Lecture 7

[SDE examples]

1. geometric Brownian motion

Consider $dx = c\,x\,dW(t)$ Itô's differential

c constant

Define $y = \log x$, then

$$dy = \underbrace{\frac{dx}{x} - \frac{dx^2}{2x^2}}_{\text{Taylor expansion of } \log(x+dx)-\log x} = c\,dW - \frac{c^2}{2}dt \quad \text{, or}$$

Itô's formula: $\begin{cases} a = 0, \\ b = cx. \end{cases}$

$$dy = \frac{c^2x^2}{2}\left(-\frac{1}{x^2}\right)dt + cx\,\frac{1}{x}\,dW =$$

$$= -\frac{c^2}{2}dt + c\,dW \quad \text{, just as above.}$$

Integrating, we obtain:

$$y(t) = y(t_0) + c\left[W(t) - W(t_0)\right] - \frac{c^2}{2}(t - t_0) \text{, or}$$

$$x(t) = x(t_0)\, e^{c[W(t)-W(t_0)] - \frac{c^2}{2}(t-t_0)}.$$

Recall that for a gaussian variable z with $\mu = 0$: $z \sim \mathcal{N}(0,\sigma)$ we have:

$$\langle e^z \rangle = e^{\langle z^2 \rangle/2}.$$

Then $$\langle x(t)\rangle = \underset{\substack{\uparrow \\ x(t_0) \text{ could be stochastic}}}{\langle x(t_0)\rangle}\, e^{\frac{c^2}{2}\underbrace{\langle [W(t)-W(t_0)]^2\rangle}_{\text{"} t-t_0}}\, e^{-\frac{c^2}{2}(t-t_0)} \quad (11)$$

$= \langle x(t_0)\rangle$, as expected since

$$d\langle x(t)\rangle = \langle dx(t)\rangle = c\langle x(t)\,dW(t)\rangle = 0.$$

With Stratonovich differential,

$$(S)\ \underbrace{dy}_{\frac{dx}{x}} = c\,dW$$ ordinary calculus rules apply

Hence $x(t) = x(t_0)\, e^{c[W(t)-W(t_0)]}$, yielding

$$\langle x(t)\rangle = \langle x(t_0)\rangle\, e^{\frac{c^2}{2}(t-t_0)}.$$

The mean is not stationary anymore.

(2.) Ornstein - Uhlenbeck process

$$dx = -\underset{k>0}{k}x\,dt + \sqrt{D}\,dW(t) \qquad \text{Itô}$$

Use $y(t) = x(t)\,e^{kt}$:

$$\begin{cases} a = -kx, \\ b = \sqrt{D} \end{cases} \Rightarrow \quad dy = \{-kx\,e^{kt}\}dt + \sqrt{D}\,e^{kt}\,dW(t) + \underbrace{x(t)\,d(e^{kt})}_{} =$$

due to explicit time dependence of $y(t)$: $\frac{\partial f(x,t)}{\partial t}dt$

$$= \sqrt{D}\,e^{kt}\,dW(t).$$

Integrate:

$$y(t) = \underset{0}{\underset{\|}{y(t_0)}} + \sqrt{D}\int_{t_0=0}^{t} e^{kt'}\,dW(t') \quad , \text{ or}$$

$$x(t) = \underset{\substack{\uparrow \\ y(0)=x(0)}}{x(0)}\, e^{-kt} + \sqrt{D}\int_0^t e^{-k(t-t')}\,dW(t').$$

Assume that $x(0)$ is not stochastic:

$$x(0) = x_0$$

Then $\langle x(t)\rangle = x_0\, e^{-kt}$, $\Leftarrow$ as before

$$\mathrm{Var}\{x(t)\} = \langle x^2\rangle - \langle x\rangle^2 =$$

$$= x_0^2 e^{-2kt} + D\left\langle\left[\int_0^t e^{-k(t-t')}\,dW(t')\right]^2\right\rangle -$$

$$- x_0^2 e^{-2kt} = D\int_0^t dt'\int_0^t dt''\, e^{-k(t-t')}\,e^{-k(t-t'')}\,\times$$

$$\times \underbrace{\langle dW(t')\,dW(t'')\rangle}_{\underbrace{\langle dW(t')^2\rangle}_{=dt'}\,\delta(t'-t'')} = D\int_0^t dt'\, e^{-2k(t-t')}\,dt'.$$

Thus, $\mathrm{Var}\{x(t)\} = D\,e^{-2kt}\left(\frac{1}{2k}\right)\left[e^{2kt} - 1\right] =$

$$= \frac{D}{2k}\left[1 - e^{-2kt}\right] \quad \Leftarrow \text{as before}$$

Correlation function:

$$\langle x(t)\,x(s)\rangle - \langle x(t)\rangle\langle x(s)\rangle =$$

$$= \underline{x_0^2 e^{-kt} e^{-ks}} + D\left\langle\int_0^t e^{-k(t-t')}\,dW(t')\int_0^s e^{-k(s-s')}\,dW(s')\right\rangle -$$

$$\underline{- x_0 e^{-kt} \times x_0 e^{-ks}} \;\; \boxed{=}$$

$$\Leftrightarrow D\int_0^{\min(t,s)} e^{-k(t+s-2t')}\,dt' =$$

$$= D e^{-k(t+s)} \begin{cases} \int_0^s e^{2kt'}dt', & t \geq s \\ \int_0^t e^{2kt'}dt', & t < s \end{cases} =$$

$$= D e^{-k(t+s)} \begin{cases} \frac{1}{2k}[e^{2ks}-1], & t \geq s \\ \frac{1}{2k}[e^{2kt}-1], & t < s \end{cases} =$$

$$= -\frac{D}{2k} e^{-k(t+s)} + \frac{D}{2k} \begin{cases} e^{k(s-t)}, & t \geq s \\ e^{k(t-s)}, & t < s \end{cases} =$$

$$= -\frac{D}{2k} e^{-k(t+s)} + \frac{D}{2k} e^{-k|t-s|}.$$

Since $k>0$, we can get SS behavior by considering $t, s \to \infty$ s.t. $|t-s|$ is finite:

$$\langle x(t)\,x(s)\rangle_s = \frac{D}{2k} e^{-k|t-s|} \Leftarrow \text{as before}$$

