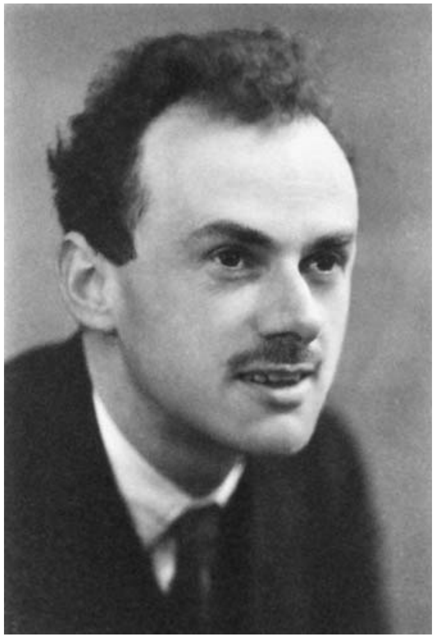
# Comments On Continuous Families Of Quantum Systems



Gregory Moore

Dirac Medal Ceremony

ICTP, Trieste, August 8, 2016

# Soviet-American Workshop On String Theory, Princeton, October 1989

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LIFE AFTER RCFT

– OR –

WHERE DO WE GO FROM HERE?

I. RCFT: GO THE DISTANCE

II. GENERALIZE

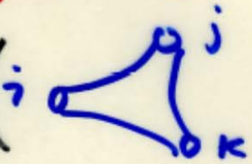
III. WHAT ABOUT EXPERIMENT?

# MODULAR TENSOR CTGRY

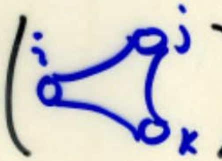
DATA:

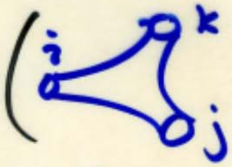
1. $I$ = LABELS, REP'S, S-SEL. SECTORS

$i \to i^\vee \to i$ ; $0^\vee = 0$

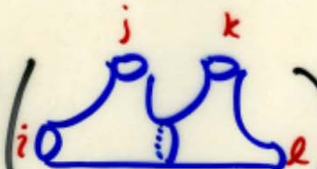
2. $V^i_{jk} = \mathcal{H}\Big($ 

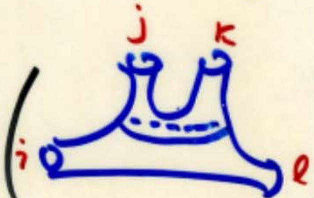
$\Big)$ "3-POINT COUPLINGS"

3. $\Omega^i_{jk} : \mathcal{H}\Big($ 

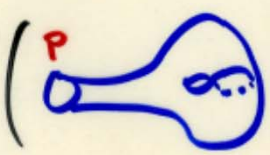
$\Big) \cong \mathcal{H}\Big($ 

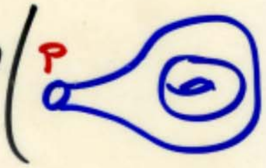
$\Big)$

4. $F: \mathcal{H}\Big($ 

$\Big) \cong \mathcal{H}\Big($ 

$\Big)$

5. $S(p): \mathcal{H}\Big($ 

$\Big) \cong \mathcal{H}\Big($ 

$\Big)$

6. $e^{2\pi i c/8}$

# PART III: EXPERIMENT ??

"REAL WORLD" APPLICATIONS IN CONDENSED MATTER PHYSICS?

2D 2nd ORDER PHASE TRNS – OF COURSE

CSW ⇒ APPL'S TO F.Q.H.E. & ANYONS

BASIC PRINCIPLES OF (NON)RELTVSTIC 2+1 QFT ⇒ DATA FOR MTC + g=0 AXIOMS

Fröhlich
Gabbiani
Marchetti

– NONABELIAN ANYONS NOT RULED OUT –

F.Q.H. SYSTEM ⇒ L.G. THEORY

Girvin-MacDonald
N. Read

$$A(z) \sim \int \frac{\langle \alpha | \psi^{+}\psi(z') | \alpha \rangle}{z - z'} d^2 z'$$

w/ CS TERM

LOW ENERGY, LONG RANGE: PURE CSW

⇒ FULL MTC !?

# NEW STATES

w/ N. Read

EXAMPLE: $\mathrm{Pf}\,\frac{1}{z_{ij}} \cdot \prod_{i<j}(z_i - z_j)^q$

$$\Psi_{Pf} = \underbrace{\langle \psi(z_1) \cdots \psi(z_{2N}) \rangle}_{\text{ISING}} \langle \prod_{1}^{2N} e^{i\sqrt{q}\phi(z_i)} \rangle$$

1. $\nu = \frac{1}{q}$ q even !

