the partition function is built up from a finite set of characters, which are modular forms under some subgroup of the modular group $SL(2,\mathbb{Z})$ [12].

The rational theories in the c=1 spectrum are found at the rational values of R^2. In particular, the gaussian model with $\frac{1}{2}R^2 = m/n$ contains the chiral vertex operators $V_{\pm}(z) = \exp\left(\pm i\sqrt{N}\varphi(z)\right)$ of spin $\frac{1}{2}N = mn$. The representations of the corresponding extension of the current algebra are built on the chiral primary fields $\psi_k = \exp\left(\frac{2\pi i k}{\sqrt{N}}\varphi\right)$, where $k \in \mathbb{Z}_N$. For these values of the momenta the operator products with $V_{\pm}$ are local. The fusion rules are simply

$$[\psi_k]\cdot[\psi_{k'}] = [\psi_{k+k'}] \quad . \tag{1.9}$$

The corresponding 1-loop characters are given by

$$\chi_k(\tau) = \frac{\vartheta\begin{bmatrix} k/N \\ 0 \end{bmatrix}(\tau)}{\eta(\tau)} \tag{1.10}$$

We’ll return to Erik’s paper
(from March 1988)

1. Prehistory: A. 2dCFT and B. String Theory
2. Three Roads To RCFT: 1986-1987
3. Princeton: Fall 1987- January 1988
4. Braiding & Fusion & S & T: Moore & Seiberg Spring 1988
5. Chern-Simons Theory: July 1988- July 1989
6. MTC & Anyons: August 1989 – November 1989

INSTITUTE
for ADVANCED STUDY

REPORT

FOR THE ACADEMIC YEARS
1987·88 AND 1988·89

PRINCETON · NEW JERSEY

Academic Activities of the School: Academic Year 1987–88

In informal collaboration with the School of Natural Sciences and with the help of a grant from the Alfred P. Sloan Foundation, the School of Mathematics held a program in string theory during the academic year 1987–88. It invited a number of senior mathematicians and mathematical physicists, and they led seminars on string theory and related topics.

There were three seminars on conformal field theory: D. Friedan and S. Shenker, term I; G. Segal, term I; and P. Goddard, term II. In addition, D. Olive held one on algebraic aspects of string theory; it was continued by J. Lepowsky and I. Frenkel, who discussed orbifold conformal field theory, aiming at the construction of a remarkable finite group called the Monster. Parallel lectures on the last topic were given by P. Deligne. M. Atiyah led a seminar on applications to 3–dimensional and 4–dimensional manifolds of ideas coming from physics.

In term II, E. Witten continued with a series of lectures that presented a more powerful heuristic approach to some of the same questions. The supergeometry seminar in term I was also related to physics, the theory it expounded being an outgrowth of the treatment by physicists of Fermi fields.

In all these seminars an effort was made to have the first lectures understandable to the non–specialist. The presence of both mathematicians and mathematical physicists in the audience and the corresponding need for communication were helpful.

November 13

Lecture: "Medieval Confraternities: A Re-Assessment" · H
ANDRÉ VAUCHEZ, *University of Paris*

Seminar on Conformal Field Theory · M
STEPHEN SHENKER, *School of Mathematics*, IAS

Seminar on Conformal Field Theory · M
GRAEME SEGAL, *University of Oxford*

November 20

Seminar on Conformal Field Theory · M
GRAEME SEGAL, *University of Oxford*

Seminar on Differential Geometry: "Fundamental group and the growth of the number of closed geodesics in rank 1 manifolds" · M
GERHARD F. KNIEPER, *School of Mathematics*, IAS

Special Lecture: "Knot polynomials, abstract tenors and the Yang-Baxter equation" · M
LOUIS H. KAUFFMAN, *University of Illinois at Chicago*

December 4

Seminar on Conformal Field Theory · M
DANIEL FRIEDAN, *School of Mathematics*, IAS

Seminar on Conformal Field Theory · M
GRAEME SEGAL, *University of Oxford*

IAS-Princeton University Lunchtime Seminar: "Ghostly Polyakov integrals; dual Peierls brackets" · N
MARK RUBIN, *Rockefeller University*

Astrophysics Seminar: "Intrinsic shapes of elliptical galaxies"
MARIJN FRANX, *University of Leiden*

Discussions with D. Friedan @ IAS inspired me to try to prove that the central charge and conformal weights in RCFT are $\in \mathbb{Q}$

IAS tea time: I remarked to Greg Anderson that we were interested in flat vector bundles: He immediately said that we should be studying ODE's with regular singularities.

Rationality in Conformal Field Theory

Greg Anderson[1] and Greg Moore[2]
[1]School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA
[2]School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540 USA

Abstract. We show that if the one-loop partition function of a modular invariant conformal field theory can be expressed as a finite sum of holomorphically factorized terms then c and all values of h are rational.

and the degeneracy of the states with $h=\bar{h}=0$ is exactly one. Another distinguishing characteristic of conformal field theory is modular invariance, which, among other things states that the one loop partition function:

$$Z(\tau) = \mathrm{Tr}\, q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} = \sum_{a,b \geqq 0} N_{ab} \chi(h_a, c) \bar{\chi}(\bar{h}_b, c) \tag{1.2}$$

is modular invariant. Here N_{ab} is the degeneracy of representations $(h_a, \bar{h}_b)$, $\chi(h,c)$ is the character of the representation $V(h,c)$, and $q = e^{2\pi i \tau}$, where $\tau \in \mathscr{H}$, the upper

Representations of what???? We don't say.

Used modular differential equations.

ON THE CLASSIFICATION OF RATIONAL CONFORMAL FIELD THEORIES

Samir D. MATHUR, Sunil MUKHI and Ashoke SEN
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

Received 19 July 1988

We propose a method for classifying rational conformal field theories in terms of the differential equation satisfied by their characters.

Towards a classification of two-character rational conformal field theories

---

A. Ramesh Chandra and Sunil Mukhi

Indian Institute of Science Education and Research,
Homi Bhabha Rd, Pashan, Pune 411 008, India

E-mail: ramesh.ammanamanchi@gmail.com, sunil.mukhi@gmail.com

ABSTRACT: We provide a simple and complete construction of infinite families of consistent

QUASICRYSTALLINE COMPACTIFICATION

J. HARVEY

Princeton University, Princeton, NJ 08540, USA

G. MOORE

Institute for Advanced Study, Princeton, NJ 08540, USA

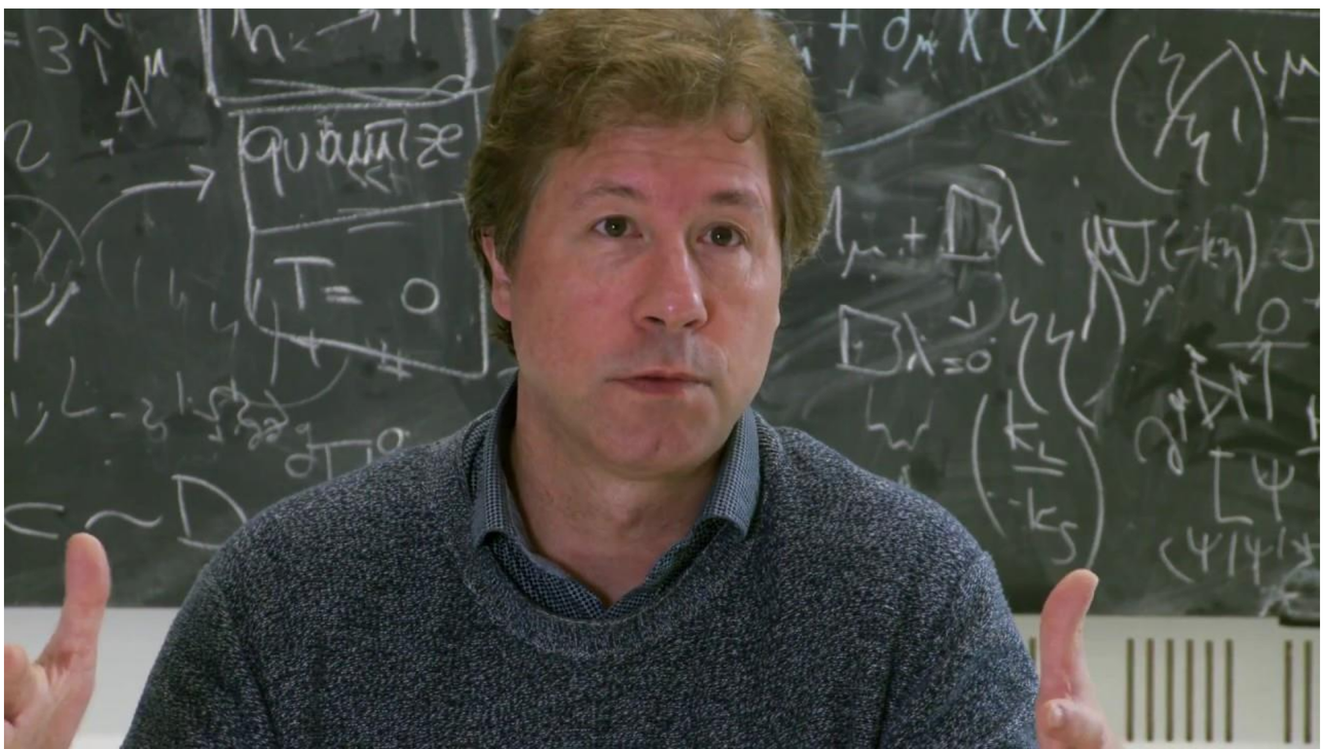
C. VAFA

Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Received 14 October 1987

A class of asymmetric orbifolds is constructed using ideas from the theory of quasicrystals. These orbifolds provide examples of solutions to string theory which are isolated in the sense that they do not belong to a continuous moduli space of solutions and moreover cannot be approximated by rational orbifolds. One of the notable features of the construction is that many aspects of the models are easily handled with the theory of cyclotomic fields.

Jan. 5, 1988: Jadwin Hall, P.U. E. Verlinde's talk on modular transformations and the fusion rule algebra



E. Verlinde, Talk at Princeton 1/5/88 G.M.

RELATION BETWEEN FUSION RULES AND MODULAR TRANSFORMATIONS

2D Conformal Field Theory

Features for $c<1$

- unitarity $\rightarrow$ c & h : rational numbers (FQS)
- null states $\rightarrow$ fusion rules (BPZ)
- characters: finite dim. rep. of modular group.

Possible approach to $c \geq 1$

Consider CFT's with extended chiral algebra with same features as Virasoro for $c<1$.
They will be called rational CFT's.

We will discuss

- relation: fusion rules $\Leftrightarrow$ rep. of modular group
- examples: Ising model
  Toroidal compact. (gaussian model)
  SU(2)-WZW models

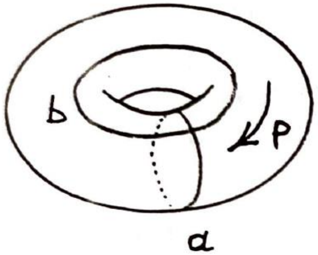
First I thought about the operators in the c=1 model constructed out of the integrals of del phi, which later became a special case of the Verlinde operators. I discussed these ideas with Robbert and Herman at the time, and this ended up in the proceedings of the Cargese school. My goal was to see if similar operators could be defined for the other conformal field theories. Holomorphic factorisation was important for this, which was already emphasised in the work of Dan and Steve. In October 1987 I made my breakthrough.

- Erik Verlinde, Sept. 2024

Partition function

$$Z = \int [dX]\, e^{-\int \partial X \cdot \bar{\partial} X}$$

$$= \sum_{p,\bar{p}} \chi_p \, \bar{\chi}_{\bar{p}}$$



$$\boxed{\chi_p = \frac{q^{\frac{p^2}{2}}}{\eta(q)^d}}$$ charact…

Momentum flux

through cycle c: $\quad P(c) = \frac{1}{2\pi} \oint_c \partial X$

Characters are eigen states of $P(a)$ but not of $P(b)$ because: $[P(a), P(b)] = \frac{1}{2\pi i}$

$$\boxed{\begin{aligned} P(a)\, \chi_p &= \left\langle \frac{1}{2\pi} \oint_a \partial X \right\rangle_p = p\, \chi_p \\ P(b)\, \chi_p &= \left\langle \frac{1}{2\pi} \oint_b \partial X \right\rangle_p = \frac{1}{2\pi i} \frac{\partial}{\partial p} \chi_p \end{aligned}}$$

Modular Transf. = Canonical Transf. !

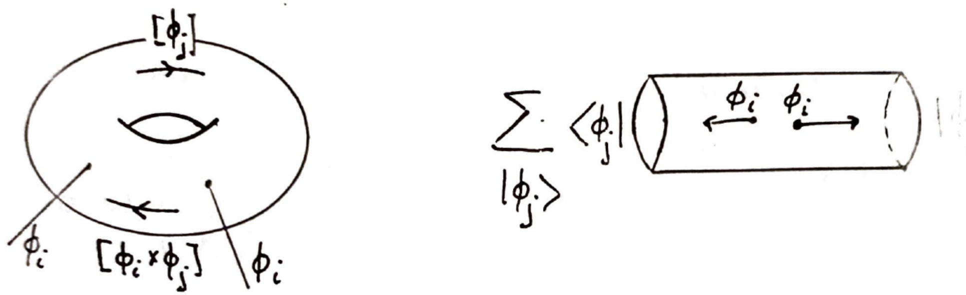
S: $\tau \to -1/\tau$ acts as Fourier …

Recalling what was done in the DVV Cargese paper

General Idea

Insert $\mathbb{1}$ into $\chi_j = \mathrm{tr}_{[\phi_j]}(q^{L_0 - \frac{c}{24}})$ and

rewrite $\mathbb{1}$ as OPE $\phi_i \times \phi_i$ (most singular p



Move ϕ_i around cycle and then $\phi_i \times \phi_i \to$

a-cycle: representation $[\phi_j]$ does not chan

χ_j is eigenstate of $\phi_i(a)$

b-cycle: $[\phi_j]$ changes according to fusion rules

$$\phi_i(b): \chi_j \to \mathrm{tr}_{[\phi_i \times \phi_j]}(q^{L_0 + \frac{c}{24}}) = \sum_{\langle \phi_i \phi_j \phi_k \rangle \neq 0} \chi_k$$

Modular transf. S interchanges a & b-cycle

S diagonalizes the fusion rules!