Note that $\langle x(t)\rangle_s = \langle x(s)\rangle_s = 0$ in this limit

[Itô's formula with explicit time dependence:]

$$df[x(t),t] = \left\{ a[x(t),t]\frac{\partial f[x(t),t]}{\partial x} + \frac{1}{2} b[x(t),t]^2 \frac{\partial^2 f}{\partial x^2} \right\} dt$$

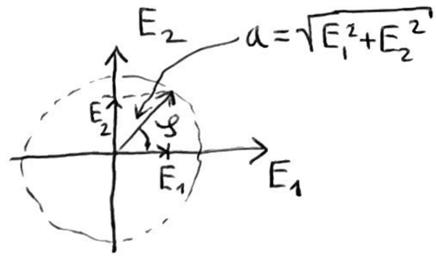
$$dx(t) = a[x(t),t]\,dt + b[x(t),t]\,dw(t)$$

$$+\; b[x(t),t]\frac{\partial f}{\partial x}\,dw(t) + \frac{\partial f}{\partial t}\,dt.$$

(3.) Random electric fields

$$\begin{cases} dE_1(t) = -\gamma E_1(t)\,dt + \xi\, dw_1(t), & \text{Re part} \\ dE_2(t) = -\gamma E_2(t)\,dt + \xi\, dw_2(t), & \text{Im part} \end{cases}$$

$\boxed{\gamma > 0}$



Convert to polar coords:

$$\begin{cases} E_1(t) = a(t) \cos\varphi(t), \\ E_2(t) = a(t) \sin\varphi(t). \end{cases}$$

Define $\mu(t) = \log a(t)$, s.t.

$$E_1 + iE_2 = \underbrace{e^{\mu}}_{a} \times e^{i\varphi}. \Rightarrow \mu + i\varphi = \log(E_1 + iE_2)$$

Then
$$d(\mu + i\varphi) = \frac{d(E_1 + iE_2)}{E_1 + iE_2} - \frac{d(E_1 + iE_2)^2}{2(E_1 + iE_2)^2} =$$

$$= -\frac{\gamma(E_1 + iE_2)}{E_1 + iE_2}dt + \frac{\xi(dw_1 + i\,dw_2)}{E_1 + iE_2} - \frac{\xi^2[dw_1 + i\,dw_2]^2}{2(E_1 + iE_2)^2} =$$

$$= -\gamma\,dt + \underbrace{\xi e^{-\mu - i\varphi}(dw_1 + i\,dw_2)}_{\xi e^{-\mu}[\cos\varphi - i\sin\varphi](dw_1 + i\,dw_2)}.$$

since
$dw_1 dw_2 = 0$
$dw_1^2 - dw_2^2 =$
$= dt - dt = 0$

Re part: $d\mu = -\gamma\, dt + \xi e^{-\mu}\left[\cos\varphi\, dW_1 + \sin\varphi\, dW_2\right] =$

$$= \underbrace{-\gamma}_{a}\, dt + \underbrace{\xi e^{-\mu}\cos\varphi}_{b_1[\mu,\varphi]}\, dW_1 + \underbrace{\xi e^{-\mu}\sin\varphi}_{b_2[\mu,\varphi]}\, dW_2 .$$

Then
$$da = -\gamma \frac{d\overbrace{e^{\mu}}^{=a}}{d\mu}\, dt + \left\{ \frac{\xi^2 \overbrace{e^{-2\mu}}^{a^{-2}} \cos^2\varphi}{2} \underbrace{\frac{d^2 e^{\mu}}{d\mu^2}}_{=a} + \right.$$

$$\left. + \frac{\xi^2 \overbrace{e^{-2\mu}}^{=a^{-2}} \sin^2\varphi}{2} \underbrace{\frac{d^2 e^{\mu}}{d\mu^2}}_{=a} \right\} dt +$$

$$+ \xi e^{-\mu}\cos\varphi \frac{de^{\mu}}{d\mu}\, dW_1 + \xi e^{-\mu}\sin\varphi \frac{de^{\mu}}{d\mu}\, dW_2 =$$

$$= \left[-\gamma a + \frac{\xi^2}{2a}\right] dt + \xi\left[\cos\varphi\, dW_1 + \sin\varphi\, dW_2\right].$$

Im part: $d\varphi = \frac{\xi}{a}\left[\cos\varphi\, dW_2 - \sin\varphi\, dW_1\right].$

Now, define

$$\begin{cases} dW_a = dW_1 \cos\varphi + dW_2 \sin\varphi , \\ dW_\varphi = dW_1(-\sin\varphi) + dW_2 \cos\varphi \end{cases}$$

orthogonal transformation

with $S = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$

Thus, $W_a(t)$ & $W_\varphi(t)$ are Wiener processes.

Finally,
$$\begin{cases} da = \left[-\gamma a + \frac{\xi^2}{2a}\right] dt + \xi\, dW_a , \\ d\varphi = \frac{\xi}{a}\, dW_\varphi . \end{cases}$$

can be simulated numerically

$\Leftarrow$ all changes in φ are random