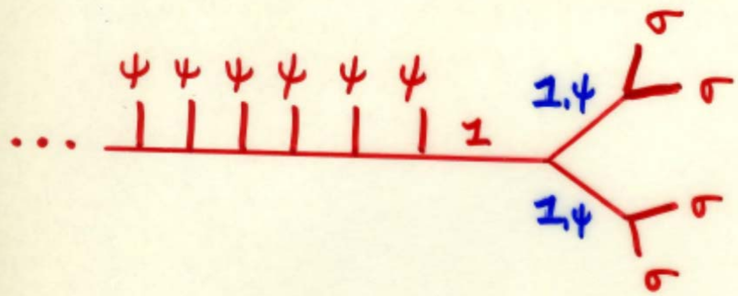
2. $\exists$ HAMILTONIAN SUCH THAT $\Psi_{Pf}$ IS NONDEGENERATE, INCOMPRESSIBLE, GROUNDSTATE

$$\mathcal{H} \sim \sum(-i\nabla_i - A)^2 + V \sum_{i<j} P_{q+1}^{ij} + \text{MANY BODY INT'S}$$

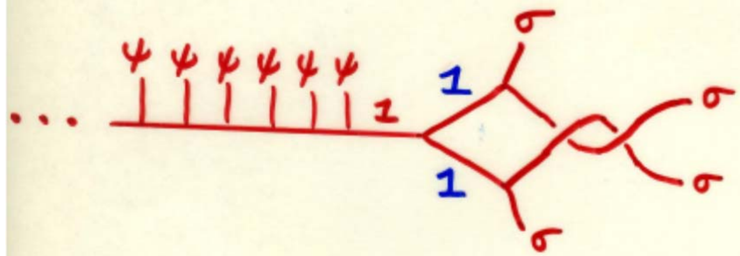
3. ORDER PARAMETER: $U^q(z)\psi^\dagger(z)\,\frac{1}{z-w}\,U^q(w)\psi^\dagger(w)$

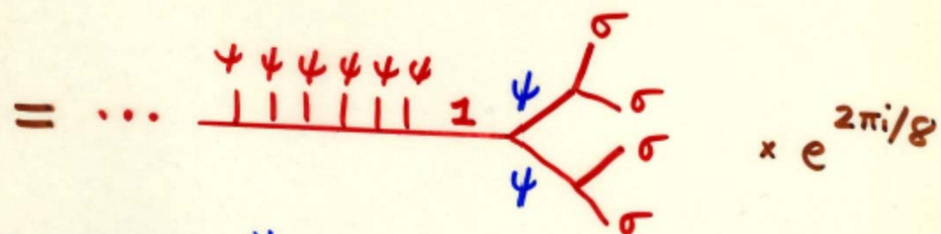
N. Read

# DOUBLY DEGENERATE:




# ANALYTIC CONTINUATION:




"PHYSICAL" REALIZATION OF

# <u>NONABELIAN</u> <u>ANYONS</u>

# Part II

# A Comment On Berry Connections

# Philosophy

If a physical result is not mathematically natural, there might well be an underlying important physical issue.

We will illustrate this with continuous families of quantum systems

i.e. quantum systems parametrized by a space X of control parameters.

In this context one naturally encounters Berry connections – an enormously successful idea.

# A Little Subtlety

Given a continuous family of Hamiltonians with a gap in the spectrum there is, in general, not one Berry connection, but rather a family of Berry connections.

Example: For band insulators there is a family of natural Berry connections, whose gauge equivalence classes are parametrized by a real-space torus.

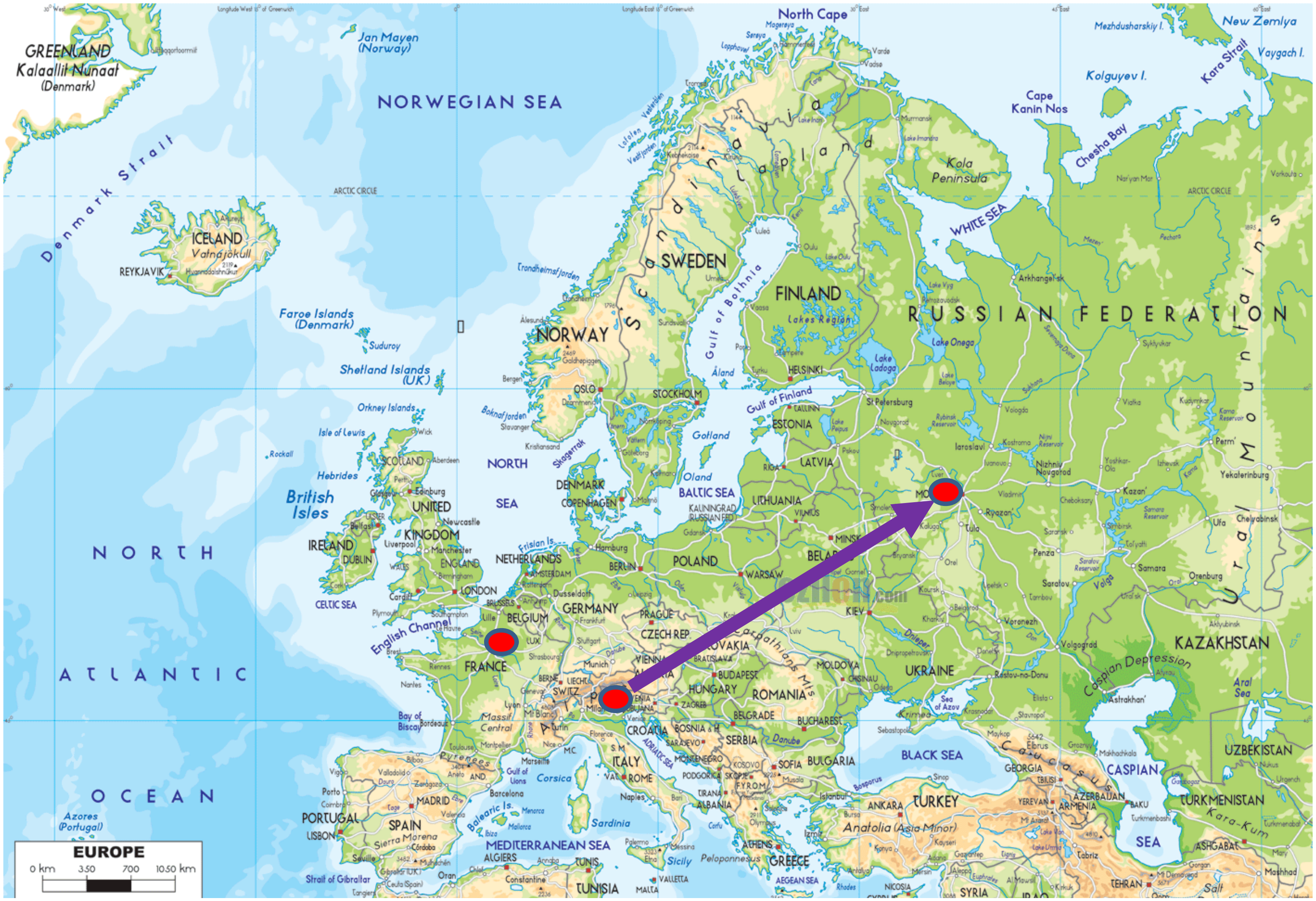
This has consequences for topological contributions to electric polarization and magneto-electric polarizability: a 3D Chern ``insulator'' has a bulk QHE.