Generalizes to all RCFTs.

Identifies fusion rules with matrix coefficients of the Verlinde operator.

Fusion rules define an ALGEBRA.

From the construction, obviously diagonalized by the S-matrix.

Next day there was an important meeting
(first of three in this talk) in Witten's office in D-building

with Kazhdan, Moore, Seiberg, and Witten.
Trying to figure out what Erik had said.
Witten: ``It's magic.''

Erik released a preliminary version of his paper during his Princeton visit.

Crucial discussion with Erik about how to prove it.
He suggests looking at 2-point functions on the torus.

FUSION RULES AND MODULAR TRANSFORMATIONS IN 2D CONFORMAL FIELD THEORY

Erik VERLINDE

Institute for Theoretical Physics, University of Utrecht, P. O. Box 80.006, 3508 TA Utrecht, The Netherlands

Received 15 March 1988

We study conformal field theories with a finite number of primary fields with respect to some chiral algebra. It is shown that the fusion rules are completely determined by the behavior of the characters under the modular group. We illustrate with some examples that conversely the modular properties of the characters can be derived from the fusion rules. We propose how these results can be used to find restrictions on the values of the central charge and conformal dimensions.

identity:

$$A_{ij}{}^{k} = \sum_{n} S_j^n \lambda_i^{(n)} S_n^{\dagger k} . \tag{3.11}$$

Note that the r.h.s. indeed satisfies the property (3.7). By using in addition to (3.11) the fact that $A_{i0}{}^{k} = \delta_i^k$ we can also express the eigenvalues $\lambda_i^{(n)}$ in terms of the entries of the matrix S_i^n

$$\lambda_i^{(n)} = S_i^n / S_0^n . \tag{3.12}$$

Together with (3.11) this allows us to compute the coefficients A_{ijk} if we know the matrix S. In this way we have calculated the coefficients $A_{ij}{}^{k}$ for many rational CFT's for which the modular properties are known, and for all of them we have indeed found that:

$$A_{ijk} = N_{ijk} . \tag{3.13}$$

We conjecture that this is true for every RCFT, but a proof of this fact requires a better understanding of the operators $\phi_i(\boldsymbol{b})$. We have shown that the coefficients

Elegant. Readable. Convincing.

David Kazhdan’s quip:

Physics is very interesting, there are many interesting theorems. Unfortunately, there are no definitions.

a.) Identification of matrix coefficients of Verlinde operators with fusion rules.

b.) Kazhdan remark: $N_{888} > 1$ in $su(3)$

c.) Missing notion of a chiral vertex operator.

1. Prehistory: A. 2dCFT and B. String Theory
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4. **Braiding & Fusion & S & T: Moore & Seiberg Spring 1988**
5. Chern-Simons Theory: July 1988- July 1989
6. MTC & Anyons: August 1989 – November 1989

January-February 1988: Discussions start with Seiberg (and Tom Banks)

1. BPZ/FQS minimal models/unitarity classification
2. Constraints of modular invariance
3. c=1 classification of DVV & Ginsparg
4. Capelli,Itzykson,Zuber classification of SU(2) WZW models
5. Verlinde's conjecture

The time for systematic classification seemed right.

Someone (Goddard, Segal, or Witten) put Tsuchiya-Kanie's paper in my hand

Vertex Operators in Conformal Field Theory on $\mathbb{P}^1$ and Monodromy Representations of Braid Group

Dedicated to Professor Hirosi Toda on his 60th birthday

Akihiro Tsuchiya and Yukihiro Kanie

Contents

Received March 4, 1987.

Our aim in this paper is to give rigorous foundations to the work of [KZ], and to reformulate and develop the operator formalism in the conformal field theory on the complex projective line $\mathbb{P}^1$. The space $\mathscr{H}$ of operands is taken to be a sum $\mathscr{H} = \sum_{j=0}^{l/2} \mathscr{H}_j$ of the integrable highest weight modules $\mathscr{H}_j$ of the affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ of type $A_1^{(1)}$. We fix the value ℓ (positive integer) of the central element c

Notice: Completely chiral, even the Hilbert space $\mathcal{H}$!

$$\mathcal{H}_{WZW} \cong \bigoplus_{0 \leq \lambda \cdot \rho \leq k} \quad V_\lambda \otimes \bar{V}_\lambda$$

Chiral vertex operators

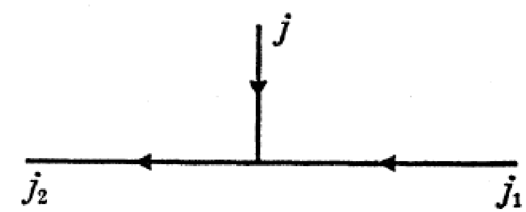
A triple $\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix}$ of nonnegative half integers j_2, j_1 and j is called a *vertex*. Put $\hat{\Delta}(\mathrm{v})=\Delta_j+\Delta_{j_1}-\Delta_{j_2}$. Then the *Clebsch-Gordan condition*

$$|j_1-j_2|\le j\le j_1+j_2 \quad \text{and} \quad j_1+j_2+j\in\mathbb{Z}$$

for a vertex v is a condition for $\mathrm{Hom}_{\mathfrak{g}}(V_j\otimes V_{j_1}, V_{j_2})\neq 0$. In this case $\mathrm{Hom}_{\mathfrak{g}}(V_j\otimes V_{j_1}, V_{j_2})=\mathbb{C}$ and v is called a CG-vertex.

For a vertex $\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix}$ with $j_2, j_1\le \ell/2$, a vertex operator $\Phi(z)$ of spin j is called *of type* v, if $\Phi(u; z)=\Pi_{j_2}\Phi(u; z)\Pi_{j_1}$ for any $u\in V_j$, where Π_i is the projection of $\mathscr{H}$ (or $\hat{\mathscr{H}}$) onto $\mathscr{H}_i$ (or $\hat{\mathscr{H}}_i$ respectively). Then we get the condition for the existence of vertex operators:

v:



Theorem 1 (Proposition 2.1 and Theorem 2.2).

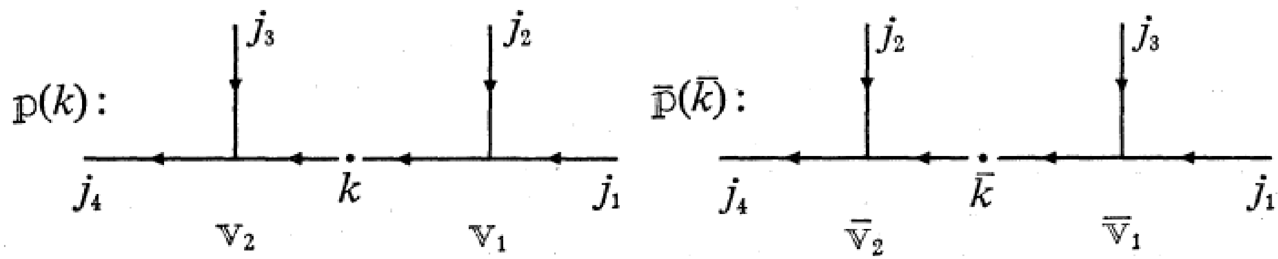
i) *A vertex operator $\Phi(z)$ of type v is uniquely determined by the form (initial term) $\Phi_0\in\mathrm{Hom}_{\mathfrak{g}}(V_{j_2}^{\dagger}\otimes V_j\otimes V_{j_1}, \mathbb{C})$ defined by*

$$\Phi_0(v, u, w)=(z^{\hat{\Delta}(\mathrm{v})}\langle v|\Phi(u; z)|w\rangle)|_{z=0} \qquad (v\in V_{j_2}^{\dagger},\ u\in V_j,\ w\in V_{j_1}).$$

ii) *There exists a nonzero vertex operator Φ of type $\mathrm{v}=\begin{pmatrix} & j & \\ j_2 & & j_1\end{pmatrix}$ on $\mathscr{H}$, if and only if the vertex v is an ℓCG-vertex, that is, it satisfies the ℓ-constrained Clebsch-Gordan condition*:

$$|j_1-j_2|\le j\le j_1+j_2,\quad j_1+j_2+j\in\mathbb{Z} \quad \textit{and} \quad j_1+j_2+j\le \ell.$$

Braiding & Yang-Baxter

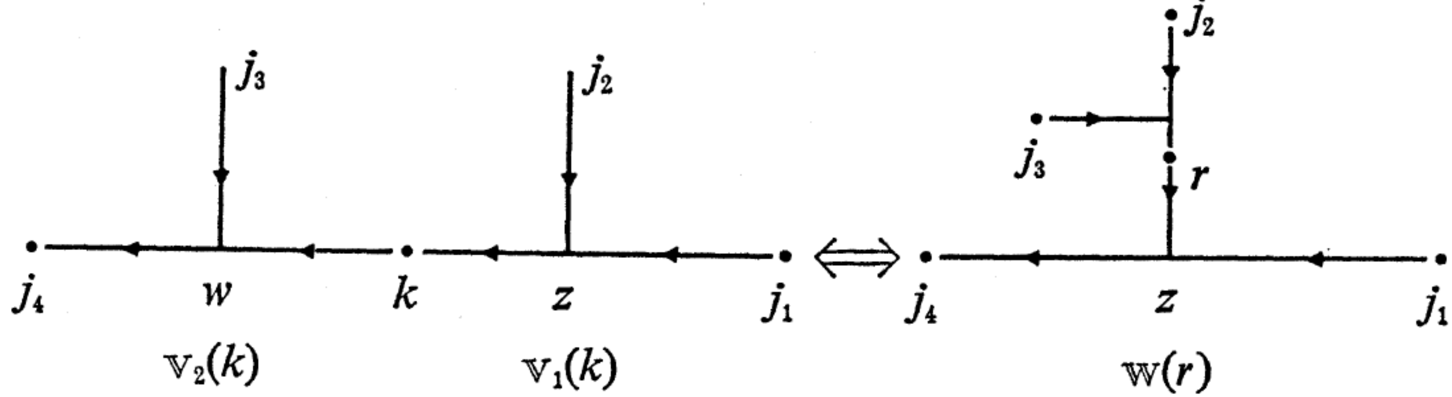


Proposition 4.2. i) *Let* $\mathbb{J}=(j_4, j_3, j_2, j_1)$ *with* $I_\ell(\mathbb{J})\neq\emptyset$. *Then for each intermediate edge* $k\in I_\ell(\mathbb{J})$ *and* $(w,z)\in I_2$,

$$T\Phi_{\mathbf{v}_2(k)}(z)\Phi_{\mathbf{v}_1(k)}(w)=\sum_{\bar{k}\in I_\ell(\mathbb{J})}\Phi_{\mathbf{v}_2(\bar{k})}(w)\Phi_{\mathbf{v}_1(\bar{k})}(z)C^{\bar{k}}_{k}(\mathbb{J}),$$

where the operator in the left hand side is considered as the analytic continuation of the composition of the vertex operators $\Phi_{\mathbf{v}_2}(w)$ *and* $\Phi_{\mathbf{v}_1}(z)$ *along the path* $b(t)$ *in the manifold* X_N.

Tsuchiya-Kanie study fusion



But end their paper with complicated explicit formulae for connection coefficients for the KZ equation in the SU(2) WZW case. They did not introduce a fusion tensor.

Important follow-up to Tsuchiya-Kanie.

Ann. Inst. Fourier, Grenoble
37, 4 (1987), 139-160

MONODROMY REPRESENTATIONS OF BRAID GROUPS AND YANG-BAXTER EQUATIONS

by Toshitake KOHNO

INTRODUCTION

The purpose of this paper is to give a description of the monodromy of integrable connections over the configuration space arising from classical Yang-Baxter equations. These monodromy representations define a series of linear representations of the braid groups $\theta : B_n \to \mathrm{End}(W^{\otimes n})$ with one parameter, associated to any finite dimensional complex simple Lie algebra $\mathfrak{g}$ and its finite dimensional irreducible representations $\rho : \mathfrak{g} \to \mathrm{End}(W)$. By means of trigonometric solutions of the quantum Yang-Baxter equations due to Jimbo ([10] and [11]), we give an explicit form of of these representations in the case of a non-exceptional simple Lie algebra and its vector representation (Theorem 1.2.8) and in the case of $\mathfrak{sl}(2,\mathbf{C})$ and its arbitrary finite dimensional irreducible representations (Theorem 2.2.4).

Our monodromy representation θ commutes with the diagonal action of the q-analogue of the universal enveloping algebra of $\mathfrak{g}$ in the sense of Jimbo [9], which was discussed as quantum groups by Drinfel'd [7]. In particular, in the case $\mathfrak{g} = \mathfrak{sl}(m,\mathbf{C})$, the representation θ gives Hecke algebra representations of B_n appearing in a recent work of Jones [14].