THE ORIGIN OF THE PROBLEM IS
THE PROBLEM OF THE ORIGIN.

# Affine Space

Like a vector space – but no natural choice of origin.

Definition: There is a transitive and free action of a vector space.



# Non-symmorphic crystals

E(n): The Euclidean group of length-preserving transformations of affine n-dimensional space.

There is a natural subgroup $\mathbb{R}^n$ of translations

But there is no natural subgroup isomorphic to O(n):

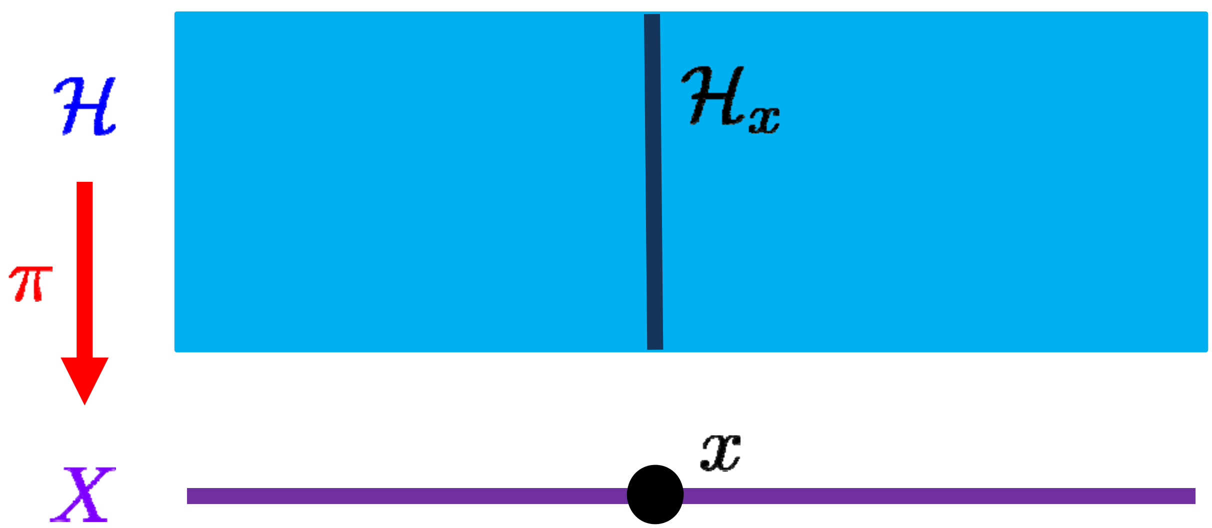
One must CHOOSE an origin to define such a group.

That's why there are non-symmorphic crystal structures.

# Hilbert Bundles

Hilbert bundle over a space
X of control parameters

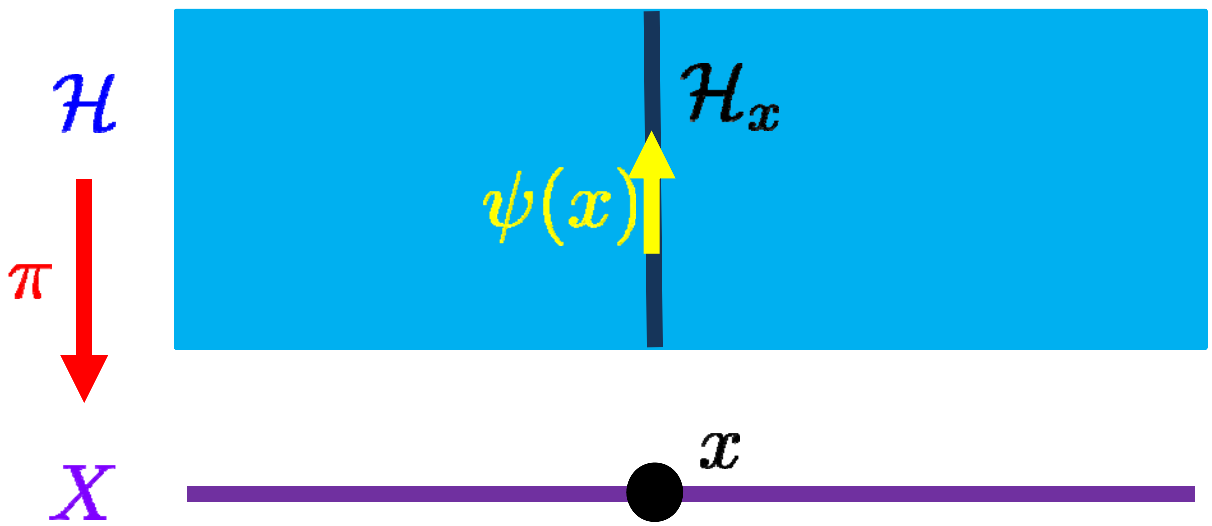
$$\begin{array}{ccc} \mathcal{H}_x & & \mathcal{H} \\ \downarrow & & \downarrow \pi \\ x & \hookrightarrow & X \end{array}$$



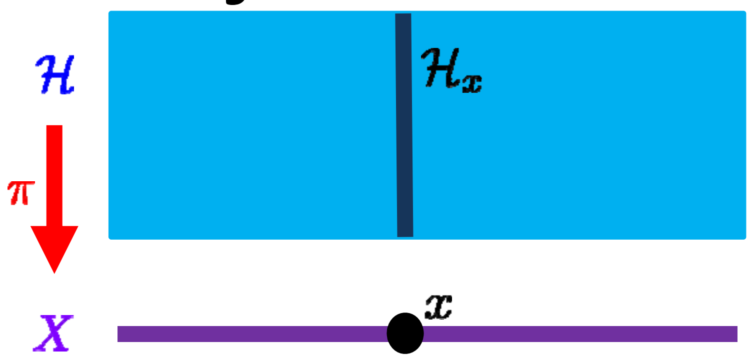
# Sections Of A Hilbert Bundle

Space of sections: $\Gamma[\mathcal{H} \to X]$

$$\Psi : x \mapsto \psi(x) \in \mathcal{H}_x$$



# Projected Bundles




Given a continuous family of projection operators: $P(x) : \mathcal{H}_x \to \mathcal{H}_x$

Projected bundle $\mathcal{V}$ : Subbundle with sections:

$$\Gamma(\mathcal{V}) := \{\psi(x) | P(x)\psi(x) = \psi(x)\} \subset \Gamma(\mathcal{H})$$

$$\mathcal{H} = S^2 \times \mathbb{C}^2 \qquad P(\hat{x}) = \frac{1}{2}(1 + \hat{x} \cdot \vec{\sigma})$$

$$\pi \downarrow$$

$$\hat{x} \in S^2$$

Definition: A *vector bundle* $\mathcal{V}$ is a projected bundle.

# Projected Connection

Connection:

$$\nabla : \Gamma(\mathcal{V}) \to \Omega^1(\mathcal{V})$$

$$\nabla(f\Psi) = df \otimes \Psi + f\nabla\Psi$$

Remark: The space of connections on a vector bundle is an affine space modeled on the vector space: $\Omega^1(\mathrm{End}(\mathcal{V}))$

If the vector bundle $\mathcal{V}$ is defined using a family of projection operators P(x)

and we choose a connection $\nabla^{\mathcal{H}}$ on $\mathcal{H}$:

we get a connection on the bundle $\mathcal{V}$ :

$$\nabla^P := P \circ \nabla^{\mathcal{H}} \circ \iota \qquad \iota : \Gamma(\mathcal{V}) \hookrightarrow \Gamma(\mathcal{H})$$

# Berry Connection

Given a continuous family of Hamiltonians $H_x$ on $\mathcal{H}_x$ , if there is a gap:

$$E_{\rm gap} \notin \cup_{x\in X} {\rm Spec}(H_x)$$

we have a continuous family of projection operators: $P(x) = \Theta(E_{\rm gap} - H_x)$

$$\nabla^B := P \circ \nabla^{\mathcal{H}} \circ \iota$$

[M. Berry (1983); B. Simon 1983) ]

Note that it requires a CHOICE of $\nabla^{\mathcal{H}}$

Commonly assumed: $\mathcal{H}$ has been trivialized:

$$\mathcal{H} = X \times \mathcal{H}_0$$

⇒ Natural choice of $\nabla^{\mathcal{H}}$ :

The trivial connection.

$$\nabla^{\mathcal{H}} \psi(x) = dx^{\mu} \frac{\partial}{\partial x^{\mu}} \psi(x)$$

$$\vec{A}^{\mathrm{Berry}} = \langle \psi | \vec{\nabla}_{\vec{R}} | \psi \rangle$$

But in general there is no natural trivialization of $\mathcal{H}$!

# Hilbert Bundle Over Brillouin Torus

Crystal in n-dimensional affine space: $C \subset \mathbb{A}^n$

Invariant under a lattice of translations: $L \subset \mathbb{R}^n$

Brillouin torus: = {unitary irreps of L}.