The study of these monodromy representations is motivated by a recent development of two dimensional conformal field theory initiated by Belavin, Polyakov and Zamolodchikov [5]. The importance of the two dimensional conformal field theory with gauge symmetry was

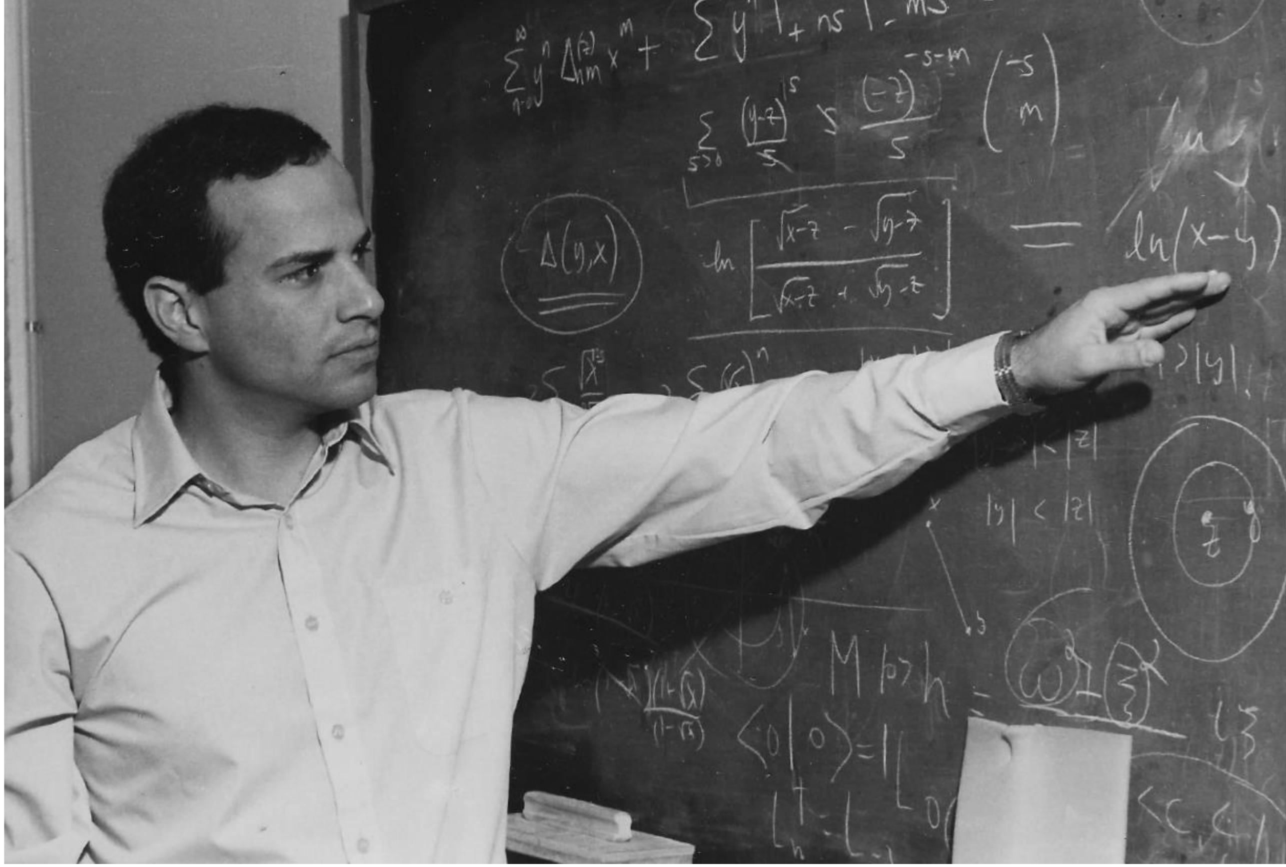
Key-words: Braid group - Yang-Baxter equation - Simple Lie algebra - Integrable connection.

Studied monodromy representations provided by the Knizhnik-Zamolodchikov equations on the plane, and the relation to the braid group representations.

It did not influence me, but it did influence others in the field: Fröhlich, Segal, Witten

Did not combine the operations of braiding and fusing or generalize to higher genus.

Discussions with Seiberg:



Polynomial constraints on braiding, fusion, and modular transformation matrices

April 22
IAS-Princeton University Lunchtime Seminar:
"Verlindemagic and modular geometry" - N
GREGORY MOORE, *School of Natural Sciences*, IAS

AIM: CLASSIFICATION OF RCFT

HOPE: SOLUTION WILL GIVE CLUES TO DEEPER NATURE OF S.T.

CLAIM: MONODROMY OF CONFORMAL BLOCKS SATISFIES RICH IDENTITIES: USEFUL FOR CLASSIFICATION

I. RCFT - DEFINITIONS - IDENTITIES

II. MODULAR GEOMETRY: $g = 0$

III. MODULAR GEOMETRY: $g = 1$

– PROOF OF VERLINDE'S CONJECTURE –

IV. SEARCH FOR EXTRATOROIDAL INTELLIGENCE

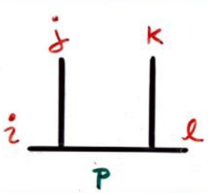
V. RECONSTRUCTION

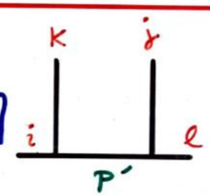
– COLLABORATION WITH NATI SEIBERG –

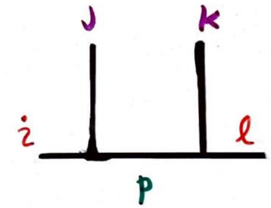
– IN PROGRESS –

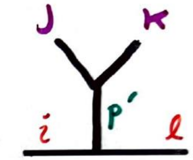
"s-t DUALITY"

$$\Phi^{j}(z)\,\Phi^{k}(w) = \sum G\;\Phi^{k}(w)\,\Phi^{j}(z)$$



$$= \sum_{p'} G_{pp'}\begin{bmatrix} j & k \\ i & l \end{bmatrix}$$




$$= \sum_{p'} N_{pp'}\begin{bmatrix} j & k \\ i & l \end{bmatrix}$$


OPE

$$\Phi^{j}_{ip}(z)\,\Phi^{k}_{pl}(w) \sim \text{sum of } \Phi^{p'}_{il}(w)$$

CFT: STATES ≈ VERTEX OPERATORS ⇒

$$\Phi^{j}_{ip}(z)\,\Phi^{k}_{pl}(w) = \sum_{p'} N_{pp'}\begin{bmatrix} j & k \\ i & l \end{bmatrix} \sum_{P'} \langle \Phi(z-w) \rangle\, \Phi^{P'}(w)$$

REPS p'

DESCENDANTS IN $\mathcal{H}_{p'}$

IMPOSE POLYNOMIAL EQS:

$$CCC = CCC \qquad S^2 = e^{-i\pi\Delta}$$

$$NNN = NN \qquad (ST)^3 = S^2$$

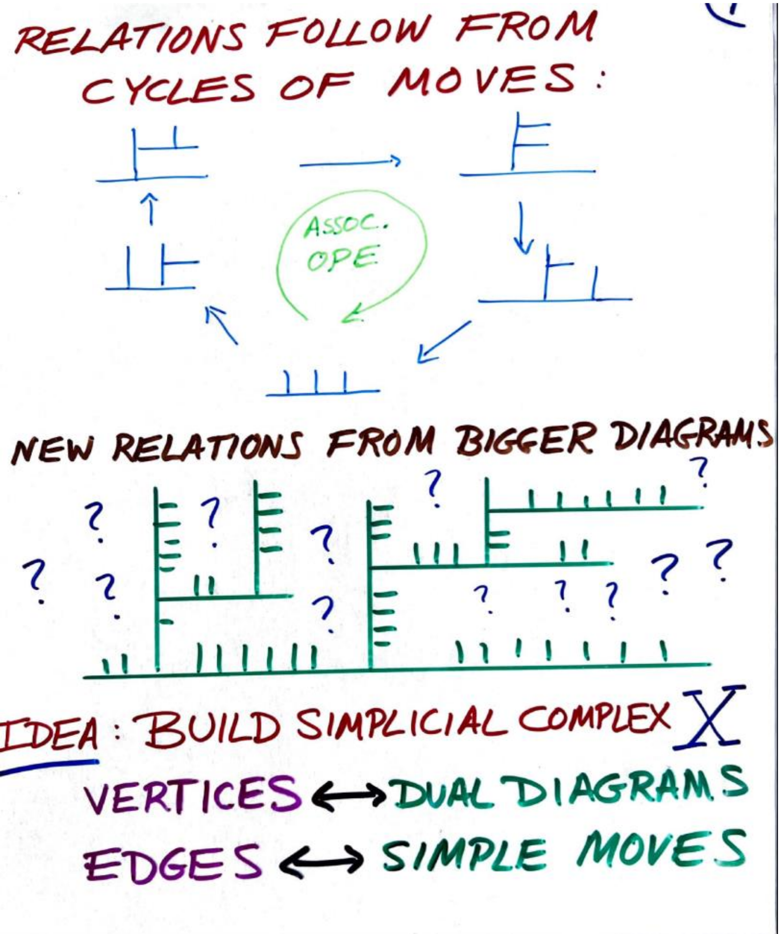
$$C \sim N$$

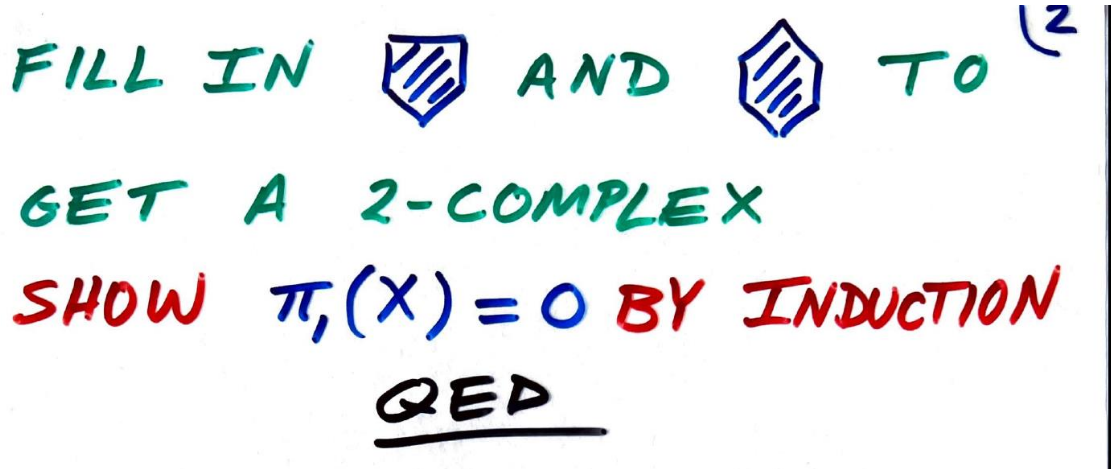
$$N\begin{bmatrix}1 & * \\ * & *\end{bmatrix} = \cdots = 1 \qquad SaS^{-1} = b$$

$$\sum cc\, G\overline{G} = cc \qquad \sum c\, S\overline{S} = c$$

CONJECTURE 2: THIS IS THE <u>FULL</u> SET OF INDEPENDENT IDENTITIES ON THE VARIABLES c, G, N, S FOLLOWING FROM THE AXIOMS OF RCFT

"ALMOST PROVED"





The talk devoted a lot of space to a proof of the Verlinde conjecture using those relations.

2^{nd} important meeting in Witten's office in D-building

The day after my IAS seminar we met with Pierre Deligne (maybe also Kazhdan) in Witten's office. Deligne explained to us about tensor categories.

Attitudes About Categories

I was aware of them, but had not studied anything in depth.

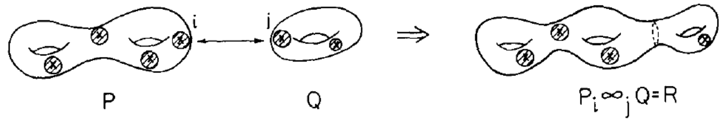
I did not take category theory seriously, regarding it as overly formal abstract nonsense with no real content.

And I generally made fun of category theory.

CONFORMAL THEORIES AND PUNCTURED SURFACES

Cumrun VAFA
Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Received 14 September 1987



We define conformal theories as realizations of certain operations involving punctured Riemann surfaces (with coordinates chosen at the punctures) in a Hilbert space. We describe the connections of our formalism with other formulations of conformal theories.

Nuclear Physics B303 (1988) 455–521
North-Holland, Amsterdam

STRINGS IN THE OPERATOR FORMALISM

L. ALVAREZ GAUMÉ
Physics Department, Boston University, Boston, MA 02215, USA

C. GOMEZ
Theory Division, CERN, CH 1211, Geneva 23, Switzerland

G. MOORE
Institute for Advanced Study, Princeton, NJ 08540, USA

C. VAFA
Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Received 4 January 1988

We consider string theories in the operator formalism. In particular we develop Polyakov strings completely in the operator language. The operator approach turns out to be economical, self-contained and manifestly modular invariant.

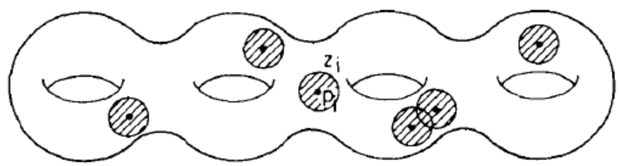


Fig. 4. A Riemann surface Σ with n punctures p_i and n coordinates in the neighborhood of punctures.

My joking response to Cumrun:
``Oh you're just defining a functor.''

We both laughed, and that was that.