Reciprocal lattice: $L^\vee \subset \mathcal{K} \cong (\mathbb{R}^n)^\vee \cong \mathbb{R}^n$

$$\bar{k} \in T^\vee = \mathcal{K}/L^\vee \quad \chi_{\bar{k}}(R) = e^{2\pi i k \cdot R} \quad R \in L$$

Bloch states define a Hilbert bundle $\mathcal{H}$ over the Brillouin torus:

$$\mathcal{H}_{\bar{k}} := \{\psi_{\bar{k}} | \psi_{\bar{k}}(x+R) = e^{2\pi i k \cdot R}\psi_{\bar{k}}(x)\}$$

# Trivializations Of $\mathcal{H}$

$\mathcal{H}$ can be trivialized by choosing Bloch functions

$$\psi_{n,\bar{k}}(x+R) = e^{2\pi i k\cdot R}\psi_{n,\bar{k}}(x) \quad n \in \mathbb{N}$$

smooth $\quad \forall \bar{k} \in T^{\vee}$

$\{\psi_{n,\bar{k}}\}$ A *basis* for Hilbert space $\quad \mathcal{H}_{\bar{k}}$

But in general there is no natural trivialization of $\mathcal{H}$!

# A Family Of Connections on $\mathcal{H}$

So: There is no such thing as ``THE'' Berry connection in the context of band structure.

But, there ***is*** a natural family of connections on $\mathcal{H}$:

[Freed & Moore, 2012]

$$\nabla^{\mathcal{H},x_0}$$

They depend on a choice of origin $x_0$ modulo L:

$$\nabla^{\mathcal{H},x_0} - \nabla^{\mathcal{H},x_0'} = \alpha$$

$$\alpha = 2\pi \mathrm{i}\, dk \cdot (x_0 - x_0') \otimes 1_{\mathcal{H}}$$

# Berry Connections For Insulators

Insulator: Projected bundle $\mathcal{F}$ of filled bands:

$$\mathcal{F}_{\bar{k}} = \Theta(E_f - H_{\bar{k}}) \cdot \mathcal{H}_{\bar{k}} \subset \mathcal{H}_{\bar{k}}$$

$$\nabla^{B,x_0} - \nabla^{B,x_0'} = \alpha$$

$$\alpha = 2\pi \mathrm{i}\, dk \cdot (x_0 - x_0') \otimes 1_{\mathcal{F}}$$

So what?

$$F(\nabla^{B,x_0}) = F(\nabla^{B,x_0'})$$

→ All Chern numbers unchanged....

# Electric polarization:

$$\langle K, P/e \rangle = \int_{T_K^\perp} \mathrm{Im}\log\det \mathrm{Hol}(\nabla^{B,x_0}, \gamma_K) \mod 2\pi$$

[ King-Smith & Vanderbilt (1993); Resta (1994) ]

# Magnetoelectric Polarizability

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{Maxwell}} \supset \int_{\mathbb{R}^4} \alpha^{ij} E_i B_j$$

``Axion angle''  $\theta(x_0) = \frac{1}{3}\alpha^i{}_i = \int_{T^\vee} CS(\nabla^{B,x_0})$

[ Qi, Hughes, Zhang; Essin, Joel Moore, Vanderbilt ]

# Dependence Of Axion Angle On $x_0$

$$CS(\nabla+\alpha)-CS(\nabla)=\mathrm{Tr}(2\alpha F+\alpha D_A\alpha+\frac{2}{3}\alpha^3)$$

$$\vec{c}:=\int_{T^\vee} c_1(\mathcal{F})\in L^\vee$$

$$\theta(x_0)-\theta(x_0')=2\pi\vec{c}\cdot(x_0-x_0')$$

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{Maxwell}}\supset\frac{1}{4\pi}\int_{\mathbb{R}^4}\langle\vec{c},d\vec{x}\rangle\wedge CS(A^{\mathrm{Maxwell}})$$

→ QHE in the *bulk* of the ``insulator'' in the plane orthogonal to $\vec{c}$

# Part III

# Born Rule For Families Of Quantum Systems Parametrized By A Noncommutative Manifold

# Quantum Systems

Set of physical ``states'' $\mathcal{S}$

Set of physical ``observables'' $\mathcal{O}$

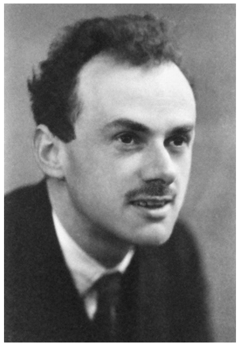
Born Rule: $BR : \mathcal{S} \times \mathcal{O} \rightarrow \mathcal{P}$

$\mathcal{P}$ Probability measures on $\mathbb{R}$.

$$m \in \mathfrak{M}(\mathbb{R}) \Longrightarrow 0 \leq \wp(m) \leq 1$$

$$m = [r_1, r_2] \subset \mathbb{R} \qquad BR(\mathrm{s}, \mathrm{O})([r_1, r_2])$$

is the probability that a measurement of the observable O in the state s has value between $r_1$ and $r_2$ .



# Standard Dirac-von Neumann Axioms

$\mathcal{S}$ Density matrices $\rho$: Positive trace class operators on Hilbert space of trace =1

$\mathcal{O}$ Self-adjoint operators T on Hilbert space

Spectral Theorem: There is a one-one correspondence of self-adjoint operators T and projection valued measures:

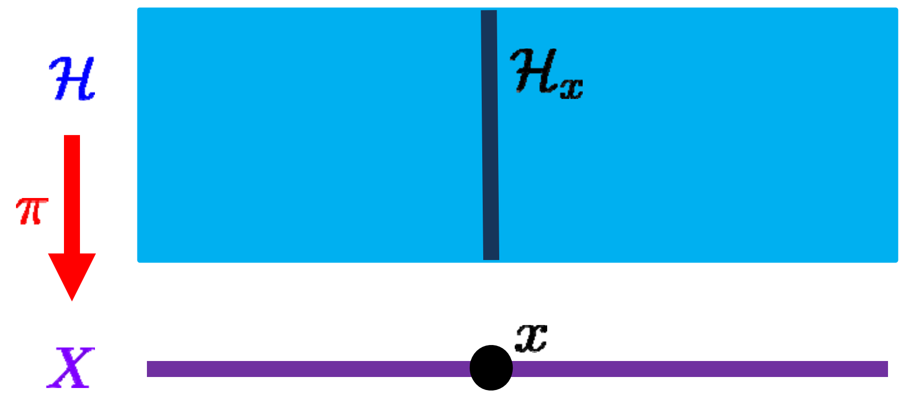
$$m \in \mathfrak{M}(\mathbb{R}) \quad \rightarrow \quad P_T(m)$$

Example: $T = \sum_\lambda \lambda P_\lambda \qquad P_T([r_1, r_2]) = \sum_{r_1 \leq \lambda \leq r_2} P_\lambda$

$$m \in \mathfrak{M}(\mathbb{R}) \qquad BR(\rho, T)(m) = \mathrm{Tr}_{\mathcal{H}} \left( \rho P_T(m) \right)$$

# Continuous Families Of Quantum Systems

Hilbert bundle over space X of control parameters.




For each x get a probability measure $\wp_x$:

$$m \in \mathfrak{M}(\mathbb{R}) \mapsto \wp_x(m) := \mathrm{Tr}_{\mathcal{H}_x}(\rho_x P_{T_x}(m))$$

$$BR : \mathcal{S} \times \mathcal{O} \times X \to \mathcal{P}$$

$$BR(\rho, T, x) = \wp_x$$

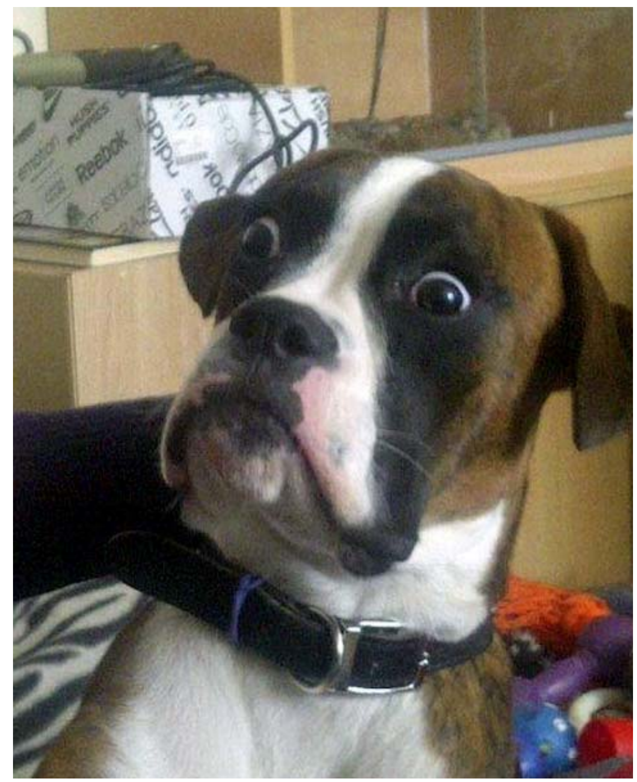
# Noncommutative Families ?

What happens when the space X of control parameters is replaced by a

*noncommutative space* ?

How does the Born rule change?

Why ask this question?



Curiosity.

(And the answer is interesting.)

With irrational magnetic flux the Brillouin torus is replaced by a noncommutative manifold. (Bellisard, Connes, Gruber,…)

NC tt* geometry (S. Cecotti, D. Gaiotto, C. Vafa)

Boundaries of Narain moduli spaces of toroidal heterotic string compactifications are NC

The ``early universe'' might be NC

# C* Algebras

A C* algebra is a (normed) algebra $\mathfrak{A}$ over the complex numbers with an involution:

$$a \in \mathfrak{A} \to a^* \in \mathfrak{A} \qquad (ab)^* = b^* a^*$$

such that ….