From writeup of my Cargese talk: Vanishing Vacuum Energies, 10/87:

Thus, the set of boundary conditions $\{(h,g)\}$ is broken up into orbits. The condition of modular invariance is that $Z(h,g)(a\tau + b/c\tau + d) = Z(h^a g^c, h^b g^d)(\tau)$ [4]. Therefore, if

[4] and so, I suppose, the Z-functor is a morphism from the "category" of boundary conditions to the "category" of partition functions.

Most other physicists I talked to didn't know about category theory, and didn't like it when it was explained.

REMARK (E. Witten) RELATED TO BIRKHOFF COHERENCE THEOREM OF CATEGORY THY.

if $z_{ij} \in \mathcal{H}_{-\epsilon}$, $B(\epsilon)$ changes γ accordingly. The relations (2.3) define the faces and as a result $\pi_1(\tilde{C}_{0,n}) = 1$. The edge-path groupoid [8] of $\tilde{C}_{0,n}$ defines the *duality groupoid* $D_{0,n}$. $\Gamma_{0,n}$ acts on $\tilde{C}_{0,n}$ so $\Gamma_{0,n}$ is a subgroup of $D_{0,n}$.

To introduce conformal blocks, thicken D to obtain a partition of the n-holed sphere into pants (a.k.a. *trinions*) together with a Fenchel-Nielsen coordinate system (ℓ_i, θ_i). For

[3] E. Witten has remarked that these statements are closely analogous to the Birkhoff coherence theorem in category theory.

associating an operator Ψ to each trinion, composing operators according to the partition of the surface. The sections $\mathscr{F}_v$ may be analytically continued from the region v to all of T. Duality states that for any pair $v,\ v' \in \tilde{\mathscr{C}}_{0,n}$ there is a duality matrix $A_{v,v'}$ with $\mathscr{F}_v = A_{v,v'} \mathscr{F}_{v'}$ throughout T. The $A_{v,v'}$ form a representa-

[#4] E. Witten has remarked that these statements are closely analogous to the Maclane coherence theorem in category theory.

I met MacLane years later. He sought me out to remind me of this and said he was very amused.

After the Deligne/Kazhdan/Witten response to my April 22, 1988 seminar we took categories seriously, and started studying

NEANTRO SAAVEDRA RIVANO

Catégories tannakiennes

Bulletin de la S. M. F., tome 100 (1972), p. 417-430

<http://www.numdam.org/item?id=BSMF_1972__100__417_0>

Catégories tannakiennes

P. DELIGNE

à A. Grothendieck en témoignage d'admiration
et de reconnaissance

1. Introduction

POLYNOMIAL EQUATIONS FOR RATIONAL CONFORMAL FIELD THEORIES

Gregory MOORE and Nathan SEIBERG [1]
Institute for Advanced Study, Princeton, NJ 08540, USA

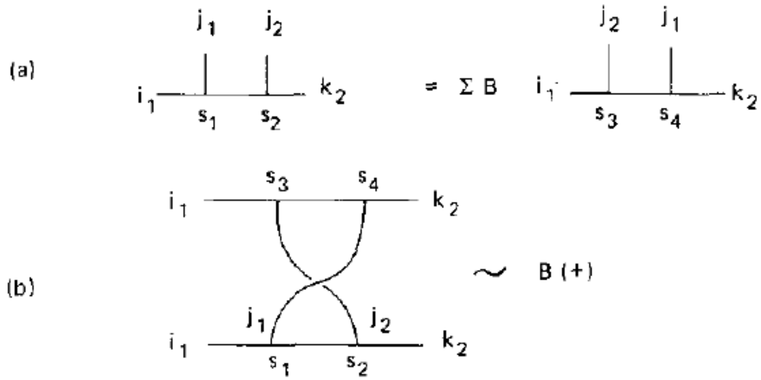
Received 24 May 1988

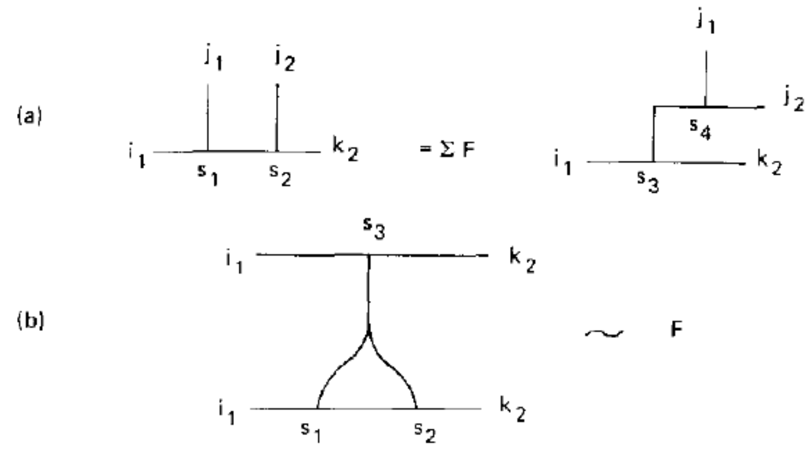
Duality of the conformal blocks of a rational conformal field theory defines matrices which may be used to construct representations of all monodromies and modular transformations in the theory. These duality matrices satisfy a finite number of independent polynomial equations, which imply constraints on monodromies allowed in rational conformal field theories. The equations include a key identity needed to prove a recent conjecture of Verlinde that the one-loop modular transformation S diagonalizes the fusion rules. Using this formalism we show that duality of the $g=0$ four-point function and modular invariance of all one-loop one-point functions guarantee modular invariance to all orders. The equations for duality matrices should be useful in the classification of conformal field theories.

No categories. (Other than footnote with blooper.)

Change C, N to B, F.

``Almost proved'' $\Rightarrow$ proved.





is symmetric and one may then deduce

$$S_{\bar{i}\bar{j}} = \kappa \exp\{-2\pi i[\Delta(i)+\Delta(j)]\}\,\frac{B^2(-)^{ij'}_{ij}}{B(-)^{ii'}_{ii}B(-)^{jj'}_{jj}}\,,$$

where κ is a normalization factor.

But we failed to appreciate the importance/ significance.

Other than the fact that it allowed us to give projective representations of mapping class groups at all genus and thereby define a (finite-dimensional) modular geometry in the sense of Friedan & Shenker.

3-step classification program for RCFTs

1. Classify solutions to the polynomial equations.
   (i.e. classify MTC’s.)

2. Use categorical viewpoint and try to imitate ``Tannakian’’ reconstruction to construct the chiral theory – up to holomorphic conformal field theories with trivial braiding/fusion/S-matrix.

3. Then classify the ways chiral parts can be glued together to make a full RCFT.

Combining Left and Right movers:

NATURALITY IN CONFORMAL FIELD THEORY

Gregory MOORE and Nathan SEIBERG*

Institute for Advanced Study, Princeton, NJ 08540, USA

Received 2 August 1988

We discuss constraints on the operator product coefficients in diagonal and nondiagonal rational conformal field theories. Nondiagonal modular invariants always arise from automorphisms of the fusion rule algebra or from extensions of the chiral algebra. Moreover, when the chiral algebra has been maximally extended a strong form of the naturality principle of field theory can be proven for rational conformal field theory: operator product coefficients vanish if and only if the corresponding fusion rules vanish; that is, if and only if the vanishing can be understood in terms of a symmetry. We illustrate these ideas with several examples. We also generalize our ideas about rational conformal field theories to a larger class of theories: "quasi-rational conformal field theories" and we explore some of their properties.

1. Automorphisms of the fusion rule algebra. (Parallel paper of Dijkgraaf-Verlinde.)

$$S_{i^\vee j} = S_{ij}^{-1} = \frac{S_{00}}{F_j}\lambda_i^{(j)} = S_{00}\frac{B^2(-)^{ij'}_{ij'}}{F_i F_j},$$

But again we failed to appreciate the importance/significance.

And we introduced more formally the idea of extensions of chiral algebras as a way to generate new RCFTs'

Add an integer spin holomorphic field to the chiral algebra (VOA) and only keep fields local with respect to that field.

Condensation

The key idea of “extensions of chiral algebras’’ led to the topic of ``anyon condensation’’ in papers of Bais and Slingerland in their study of spontaneous symmetry breaking of quantum group symmetry

Condensate induced transitions between topologically ordered phases

F. A. Bais[1,2] and J.K.Slingerland[3]

[1]*Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands*
[2]*Santa Fe Institute, Santa Fe, NM 87501, USA**
[3]*Dublin Institute for Advanced Studies, School of Theoretical Physics, 10 Burlington Rd, Dublin, Ireland.*†

(Dated: May 28, 2018)

We investigate transitions between topologically ordered phases in two spatial dimensions induced by the condensation of a bosonic quasiparticle. To this end, we formulate an extension of the theory of symmetry breaking phase transitions which applies to phases with topological excitations described by quantum groups or modular tensor categories. This enables us to deal with phases whose quasiparticles have non-integer quantum dimensions and obey braid statistics. Many examples of such phases can be constructed from two-dimensional rational conformal field theories and we find that there is a beautiful connection between quantum group symmetry breaking and certain well-known constructions in conformal field theory, notably the coset construction, the construction of orbifold models and more general conformal extensions. Besides the general framework, many representative examples are worked out in detail.

Condensation

Nati and I have been very critical of the use of the word "condensation" initiated by Bais and Slingerland

(and it has since become widespread and standard)

But… Preparing for this talk I found an unpublished M&S paper …

effect. Since these Wilson lines correspond to the effects of insertions of 't Hooft operators we may say that the 2+1 dimensional objects created by the 't Hooft operators have "condensed." It is for this reason that the charge λ of the Wilson line operators has no physical meaning: only the charge modulo the charge of the condensing objects has any meaning. The above conditions (3.4) and (3.5) on the central charge and representations, derived in the context of 2D CFT gauarantee the invisibility of the 't Hooft loops and the gauge invariance of the allowed Wilson loops.

For example in the $SO(3)$ theory the operators $W_{k/2}$ have condensed. Demanding that

Anyon condensation and tensor categories

Liang Kong[a,b,1]

[a] Institute for Advanced Study (Science Hall)
Tsinghua University, Beijing 100084, China

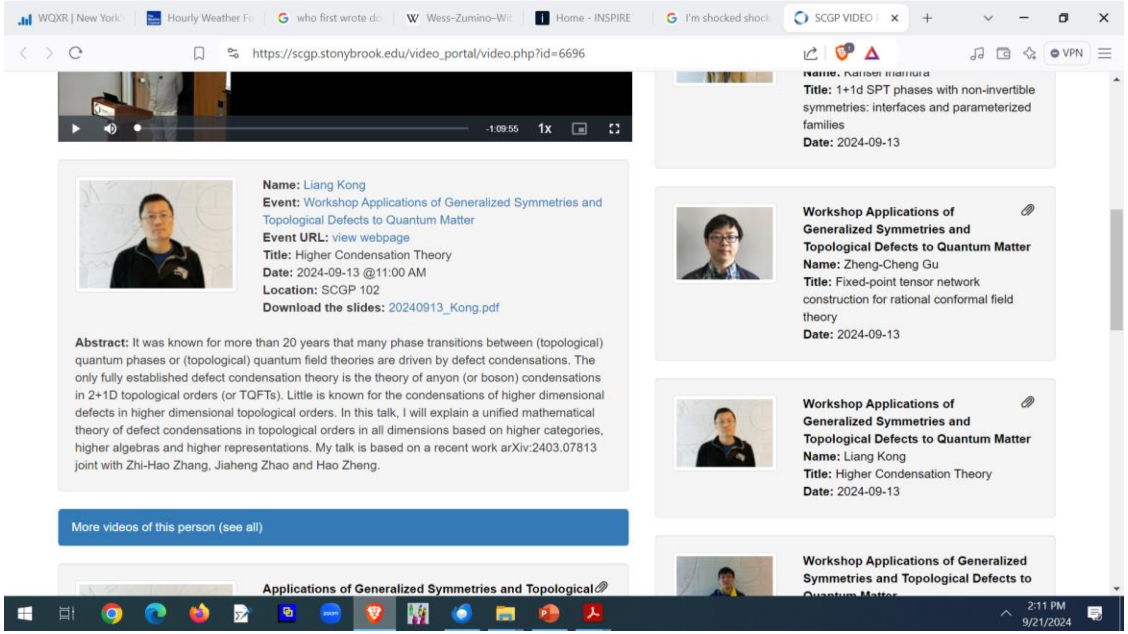
[b] Department of Mathematics and Statistics
University of New Hampshire, Durham, NH 03824, USA

1307.8244

Higher Gauging and Non-invertible Condensation Defects

Konstantinos Roumpedakis[1], Sahand Seifnashri[2,3], and Shu-Heng Shao[2]

2204.02407



SCGP Workshop
20240913

Some Modern Developments:

Combining left-movers and right-movers is now understood in terms of topological defect operators:

Frohlich, Fuchs, Runkel, Schweigert, 0909.5013

Kapustin-Saulina, 1012.0911

A. Davydov, 1312.7466,1412.8505

But a systematic classification like that of CIZ seems out of reach.

Classical and Quantum Conformal Field Theory

Gregory Moore and Nathan Seiberg*
Institute for Advanced Study, Princeton, NJ 08540, USA

Abstract. We define chiral vertex operators and duality matrices and review the fundamental identities they satisfy. In order to understand the meaning of these equations, and therefore of conformal field theory, we define the classical limit of a conformal field theory as a limit in which the conformal weights of all primary fields vanish. The classical limit of the equations for the duality matrices in rational field theory together with some results of category theory, suggest that (quantum) conformal field theory should be regarded as a generalization of group theory.