Example 1: $\mathfrak{A} = C(X) := \{f : X \to \mathbb{C}\}$

Example 2: $\mathfrak{A} = Mat_n(\mathbb{C})$

Self-adjoint: $a^* = a$

Positive: $a = b^* b$

# Gelfand's Theorem

The topology of a (Hausdorff) space X is completely captured by the C*-algebra of continuous functions on X:

$$C(X) := \{f : X \to \mathbb{C}\}$$

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad (f_1 \cdot f_2)(x) := f_1(x) f_2(x)$$

``Points'' become 1D representations:

$$\mathrm{ev}_{x_0} : f \in C(X) \mapsto f(x_0) \in \mathbb{C}$$

Commutative C* algebra: $\mathfrak{A} \longrightarrow \mathrm{Irrep}(\mathfrak{A})$ A topological space

$$\mathfrak{A} \cong C(\mathrm{Irrep}(\mathfrak{A}))$$

# Noncommutative Geometry

Statements about the topology/geometry of X are equivalent to algebraic statements about C(X)

Replace C(X) by a noncommutative C* algebra $\mathfrak{A}$

Interpret $\mathfrak{A}$ as the ``algebra of functions on a noncommutative space''  …

… even though there are no points.

``pointless geometry''

Example: Noncommutative torus:

$$U_i U_i^* = U_i^* U_i = 1 \qquad U_i U_j = e^{2\pi \mathrm{i} \phi_{ij}} U_j U_i$$

# Noncommutative Control Parameters

We would like to define a family of quantum systems parametrized by a NC manifold whose ``algebra of functions'' is a general C* algebra $\mathfrak{A}$

What are observables?

What are states?

What is the Born rule?

What replaces the Hilbert bundle?

# Noncommutative Hilbert Bundles

Definition: Hilbert C* module $\mathcal{E}$ over C*-algebra $\mathfrak{A}$.

Complex vector space $\mathcal{E}$ with a right-action of $\mathfrak{A}$
and an ``inner product'' valued in $\mathfrak{A}$

$$\Psi_1, \Psi_2 \in \mathcal{E} \qquad (\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A}$$

$$(\Psi_1, \Psi_2)^*_{\mathfrak{A}} = (\Psi_2, \Psi_1)_{\mathfrak{A}}$$

$$(\Psi, \Psi)_{\mathfrak{A}} \geq 0$$ (Positive element of the C* algebra.)

such that .....

Like a Hilbert space, but ``overlaps'' are valued in a (possibly) noncommutative algebra.

# Quantum Mechanics With Noncommutative Amplitudes

Basic idea: Replace the Hilbert space by a Hilbert C* module

$$\mathcal{H} \to \mathcal{E}$$

$$\Psi_1, \Psi_2 \in \mathcal{E} \qquad (\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A}$$

Overlaps are valued in a possibly noncommutative algebra.

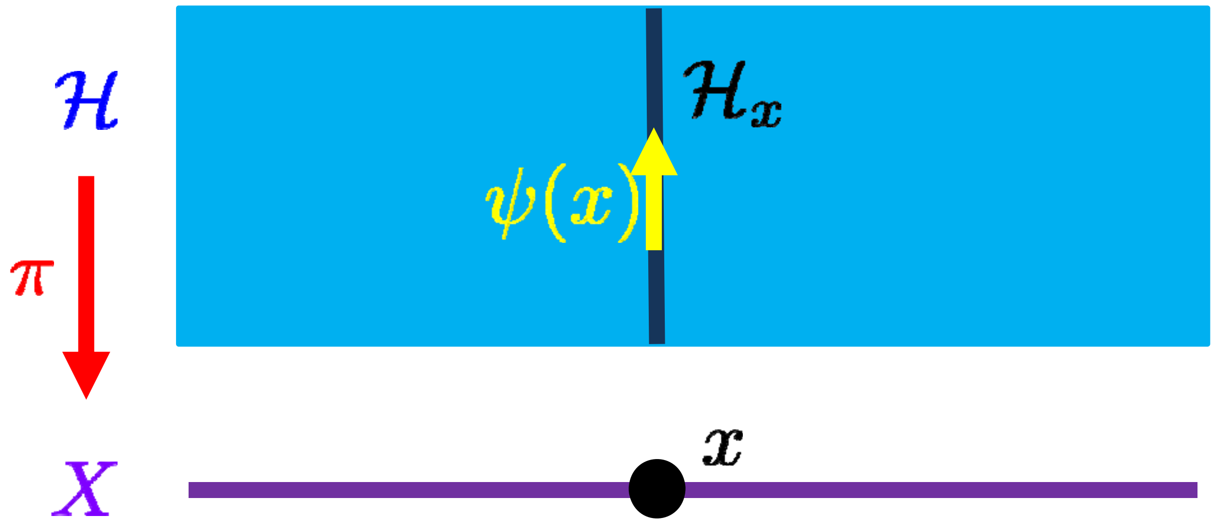
QM: $$0 \leq \ \wp(\lambda) = (\psi_\lambda, \psi)(\psi_\lambda, \psi)^* \ \leq 1$$

QMNA: $$(\Psi_\lambda, \Psi)(\Psi_\lambda, \Psi)^* \in \mathfrak{A}$$

# Example 1: Hilbert Bundle Over A Commutative Manifold

$$\mathcal{E} = \Gamma[\mathcal{H} \to X] \qquad \mathfrak{A} = C(X)$$

$$\Psi : x \mapsto \psi(x) \in \mathcal{H}_x$$




$$(\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A} := C(X)$$

$$(\Psi_1, \Psi_2)_{\mathfrak{A}}(x) := (\psi_1(x), \psi_2(x))_{\mathcal{H}_x} \in \mathbb{C}$$

# Example 2: Hilbert Bundle Over A Fuzzy Point

Def: ``fuzzy point'' has $\mathfrak{A} \cong \mathrm{Mat}_{a\times a}(\mathbb{C})$

$$\mathcal{E} = \mathrm{Mat}_{b\times a}(\mathbb{C})$$

$$(\Psi_1, \Psi_2)_{\mathfrak{A}} = \Psi_1^\dagger \Psi_2$$

# Observables In QMNA

Consider ``adjointable operators'' $T : \mathcal{E} \to \mathcal{E}$

$$(\Psi_1, T\Psi_2)_{\mathfrak{A}} = (T^*\Psi_1, \Psi_2)_{\mathfrak{A}}$$

The adjointable operators $\mathfrak{B}$ are another C* algebra.

Definition: *QMNA* *observables* are self-adjoint elements of $\mathfrak{B}$

(Technical problem: There is no spectral theorem for self-adjoint elements of an abstract C* algebra. )

# C* Algebra States

Definition: A *C*-algebra state* $\omega \in \mathcal{S}(\mathfrak{A})$ is a positive linear functional

$$\omega : \mathfrak{A} \to \mathbb{C} \qquad \omega(\mathbf{1}) = 1$$

$$\mathfrak{A} = C(X) \qquad \omega \in \mathcal{S}(\mathfrak{A})$$

$$\omega(f) = \int_X f d\mu$$ $d\mu$ = a positive measure on X:

$$\mathfrak{A} \cong \mathrm{Mat}_{a\times a}(\mathbb{C}) \qquad \omega \in \mathcal{S}(\mathfrak{A})$$

$$\omega(T) = \mathrm{Tr}_{\mathcal{H}}(\rho T)$$ $\rho$ = a density matrix

# QMNA States

Definition: A ***QMNA state*** is a completely positive unital map $\varphi : \mathfrak{B} \to \mathfrak{A}$

``Completely positive'' comes up naturally both in math and in quantum information theory.

Positive: $\varphi : \mathfrak{B}_{\geq 0} \to \mathfrak{A}_{\geq 0}$

Unital: $\varphi(1_{\mathfrak{B}}) = 1_{\mathfrak{A}}$

Completely positive

$$\varphi \otimes 1 : (\mathfrak{B} \otimes \mathrm{Mat}_n(\mathbb{C}))_{\geq 0} \to (\mathfrak{A} \otimes \mathrm{Mat}_n(\mathbb{C}))_{\geq 0}$$

# QMNA Born Rule

Main insight is that we should regard the Born Rule as a map

$$BR : \mathcal{S}^{\mathrm{QMNA}} \times \mathcal{O}^{\mathrm{QMNA}} \times \mathcal{S}(\mathfrak{A}) \to \mathcal{P}$$

For general $\mathfrak{A}$ the datum $\omega \in \mathcal{S}(\mathfrak{A})$ together with complete positivity of $\varphi$ give just the right information to state a Born rule in general:

$$BR(\varphi, T, \omega) \in \mathcal{P}$$

# Family Of Quantum Systems Over A Fuzzy Point

$$\mathcal{E} = \mathrm{Mat}_{b\times a}(\mathbb{C}) = \mathbb{C}^b \otimes \mathbb{C}^a = \mathcal{H}_{\mathrm{Bob}} \otimes \mathcal{H}_{\mathrm{Alice}}$$

$$\mathfrak{A} = Mat_a(\mathbb{C}) = \mathrm{End}(\mathcal{H}_{\mathrm{Alice}})$$

$$\mathfrak{B} = Mat_b(\mathbb{C}) = \mathrm{End}(\mathcal{H}_{\mathrm{Bob}})$$

$$BR(\varphi, T, \omega)(m) = \mathrm{Tr}_{\mathcal{H}_A} \rho_A \varphi(P_T(m))$$

``A NC measure $\omega \in \mathcal{S}(\mathfrak{A})$'' is equivalent to a density matrix $\rho_A$ on $\mathcal{H}_A$

QMNA state: $\varphi(T) = \sum_\alpha E_\alpha^\dagger T E_\alpha \quad \sum_\alpha E_\alpha^\dagger E_\alpha = 1$

# Quantum Information Theory & Noncommutative Geometry

$$BR(\varphi, T, \omega)(m) = \mathrm{Tr}_{\mathcal{H}_A} \rho_A \varphi(P_T(m))$$

$$= \sum_\alpha \mathrm{Tr}_{\mathcal{H}_A} \rho_A E_\alpha^\dagger (P_T(m)) E_\alpha$$

$$= \sum_\alpha \mathrm{Tr}_{\mathcal{H}_B} E_\alpha \rho_A E_\alpha^\dagger P_T(m)$$

$$= \mathrm{Tr}_{\mathcal{H}_B} \mathcal{E}(\rho_A) P_T(m)$$

Last expression is the measurement by Bob of T in the state $\rho_A$ prepared by Alice and sent to Bob through quantum channel $\mathcal{E}$.

# Three Conclusions

Don’t be discouraged by negative response.

(Within reason)

Please say, ``a Berry connection,’’ not ``the Berry connection.’’

(Unless you have specified $\nabla^{\mathcal{H}}$)

QM can be generalized to QMNA.

(Is it really a generalization?)

(Is it useful?)