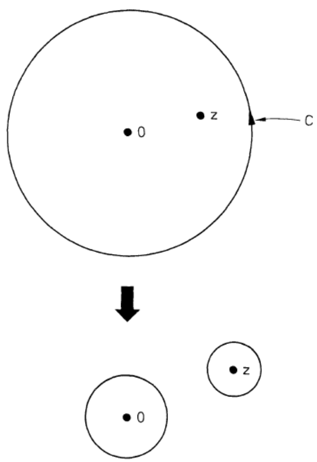


Fig. 5. Deformation of contours used to obtain a "tensor product" of representations

representations to a third representation. Hence, to begin with the general definition, one would like to make sense of $\mathcal{H}_j\otimes\mathcal{H}_k$ as a representation space of $\mathcal{A}$. One cannot take the standard tensor product representation because of the central terms in $\mathcal{A}$. Rather, one uses contour deformation to write a deformed tensor product representation parametrized by $z\neq 0,\ \infty$ in the complex plane $\Delta_{z,0}:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$. To do this define:

$$\begin{aligned}\Delta_{z,0}(\mathcal{O}^i_n)&=\oint_z \zeta^{n+\Delta_i-1}\Big(\sum_m(\zeta-z)^{-m-\Delta_i}\mathcal{O}^i_m\Big)\otimes 1+1\otimes\mathcal{O}^i_n\\&=\sum_{k=0}^{\infty}\binom{n+\Delta_i-1}{k} z^{n+\Delta_i-1-k}\mathcal{O}_{1+k-\Delta_i}\otimes 1+1\otimes\mathcal{O}^i_n.\end{aligned}\tag{2.4}$$

To summarize, we have shown that every compact group (discrete or continuous) leads to a classical conformal field theory on the plane. The correspondence between familiar concepts in group theory and conformal field theory is the following:

Group	Chiral algebra
Representations	Representations
Clebsch-Gordan coefficients/intertwiners	Chiral vertex operators
Invariant tensors	Conformal blocks
Symmetry of couplings	Ω
Racah coefficients ($6j$ symbols)	Fusion matrix
Functions on the group	Physical fields
Product of functions on the group	Operator product expansion
Average over the group of a product of functions	Physical correlation function

Appendix B. The Completeness Theorem

In this appendix we give the proof that the set of equations given in Sect. 4 give a complete set of relations for the duality groupoid. We divide the problem into three steps considering genus zero, one and larger than one in turn.

$\Gamma_{0,n}$; $\Gamma_{1,n}$; $\Gamma_{g,0}$ but not for $\Gamma_{g,n}$ for $g > 1$, for which we were severely arraigned in 1998 by Bakalov and Kirillov:

These questions were studied in a series of pioneering papers of Moore and Seiberg [MS1, MS2]. These authors used spheres with 3 holes (trinions) as their building blocks, and they gave a complete set of simple moves and relations among them. However, their paper [MS2] has some serious flaws. First of all, they use the language of chiral vertex operators, which is important for applications to conformal field theory, but which is not really relevant for finding the set of simple moves and relations, since this question is of purely topological nature. This lead them to miss some "obvious" axioms which are automatically satisfied in any conforaml field theory. What is worse, their proof contains some gaps, the most serious of them being a completely inadequate treatment of the case of surfaces of higher genus with $n > 1$ holes. The reason is that they used an explicit presentation of the mapping class group $\Gamma(\Sigma)$ by generators and relations, found by Wajnryb [Waj], and such a presentation for surfaces of higher genus was known only for surfaces with ≤ 1 holes. (For surfaces with arbitrary number of holes, a presentation of the

Parallel Developments At The Time

Graeme Segal: The definition of conformal field theory

THE DEFINITION OF CONFORMAL FIELD THEORY

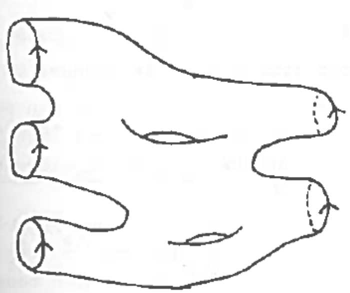
G. B. Segal
Mathematical Institute
24-29 St. Giles
Oxford OX1 3LB
England

I shall propose a definition of 2-dimensional conformal field theory which I believe is equivalent to that used by physicists.

1. THE CATEGORY $\mathcal{C}$

The category $\mathcal{C}$ is defined as follows. There is a sequence of objects $\{C_n\}_{n\geq 0}$, where C_n is the disjoint union of a set of n parametrized circles.

A morphism $C_n \to C_m$ is a Riemann surface X with boundary ∂X, together with an identification $i : C_m - C_n \to \partial X$. (We identify morphisms $(X,i),(X',i')$ if there is an isomorphism $f : X \to X'$ such that $f \circ i = i'$. Notice that the boundary of a Riemann surface is canonically oriented. The identifications i are supposed to be orientation-preserving, and $C_m - C_n$ means the union $C_m \amalg C_n$ with the orientation of C_n reversed.)



A morphism $C_3 \to C_2$.

Como, August 24–29, 1987.

The Definition of Conformal Field Theory

Graeme Segal

St. Catherine's College, Oxford.

The object of this work is to present a definition of a two-dimensional conformally invariant quantum field theory in mathematical language, and to describe the basic examples. I hope this will be helpful to mathematicians who are interested in physics; but apart from that there are several areas of pure mathematics where conformal field theories seem to play a fundamental but quite unexpected role. I shall give five examples.

(i) The "monster" group of Griess-Fischer is the group of automorphisms of a fairly simple and natural conformal field theory. The graded representation of the monster group whose Poincaré series is the modular function j is the basic Hilbert space of the field theory,

§3. Modular functors

We start with a finite set I of labels, containing a distinguished label called 1. There is an operation of "conjugation" $\alpha \mapsto \bar{\alpha}$ on I such that $\bar{1} = 1$.

A modular functor based on I is a rule which associates a finite dimensional vector space $V_{\Sigma,\alpha}$ to each Riemann surface Σ with boundary, where each boundary component of Σ is equipped with a parametrization and also a label from I. (Here $\alpha = (\alpha_1,\ldots,\alpha_k)$ is a multi-index, where $\alpha_i \in I$ is the label of the i^{th} boundary circle.) The spaces $V_{\Sigma,\alpha}$ are required to have the following four properties.

(3.1) $$V_{\Sigma_1 \amalg \Sigma_2, \alpha_1 \amalg \alpha_2} = V_{\Sigma_1,\alpha_1} \otimes V_{\Sigma_2,\alpha_2},$$

where $\Sigma_1 \amalg \Sigma_2$ denotes the disjoint union.

(3.2) If $\check{\Sigma}$ is obtained from Σ by identifying two boundary circles S_1 and S_2 (using their parametrizations) then

$$V_{\check{\Sigma},\beta} = \bigoplus_{\alpha} V_{\Sigma,\beta\alpha\bar{\alpha}},$$

where the sum is over all labellings $\beta\alpha\bar{\alpha}$ of $\partial\Sigma$ which agree with the labelling β of $\partial\check{\Sigma}$ and give conjugate labels $\alpha,\bar{\alpha}$ to S_1 and S_2.

(3.3) $$V_{D,\alpha} = \begin{cases} \mathbb{C} & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha \neq 1, \end{cases}$$

where D is the standard disc.

(3.4) $V_{\Sigma,\alpha}$ depends holomorphically on Σ, in the sense that if $\{\Sigma_t\}$ is a holomorphic family of surfaces parametrized by $t \in T$ then $\{V_{\Sigma_t,\alpha}\}$ is a holomorphic vector bundle on T.

IXth International Congress on Mathematical Physics, Swansea, July 1988. More on that later.

Parallel Developments At The Time Constructive/Algebraic QFT Community:

STATISTICS OF FIELDS, THE YANG-BAXTER EQUATION, AND THE THEORY OF KNOTS AND LINKS

Jürg Fröhlich
Theoretical Physics
ETH-Hönggerberg, CH-8093 Zürich

1. Introduction

In these notes we describe an analysis of the statistics problem in quantum field theory. It has been known for some time that in two space-time dimensions there is more to the problem of statistics of local fields than Bose- or Fermi statistics [1]. In three-dimensional space-time, local quantum fields obey Bose- or Fermi statistics, but "extended particles", coupled to the vacuum by fields localized on cones, may exhibit intermediate statistics, so called Θ-statistics [2].

There is a fairly close, and perhaps somewhat surprising, connection between the statistics problem in two-dimensional quantum field theory and the following three topics:

(1) Soliton quantization in two-dimensional quantum field models; [3,4,5,6].

(2) Two-dimensional, conformal quantum field theory; some aspects of the covariant quantization of superstring theory (in particular, the construction of fermion emission vertices); [7,8,9,10].

(3) Polynomial invariants for knots and links [11]. Some of these connections will be sketched briefly, but there is no room here for a detailed treatment. The results discussed in these notes may be viewed as comments on the beautiful and deep lectures by G. Mack, D. Friedan and K.Gawedzki. Some of my results, next to many other things, are also sketched in the notes of B. Schroer.

The first examples of statistics different from standard Bose- or Fermi statistics in the context of simple, two- dimensional quantum field models were encountered in [1]. A related issue, the construction of order- and disorder variables in two-dimensional classical spin systems such as the Ising model, was analyzed by Kadanoff and Ceva [12]. Since the Euclidean description of quantum field theory (see e.g. [13]) was not a well known thing, at the time, Kadanoff and Ceva do, however, not seem to have realized the implications of their work for the statistics problem in two-dimensional quantum field theory. That problem was studied, independently

NEW METHODS AND RESULTS IN CONFORMAL QFT_2 AND THE "STRING IDEA"

Bert Schroer and FU Berlin
Institut für Theorie der Elementarteilchen
Arnimallee 14, 1000 Berlin 33

Abstract Causal fields in conformal QFT_2 yield a new algebra structure: the exchange algebra in which "braid" matrices (special Yang-Baxter structures) appear. They are directly related to the possible dimensional spectra of local fields. The Virasoro structure, i.e. the central extension charge (=Casimir energy) is part of the representation theory. An extension of Unruh's idea allows us to calculate correlation functions of arbitrary conformal QFT's on higher genus Riemann surfaces directly in terms of the flat space correlations. As a generalization of Hawking's temperature, the new positive definite states are characterized by $3g-3$ "temperatures".

I. Introduction and History

Conformal QFT was introduced[1] in the 60^s. However, the field theory community in those days largely ignored or rejected conformal invariance since apart from free fields it seemed to be at odds with the principle of Einstein causality. In those days the consequences of relativistic causality combined with assumptions on the energy-momentum spectrum were the main research topic. Apart from certain structural theorems[2,3] such as TCP and the connection of spin with statistics, the main experimentally testable results were dispersion relations and certain spectral sum rules.

At the beginning of the 70^s, when the intimate relation between relativistic quantum field theory and statistical mechanics through the method of analytic continuation ("Euclidean quantum field theory") became widely appreciated, the issue of conformal invariance reappeared on the statistical mechanics side, where the "causality paradox" created less of a nuisance. In this spirit of statistical mechanics Migdal[4] and Polyakov[5] proposed the so-called "conformal bootstrap", i.e. the study of the Euclidean Schwinger-Dyson equations with conformal invariant boundary conditions. This programme was mathematically refined with the help of group theoretical ideas and techniques.[6]. However, even with all these refinements it remained too structureless as far as non-perturbative solutions are concerned. On the conceptual side the already mentioned paradox remained a sore, particularly in view of the failure of all attempts (even up to our present days) to soften the postulate of Einstein causality i.e. to replace it by a less stringent (macro) causality postulate. The causality paradox of global conformal transformation was clearly formulated in 1972[7]. After its partial understanding in a very limited setting[8], it was finally solved in a series of papers[9,10,11].

The discovery of a classification scheme for conformal QFT_2 by Belavin, Polyakov and Zamolodchikov[12] (BPZ) with additional important later contributions[13,14,15] did not rely on any of the global operator methods of the 70^s and therefore one may ask why should one be

MS Naturality paper cite credit for chiral vertex operators:

[10] B. Schroer, Nucl. Phys. B295 (1988) 4; Algebraic aspects of non-perturbative quantum field theories, Como lectures;
K.-H. Rehren, Comm. Math. Phys. 116 (1988) 675;
J. Frohlich, Statistics of fields, the Yang-Baxter equation, and the theory of knots and links, lectures at Cargèse 1987, *in* Nonperturbative quantum field theory (Plenum, New York), to be published;
G. Felder and J. Frohlich, unpublished lecture notes;
K.-H. Rehren and B. Schroer, Einstein causality and artin braids, FU preprint;
A. Tsuchiya and Y. Kanie, *in* Conformal field theory and solvable lattice models, Advanced Studies in Pure Mathematics 16 (1988) p. 297; Lett. Math. Phys. 13 (1987) 303

Using the approach to conformal blocks suggested by algebraic quantum field theory one is led to a concept of an ``exchange algebra'' – very similar to the braiding algebra of chiral vertex operators.

ADVANCES IN MATHEMATICS 102, 20–78 (1993)

Parallel Developments In Math

Braided Tensor Categories

ANDRÉ JOYAL

Université du Québec à Montréal, Montréal, Québec H3C 3P8, Canada

AND

ROSS STREET

Macquarie University, North Ryde, New South Wales 2113, Australia

Contents.

INTRODUCTION

Categories enriched with tensor products, here called tensor categories, but also called monoidal categories, have been studied and used extensively in the literature [ML1, EK, ML2, SR, DM]. Large examples such as the categories of Abelian groups and of Banach spaces are important for studying mathematical structures. Small examples, as found in particular in algebraic topology, are important as mathematical structures in their own right.

Some tensor products behave like composition and so are not generally expected to be commutative. Yet categories with a "commutative" tensor product deserve special attention in the same way that commutative rings do in ring theory. Natural examples of commutativity are not strict in the sense of an equality $A \otimes B = B \otimes A$. Rather, natural isomorphisms $c_{A,B}: A \otimes B \to B \otimes A$ exist. In the case of categories of sets with structure $c_{A,B}$ is given on elements by a simple switch in order. It seemed reasonable,

Braided ribbon tensor categories.

Motivations came from topology and higher category theory.

Two talks in 1986 led to a widening interest in braided tensor categories. One was the talk of P. Freyd at the Category Theory Conference at Cambridge University; he described joint work with D. Yetter [FY1] in which they had discovered that their category of tangles was braided and autonomous (that is, each object has a dual; such tensor categories are also called "compact" and "rigid"), and was "free" in some sense. The other was V. Drinfel'd's International Congress talk on quantum groups where the "quasitriangular" bialgebras are examples of braided protensor categories. These talks provided classes of new examples of our structure:

In mid-1987, we were led to deal with this problem in a different way: by looking at tangles of ribbons instead of tangles of strings. This led us to the notion of *balanced tensor categories* which, as well as a braiding, have a *twist* on each object; the free category as such is the category of braids on ribbons. Incorporating dual objects, we were led to define *tortile tensor categories*.

1 Prehistory: A. 2dCFT and B. String Theory

2 Three Roads To RCFT: 1986-1987

3 Princeton: Fall 1987- January 1988

4 Braiding & Fusion & S & T: Moore & Seiberg Spring 1988

5 Chern-Simons Theory: July 1988- July 1989

6 MTC & Anyons: August 1989 – November 1989

Chern-Simons Theory

Complicated and intricate history –
deserving of an entirely separate talk.

1. Witten: Swansea and the Jones Polynomial
2. My trip to the Soviet Union, August 1988
3. Taming the conformal zoo
4. Explicit quantization:

Eliztur et. al. & Axelrod-Della Pietra-Witten

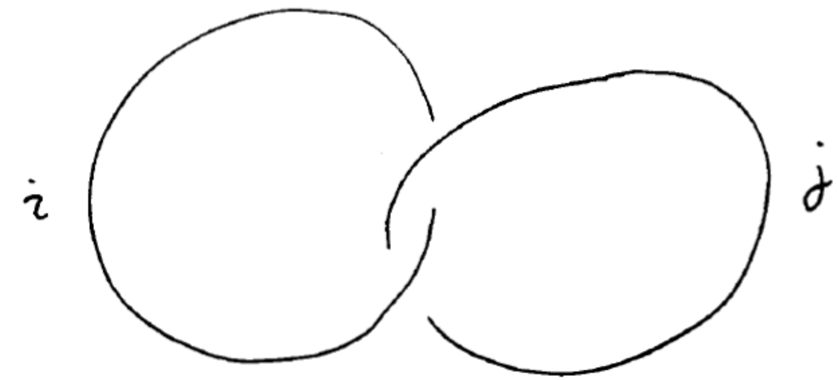
IXth International Congress on Mathematical Physics 17-27 July 1988, Swansea, UK

Atiyah, Segal, and Witten go to dinner at Annie's restaurant.

All the pieces for the CS interpretation of the Jones polynomial fall into place for Witten. He changes his talk for the next day.

3rd Important meeting in Witten's office in D-building. Nati and I are stunned.

$$\frac{S_{ji}}{S_{00}} = \frac{\left(B\begin{bmatrix} j & i \\ j & i \end{bmatrix} B\begin{bmatrix} i & j \\ j & i \end{bmatrix} \right)_{00}}{F_i F_j}$$



Quantum Field Theory and the Jones Polynomial *

Edward Witten **

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. It is shown that $2+1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from S^3 to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in $1+1$ dimensions.

Received September 12, 1988; in revised form September 27, 1988

My trip to the Soviet Union: 2nd half of August 1988

1. Going to the Gelfand seminar in Moscow (and risking my life).
2. Going to Faddeev’s institute in Leningrad (and almost getting arrested)
3. Discussions with Reshetikhin and Turaev: They were working on both quantum group representation theory and 3-manifold invariants.
4. Smuggling out manuscripts: The plane takes off

TAMING THE CONFORMAL ZOO

Gregory MOORE and Nathan SEIBERG [1]
Institute for Advanced Study, Princeton, NJ 08540, USA

Received 18 January 1989

All known rational conformal field theories may be obtained from (2+1)-dimensional Chern–Simons gauge theories by appropriate choice of gauge group. We conjecture that all rational field theories are classified by groups via (2+1)-dimensional Chern–Simons gauge theories.

Following up on a paragraph in Witten's paper ``Taming'' explained how edge states naturally follow from the CSW theory on 3-fold with boundary.

(edge states will be relevant to the FQHE application)

Modern viewpoint is quite different: Boundary field theory & anomaly inflow

The paper also showed that all known RCFT's, and all known methods of generating new RCFT's fit into the framework of CSW theory with a compact Lie group.

Classification conjecture: Compact Lie groups.

Since the known RCFT's are so well organized by CSGT, we conjecture that *all chiral algebras of RCFT arise from the quantization of the 3D CSGT for some compact Lie group*. This conjecture is in accord with the philosophy of ref. [3] which emphasized that RCFT should be viewed as a generalization of group theory. If our conjecture is correct, the classification

Finite groups are compact Lie groups!

Beautiful work by Freed and Dijkgraaf-Witten led to the very fruitful subject of Dijkgraaf-Witten theories

…. Which generalize to π $-$finite TQFT's - a very useful class of TQFTs.

Solving Chern-Simons theory with geometric quantization:

Nuclear Physics B326 (1989) 108–134
North-Holland, Amsterdam

REMARKS ON THE CANONICAL QUANTIZATION OF THE CHERN–SIMONS–WITTEN THEORY

Shmuel ELITZUR

Racah Institute of Physics, Hebrew University, Jerusalem, Israel

Gregory MOORE

Institute for Advanced Study, Princeton, NJ 08540 USA

Adam SCHWIMMER

Department of Physics, Weizmann Institute of Science, Rehovot 76100, Israel

Nathan SEIBERG*

Institute for Advanced Study, Princeton, NJ 08540, USA

Received 6 April 1989

We comment on some aspects of the canonical quantization of the Chern–Simons–Witten theory We carry out explicitly the quantization on several interesting surfaces The connection to the related two dimensional theory is illustrated from different points of view

J. DIFFERENTIAL GEOMETRY
33 (1991) 787–902

GEOMETRIC QUANTIZATION OF CHERN-SIMONS GAUGE THEORY

SCOTT AXELROD, STEVE DELLA PIETRA & EDWARD WITTEN

Abstract

We present a new construction of the quantum Hilbert space of Chern-Simons gauge theory using methods which are natural from the three-dimensional point of view. To show that the quantum Hilbert space associated to a Riemann surface Σ is independent of the choice of complex structure on Σ, we construct a natural projectively flat connection on the quantum Hilbert bundle over Teichmüller space. This connection has been previously constructed in the context of two-dimensional conformal field theory where it is interpreted as the stress energy tensor. Our construction thus gives a $(2+1)$-dimensional derivation of the basic properties of $(1+1)$-dimensional current algebra. To construct the connection we show generally that for affine symplectic quotients the natural projectively flat connection on the quantum Hilbert bundle may be expressed purely in terms of the intrinsic Kähler geometry of the quotient and the Quillen connection on a certain determinant line bundle. The proof of most of the properties of the connection we construct follows surprisingly simply from the index theorem identities for the curvature of the Quillen connection. As an example, we treat the case when Σ has genus one explicitly. We also make some preliminary comments concerning the Hilbert space structure.

… and many other related papers in at the time …

Starting in the early 1980's collaboration via email began. Here's an example:

From: SNSVAX::NATHAN 5-NOV-1988 11:56
To: ADAM, NATHAN
Subj: k+2

Dear Shmuel and Adam

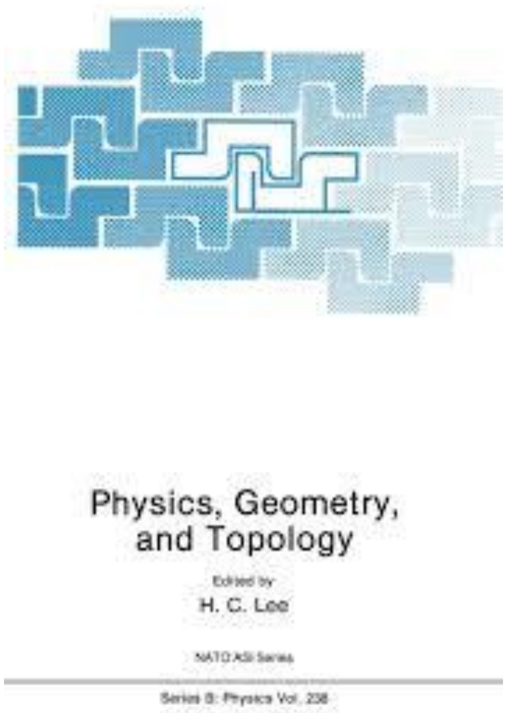
we have an idea for the k+2. It relies on two assumptions which we do not yet know how to prove. Given these two assumptions everything works beautifully.

Assumption 1: in all your expressions k should be replaced by k+2. We think that this can be justified as follows. if one first imposes the constraints and then quantizes, he should use the classical const
with the bare fields and therefore with the constant k. However, if
first quantize and then impose the constraint, you should use the ren
fields and hence with k+2. This should clearly be made more precise.

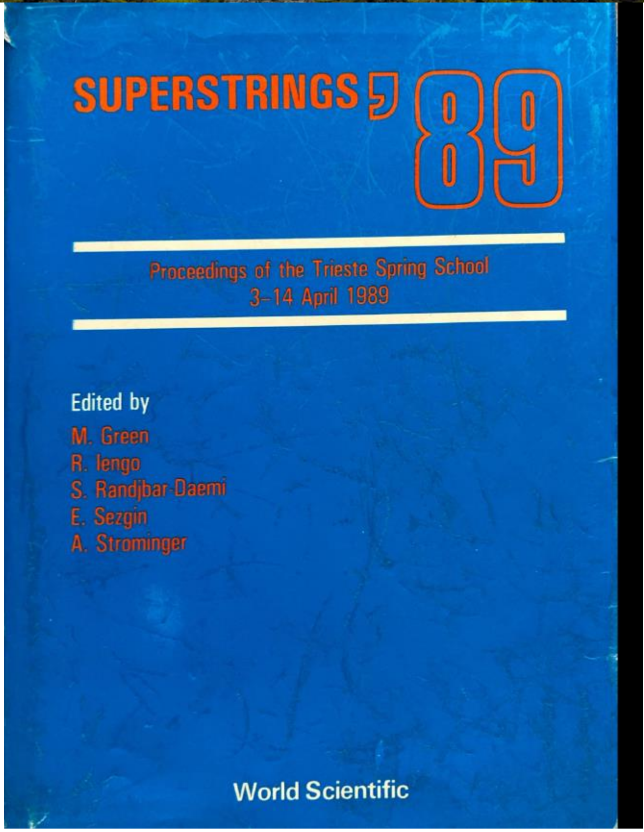
Now when you quantize, you seem to have too many states.

Assumption 2: the states with A_1=0 and A_1= 1/2 (in my normalizati
are in a one dimensional rep of the Weyl group (all those with 0 < A_

MANDATORY PHOTO CREDIT: TransCanada PipeLines Pavilion. Photo courtesy of The Banff Centre



Moore Trieste 1989 Lectures
Seiberg 1989 Banff Lectures



Lectures on RCFT *

Gregory Moore

Institute for Advanced Study

Princeton, NJ 08540, USA

and

Department of Physics

Yale University, New Haven, CT 06511-8167,USA

and

*Nathan Seiberg***

Institute for Advanced Study

Princeton, NJ 08540, USA

and

Department of Physics and Astronomy

Rutgers University, Piscataway, NJ 08855-0849, USA

---

* Given by G. Moore in the Trieste spring school 1989 and by N. Seiberg in the Banff ner school 1989.

On leave of absence from the Department of Physics, Weizmann Institute of Science, vot 76100, ISRAEL.

representations of the Teichmüller modular group. Graeme Segal abstracted the concept, implicitly used from the earliest days of dual model theory and somewhat more precisely described in [1][24][21][13] to the notion of a *modular functor*. A modular functor may be specified by the following data and axioms:

Axioms for a Modular Functor

Data:

1. Representation labels: A finite set I of labels (i.e. the representations of the chiral algebra) with a distinguished element $0 \in I$ and an involution $i \to i^{\check{}}$ such that $0^{\check{}} = 0$.

2. Conformal blocks: A map

$$(\Sigma, (i_1, v_1, P_1), \ldots (i_n, v_n, P_n)) \to \mathcal{H}(\Sigma; (i_1, v_1, P_1), \ldots (i_n, v_n, P_n)))$$

from oriented surfaces with punctures, each puncture P_r being equipped with a direction v_r and a label i_r, to vector spaces.

3. Duality transformations: A linear transformation $\mathcal{H}(f) : \mathcal{H}(\Sigma_1) \to \mathcal{H}(\Sigma_2)$ associated to an automorphism $\Sigma_1 \to \Sigma_2$ (and similarly for punctures).

Conditions:

1. Functoriality: $\mathcal{H}(f)$ depends only on the isotopy class of f. Thus the mapping class group acts on $\mathcal{H}(\Sigma)$, (and similarly for punctures).

2. Involution: If bar denotes reversal of orientation and application of the involution to the representations then $\mathcal{H}(\bar{\Sigma}) \cong \mathcal{H}(\Sigma)^{\check{}}$.

3. Multiplicativity: $\mathcal{H}(\Sigma_1 \coprod \Sigma_2) \cong \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2)$.

4. Gluing: Pinching $(\Sigma, (i_1, v_1, P_1), \ldots (i_n, v_n, P_n))$ along a cycle to obtain a surface (possibly connected or disconnected) $(\tilde{\Sigma}, (i_1, v_1, P_1), \ldots (i_n, v_n, P_n), (j, v, P), (j^{\check{}}, v, \tilde{P}))$ with a pair of identified punctures $P, \tilde{P}$ defines vector spaces related by

$$\mathcal{H}(\Sigma; (i_1, v_1, P_1), \ldots (i_n, v_n, P_n)) \cong \oplus_{j \in I} \mathcal{H}(\tilde{\Sigma}; \ldots (i_n, v_n, P_n), (j, v, P), (j^{\check{}}, v, \tilde{P})).$$

5. Normalization. $\mathcal{H}(S^2; (j, P)) \cong \delta_{j,0} \cdot \mathbf{C}$.

Axioms for a Modular Tensor Category

Data:

1. A finite index set I with a distinguished element 0 and a bijection of I to itself written $i \mapsto i^{\check{}}$.

2. Vector spaces: V^i_{jk} $i,j,k \in I$, with $dim V^i_{jk} = N^i_{jk} < \infty$

3. Isomorphisms:

$$\Omega^i_{jk} : V^i_{jk} \cong V^i_{kj}$$
$$F\begin{bmatrix} j_1 & j_2 \\ i_1 & k_2 \end{bmatrix} : \oplus_r V^{i_1}_{j_1 r} \otimes V^r_{j_2 k_2} \cong \oplus_s V^{i_1}_{s k_2} \otimes V^s_{j_1 j_2} \tag{5.7}$$

4. A constant $S_{00}(0)$.

Conditions:

1. $(i^{\check{}})^{\check{}} = i$, $0^{\check{}} = 0$.

2. $V^i_{0j} \cong \delta_{ij} C \quad V^0_{ij} \cong \delta_{ij} C \quad V^i_{jk} \cong V^{k^{\check{}}}_{j i^{\check{}}} \quad (V^i_{jk})^* \cong V^{i^{\check{}}}_{j^{\check{}} k^{\check{}}}$

3. $\Omega^i_{jk}\Omega^i_{kj} \in End(V^i_{jk})$ is multiplication by a phase.

4. The identities:

$$F(\Omega^\epsilon \otimes 1)F = (1 \otimes \Omega^\epsilon)F(1 \otimes \Omega^\epsilon)$$
$$F_{23}F_{12}F_{23} = P_{23}F_{13}F_{12}$$

for $\epsilon = \pm 1$.

5. The identities

$$S^2(p) = \pm e^{-i\pi\Delta_p} C$$
$$(ST)^3 = S^2$$

where $S(p) \in End(\oplus V^i_{pi})$ is defined by

$$S_{ij}(p) = S_{00}(0) e^{-i\pi\Delta_p} \frac{F_{i0}\begin{bmatrix} i & i \\ p & p \end{bmatrix}}{F_p F_{p0}\begin{bmatrix} j & j \\ j & j \end{bmatrix} F_{p0}\begin{bmatrix} i & i \\ i & i \end{bmatrix}} \sum_r B_{pr}\begin{bmatrix} i & j \\ i & j \end{bmatrix}(-) B_{r0}\begin{bmatrix} j & i \\ i & j \end{bmatrix}(-)$$

C represents the action of $\check{}$, the numbers $\pm e^{-i\pi\Delta_p}$ may be deduced from Ω, and $T : V^i_{ji} \to V^i_{ji}$ is scalar multiplication by $e^{2\pi i(\Delta_i - c/24)}$ for a constant c. (For more details see [15].)

These are the same as the modern axioms for an MTC expressed in terms of tensors derived using a set of simple objects.

Except:

Imposed modular identities on S.

The name modular tensor category was invented by Igor Frenkel

The first time it appeared in print

2. $\exists$ INVERTIBLE F.T. $S: \mathfrak{A} \to \mathfrak{A}$

$$S(f * g) = S(f) \cdot S(g)$$

$$\Rightarrow \quad S' \sim B^2/_{F \cdot F}$$

4. (2) & (3) ABOVE, VIEWED AS CONDITIONS ON S, ARE EQUIVALENT TO 3 TORUS EQS.

Modern Definition (Turaev 92)

MTC: = Semisimple linear ribbon category with finitely many simple objects and nondegenerate matrix:

$$S_{ij} := Tr\, b_{i,j} \circ b_{j,i}$$

Ribbon category = braided tensor category with twists and compatible duals.

Modern Definition

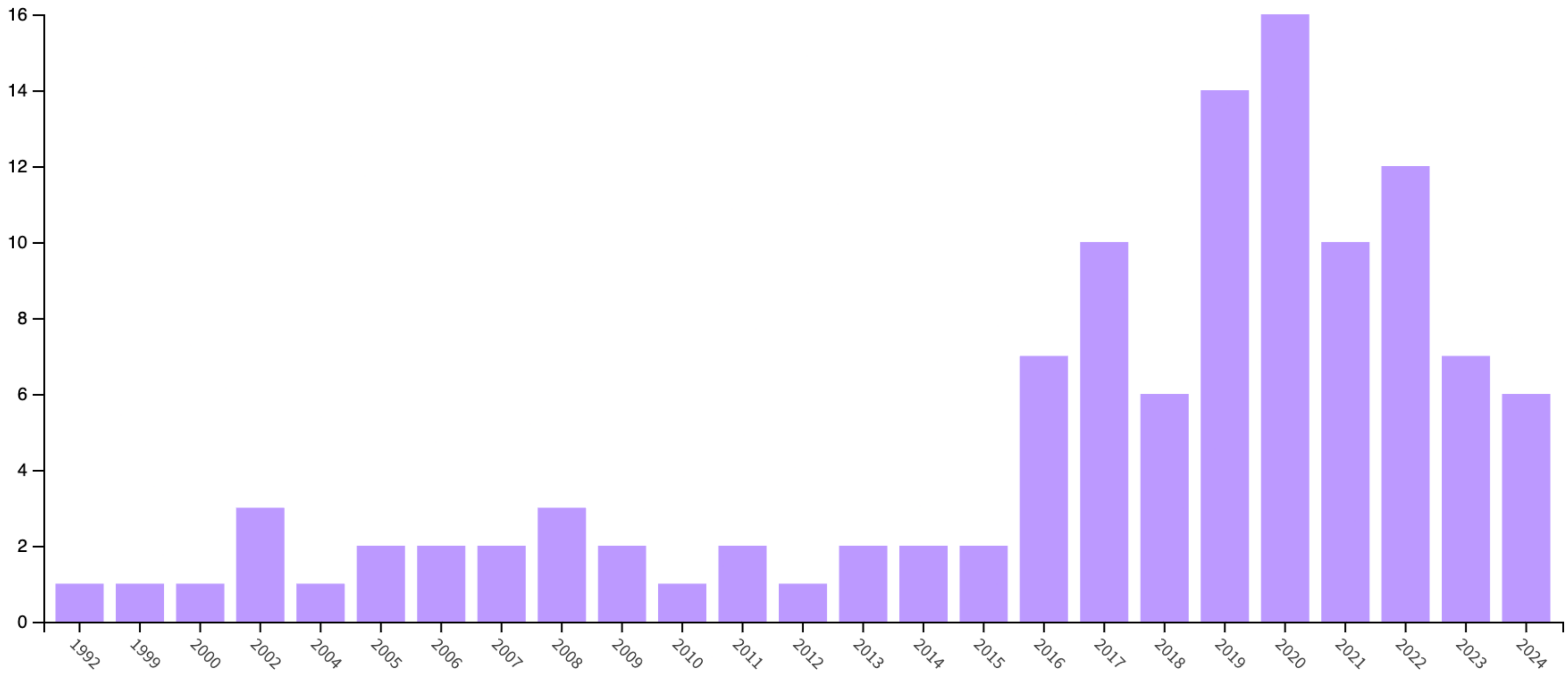
Emerged from works:

M&S, Lectures on RCFT

Reshetikhin-Turaev: 2 famous papers on quantum groups and invariants of links in 3-folds

Turaev, Modular categories and 3-manifold invariants, Int. J. Mod. Phys. 1992

Lyuboshenko, …



Web of Science search (20240922) for ``modular tensor category'' in the abstract of published papers in its data base, as a function of time: 116 results. Note: zero between 1992 and 1999.

The classification conjecture:

Conjecture 1: The modular functor of any unitary RCFT is equivalent to the modular functor of some CSW theory defined by the pair (G, λ) with G a compact group and $\lambda \in H^4(BG; \mathbb{Z})$.

that there are substantial reasons for believing conjecture 1 is correct. As we have discussed, one might imagine a proof to proceed along lines very similar to the theorems of Deligne and Doplicher-Roberts. On the other hand, it would be fascinating if there were examples of "sporadic" modular tensor categories arising from conformal field theories. In the introduction we pointed out that an alternative statement of the conjecture says that all RCFT's have already been found. It was probably first stated by Emil Martinec [7] that the nontrivial RCFT's are essentially exhausted by the coset construction, and this was repeated in [9]. It has been reiterated many times in private by Bazhanov, Fröhlich, Gawedzki, Goddard, Reshetikhin, and perhaps others.

With most straightforward interpretation of ``equivalence'' the conjecture is false: Teleman 2021

The Haagerup TQFT is not a gauge theory

A recurring conjecture in the math literature, inspired by Moore and Seiberg, states that the Witt equivalence class of every modular tensor category contains a representative Chern-Simons gauge theory of some compact group. While a different source of fusion categories has long been known (the Haagerup construction), a stronger version of the conjecture, asserting braided equivalence instead of Witt equivalence, appears to still circulate. While this stronger conjecture is easily falsified by a numerical check of Frobenius-Perron dimensions, this note gives a human-readable proof that the Haagerup category is a counterexample.

https://math.berkeley.edu/~teleman/paperlist.html

Nov. 11, 2021

The classification conjecture: Current Status

Correct version of ``equivalence'' is Witt equivalence

Importance of Witt equivalence was not appreciated (by me) in 1989.

Witt equivalence $\sim$ Existence of topological interface

Kitaev, arXiv:cond-matt/0506438

Kapustin-Saulina 1012.0911

Witt equivalence $\sim$ $\otimes$ Drinfeld center of fusion category

Davydov, Müger, Nikschych, and Ostrik, 1009.2117

Davydov, Nikschych, and Ostrik, 1109.5558

Does every Witt equivalence class contain a WCS(G,k) for compact G ?

(Yes, but in a trivial and unsatisfying way with $G = 1.$)

MTC = Drinfeld center of the Haagerup fusion category.

Is it the MTC of some RCFT?

All three steps of the proposed classification program from 1988-1989 are far from complete.

1 Prehistory: A. 2dCFT and B. String Theory

2 Three Roads To RCFT: 1986-1987

3 Princeton: Fall 1987- January 1988

4 Braiding & Fusion & S & T: Moore & Seiberg Spring 1988

5 Chern-Simons Theory: July 1988- July 1989

6 MTC & Anyons: August 1989 – November 1989

Nonabelions

1. Prehistory: See G. Goldin article.
2. Nick Read and the Pfaffian state
3. Talk at Soviet-American conference:

[October or November 1989

4. Kitaev 1997, 2006: It suddenly becomes relevant
5. Current experimental status.

Recent work on several platforms

The Prediction of Anyons: Its History and Wider Implications

Gerald A. Goldin

Dept. of Mathematics, Dept. of Physics & Astronomy,
Rutgers University, New Brunswick NJ, USA

geraldgoldin@dimacs.rutgers.edu

December 27, 2022

Abstract

Prediction of "anyons," often attributed exclusively to Wilczek, came first from Leinaas & Myrheim in 1977, and independently from Goldin, Menikoff, & Sharp in 1980-81. In 2020, experimentalists successfully created anyonic excitations. This paper discusses why the possibility of quantum particles in two-dimensional space with intermediate exchange statistics eluded physicists for so long after bosons and fermions were understood. The history suggests ideas for the preparation of future researchers. I conclude by addressing failures to attribute scientific achievements accurately. Such practices disproportionately hurt women and minorities in physics, and are harmful to science.

1 Introduction

"Anyons" are quantum particles or excitations, theoretically possible in two space dimensions, with exchange statistics intermediate between bosons and fermions. They are associated with surface phenomena in the presence of magnetic flux. Theoretical applications include explaining the quantum Hall effect, describing quantum vortices in superfluids, and their relevance to quantum computing. In 2020, more than forty years after they were first suggested [1], experimentalists succeeded in creating anyonic excitations. The experimental confirmation of their prediction attracted considerable new attention to these fascinating possibilities.

Predicting the anyon required basic changes in our understanding of quantum statistics. The prediction is often incorrectly attributed exclusively to Frank Wilczek, while the first clear predictions were by Leinaas and Myrheim in 1977 [1] and by Menikoff, Sharp, and myself in 1980-1981 [2, 3], from different theoretical perspectives. Wilczek's 1982 work [4, 5] took still a third path to the prediction. He also coined the name "anyons" to describe such particles. This article describes the early history of intermediate quantum statistics, including

PHYSICAL REVIEW LETTERS

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Quantum Mechanics of Fractional-Spin Particles

Frank Wilczek
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106

(Received 22 June 1982)

Composites formed from charged particles and vortices in (2+1)-dimensional models, or flux tubes in three-dimensional models, can have any (fractional) angular momentum. The statistics of these objects, like their spin, interpolates continuously between the usual boson and fermion cases. How this works for two-particle quantum mechanics is discussed here.

PACS numbers: 03.65.Ca, 03.65.Ge, 05.30.-d

In a recent note[1] I showed that charged particles orbiting around magnetic flux tubes have orbital angular momentum integer $+q\Phi/2\pi$; this phenomenon is realized for example in the vortices of a type-II superconductor and in string solutions of gauge theories.[1] Closely related observations were made previously by Hasenfratz,[2] and recently by Goldin and Simon.[3] See also the discussion by Peshkin.[4] If there is a generalized spin-statistics connection, we must expect that the flux-tube–particle composites have unusual statistics, interpolating between bosons and fermions. Since interchange of two of these par-

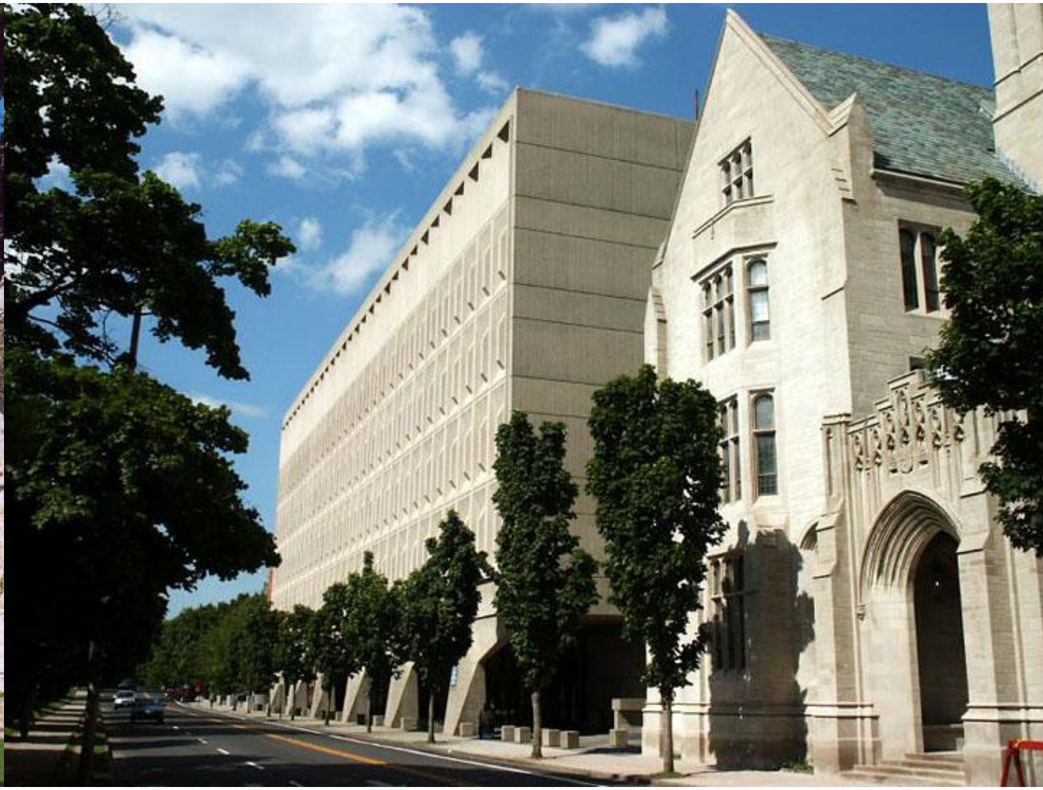
One anyon.—Let us recall how the fractional L_z arises. Charged particles orbiting around a flux tube carrying flux Φ are subject to an azimuthal vector potential

$$A_\varphi = \Phi/2\pi r. \tag{1}$$

Although the potential gives vanishing magnetic field strength, and therefore is negligible in classical physics, it does play a role in quantum mechanics.[7] It is convenient to eliminate A_φ by a gauge transformation:

$$A_i' = A_i - \partial_i \Lambda = 0, \quad \Lambda = \Phi\varphi/2\pi. \tag{2}$$

August 1989: Camping out at Yale

I wanted to apply MTC's to anyon physics:

So, first thing upon arrival at Yale, August 1989 was to go see Nick Read.

In our very first conversation we had the Pfaffian state. One motivation for the Pfaffian state was the conformal blocks of the Ising model. This fit nicely with ideas Nick had about pairing and condensation.

NONABELIONS IN THE FRACTIONAL QUANTUM HALL EFFECT

Gregory MOORE

Department of Physics, Yale University, New Haven, CT 06511, USA

Nicholas READ

Departments of Applied Physics and Physics, Yale University, New Haven, CT 06520, USA

Received 31 May 1990
(Revised 5 December 1990)

Appiications of conformal field theory to the theory of fractional quantum Hall systems are discussed. In particular, Laughlin's wave function and its cousins are interpreted as conformal blocks in certain rational conformal field theories. Using this point of view a hamiltonian is constructed for electrons for which the ground state is known exactly and whose quasihole excitations have nonabelian statistics; we term these objects "nonabelions". It is argued that universality classes of fractional quantum Hall systems can be characterized by the quantum numbers and statistics of their excitations. The relation between the order parameter in the fractional quantum Hall effect and the chiral algebra in rational conformal field theory is stressed, and new order parameters for several states are given.

these "particles" gives a matrix, i.e. nonabelian action on this vector. It is interesting to ask whether there exist in nature exotic two-dimensional systems whose elementary excitations include some transforming as nonabelian representations of $\mathscr{B}_n$. Particles defining nontrivial abelian representations of $\mathscr{B}_n$ are known as "anyons" and it seems apt to call these new objects "nonabelions". Fractional

Probably the first occurrence of the term``nonabelion''.

the properties of higher genus surfaces. Indeed, it seems to us highly likely that a FQHE system should define a "modular functor" or a "modular tensor category" (see ref. [1]; actually what we really require here is an extension of these concepts to chiral superalgebras as opposed to algebras).

Anyons in an exactly solved model and beyond

Alexei Kitaev

California Institute of Technology, Pasadena, CA 91125, U.S.A.

e-mail: kitaev@iqi.caltech.edu

January 1, 2008

Abstract

A spin 1/2 system on a honeycomb lattice is studied. The interactions between nearest neighbors are of XX, YY or ZZ type, depending on the direction of the link; different types of interactions may differ in strength. The model is solved exactly by a reduction to free fermions in a static $\mathbb{Z}_2$ gauge field. A phase diagram in the parameter space is obtained. One of the phases has an energy gap and carries excitations that are Abelian anyons. The other phase is gapless, but acquires a gap in the presence of magnetic field. In the latter case excitations are non-Abelian anyons whose braiding rules coincide with those of conformal blocks for the Ising model. We also consider a general theory of free fermions with a gapped spectrum, which is characterized by a spectral Chern number ν. The Abelian and non-Abelian phases of the original model correspond to $\nu = 0$ and $\nu = \pm 1$, respectively. The anyonic properties of excitation depend on ν mod 16, whereas ν itself governs edge thermal transport. The paper also provides mathematical background on anyons as well as an elementary theory of Chern number for quasidiagonal matrices.

Contents

Soviet-American Workshop On String Theory, Princeton University, Oct. 30 - Nov. 2, 1989

LIFE AFTER RCFT

– OR –

WHERE DO WE GO FROM HERE?

I. RCFT: GO THE DISTANCE

II. GENERALIZE

III. WHAT ABOUT EXPERIMENT?

PART III: EXPERIMENT ??

"REAL WORLD" APPLICATIONS IN
CONDENSED MATTER PHYSICS?

2D 2$^{\text{nd}}$ ORDER PHASE TRNS – OF COURSE

CSW $\Longrightarrow$ APPL'S TO F.Q.H.E. & ANYONS

BASIC PRINCIPLES OF (NON)RELTVSTIC 2+1 QFT $\Longrightarrow$ DATA FOR MTC + g=0 AXIOMS

Fröhlich
Gabbiani
Marchetti

– NONABELIAN ANYONS NOT RULED OUT –

F.Q.H. SYSTEM $\Longrightarrow$ L.G. THEORY

Girvin-MacDonald
N. Read

$$A(z) \sim \int \frac{\langle \alpha | \psi^{+}\psi(z') | \alpha \rangle}{z - z'} \, d^2 z'$$

w/ CS TERM

LOW ENERGY, LONG RANGE: PURE CSW

$\Longrightarrow$ FULL MTC !?

NEW STATES

w/ N. Read

(5

EXAMPLE: $\mathrm{Pf}\,\frac{1}{z_{ij}} \cdot \prod_{i<j}(z_i - z_j)^q$

$$\Psi_{Pf} = \underbrace{\langle \psi(z_1) \cdots \psi(z_{2N}) \rangle}_{\text{ISING}} \langle \prod_{1}^{2N} e^{i\sqrt{q}\,\phi(z_i)} \rangle$$

1. $\nu = \frac{1}{q}$ q even !

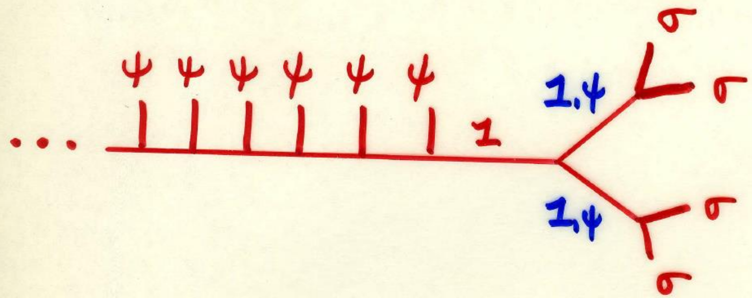
2. $\exists$ HAMILTONIAN SUCH THAT Ψ_{Pf} IS NONDEGENERATE, INCOMPRESSIBLE, GROUNDSTATE

$$\mathcal{H} \sim \sum(-i\nabla_i - A)^2 + V \sum_{i<j} P^{ij}_{q+1} + \text{MANY BODY INT'S}$$

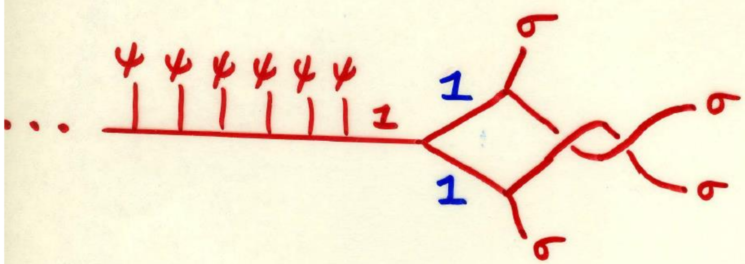
3. ORDER PARAMETER : $U^q(z)\psi^\dagger(z)\frac{1}{z-w}U^q(w)\psi^\dagger(w)$

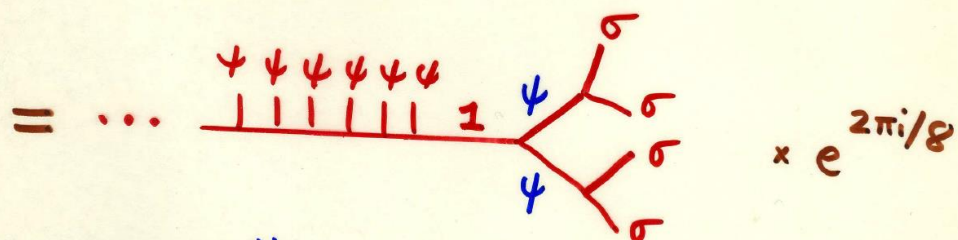
N. Read

DOUBLY DEGENERATE:



ANALYTIC CONTINUATION:





"PHYSICAL" REALIZATION OF

<u>NONABELIAN</u> <u>ANYONS</u>

Reception:

One problem:

Overt hostility to the use of category theory.

For better or worse, I completely dropped this line of research and started working on matrix models of 2d gravity…

From Wikipedia article on "Anyon"

While at first non-abelian anyons were generally considered a mathematical curiosity, physicists began pushing toward their discovery when Alexei Kitaev showed that non-abelian anyons could be used to construct a topological quantum computer.

Fault-tolerant quantum computation by anyons

A. Yu. Kitaev

L.D.Landau Institute for Theoretical Physics,
117940, Kosygina St. 2
e-mail: kitaev@itp.ac.ru

February 1, 2008

Abstract

A two-dimensional quantum system with anyonic excitations can be considered as a quantum computer. Unitary transformations can be performed by moving the excitations around each other. Measurements can be performed by joining excitations in pairs and observing the result of fusion. Such computation is fault-tolerant by its physical nature.

A quantum computer can provide fast solution for certain computational problems (e.g. factoring and discrete logarithm [1]) which require exponential time on an ordinary computer. Physical realization of a quantum computer is a big challenge for scientists. One important problem is decoherence and systematic errors in unitary transformations which occur in real quantum systems. From the purely theoretical point of view, this problem has been solved due to Shor's discovery of fault-tolerant quantum computation [2], with subsequent improvements [3, 4, 5, 6]. An arbitrary quantum circuit can be simulated using imperfect gates, provided these gates are close to the ideal ones up to a constant precision δ. Unfortunately, the threshold value of δ is rather small[1]; it is very difficult to achieve this precision.

Experimental realization of nonabelions has not yet (20240926) been achieved.

But several encouraging experimental results exist – especially for the Majorana edge mode, in different platforms (e.g. GsAs, graphene)

And several groups are getting extremely close....

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Steve Simon, Erik Verlinde, Edward Witten

for help providing relevant material
and correcting errors


NOT
That's all Folks!