Lecture 6

[Stochastic differential equations (SDE)]

Recall that $\dfrac{dx}{dt} = a(x,t) + b(x,t)\,\xi(t)$

is better interpreted as

$$x(t) - x(t_0) = \int_{t_0}^{t} dt'\, a[x(t'),t'] + \int_{t_0}^{t} dW(t')\, b[x(t'),t'] .$$

The corresponding Itô SDE is written as:

$$dx(t) = a[x(t),t]\,dt + b[x(t),t]\,dW(t) .$$

Using a discretized mesh of time points

$t_0 < t_1 < t_2 < \ldots < t$, [n subintervals]

we can write

$$x_{i+1} = x_i + a(x_i,t_i)\underbrace{\Delta t_i}_{t_{i+1}-t_i} + b(x_i,t_i)\underbrace{\Delta W_i}_{W(t_{i+1})-W(t_i)} .$$

Note that ΔW_i is an independently sampled increment of the Wiener process.

This is the Cauchy-Euler method for solving SDEs numerically.

One can show that, given a sample function $\widetilde{W}(t)$ of the Wiener process $W(t)$, there is a unique solution to the Itô SDE under fairly mild conditions on $a(x,t)$ & $b(x,t)$.

Numerical solutions converge to this unique solution as $n \uparrow$.

Note that $x(t)$ [SDE solution] is a Markov process. Indeed, $x(t)$ for $t > t_0$ depends only on: (i) $x(t_0)$; (ii) the particular sample path $W(t)$ for $t > t_0$, which is independent of $x(t)$ for $t < t_0$. Thus, $x(t)$ is a Markov process.

Change of variables (Itô's formula)

Consider $f[x(t)]$:

$\uparrow$ arbitrary f'n

$$df[x(t)] = f[x(t) + dx(t)] - f[x(t)] =$$

$$= f'[x(t)]\,dx(t) + \frac{1}{2} f''[x(t)]\,dx(t)^2 + \ldots =$$

$$= f'[x(t)] \left\{ a[x(t),t]\,dt + b[x(t),t]\,dW(t) \right\} +$$

$$+ \frac{1}{2} f''[x(t)]\, b[x(t),t]^2 \underbrace{dW(t)^2}_{\text{"}dt} =$$

$$= \left\{ a[x(t),t]\, f'[x(t)] + \frac{1}{2} b[x(t),t]^2 f''[x(t)] \right\} dt +$$

$$+ b[x(t),t]\, f'[x(t)]\,dW(t).$$

Itô's formula, 1D

Multi-D:

$$\begin{cases} dW_i(t)\, dW_j(t) = \delta_{ij}\, dt, \\ dW_i(t)^{N+2} = 0, \quad N>0 \\ dW_i(t) \times dt = 0, \\ dt^{1+N} = 0, \quad N>0 \end{cases}$$

Then, if $dx_i = A_i(\vec{x},t)\,dt + \underbrace{\overbrace{B_{ij}(\vec{x},t)}^{\substack{B(\vec{x},t) \\ \text{matrix}}}\, dW_j(t)}_{\text{sum over } j}$:

$$df(\vec{x}) = \Big\{ A_i\, \partial_i f(\vec{x}) + \frac{1}{2} \underbrace{B_{ik}\, \overbrace{B_{jk}}^{B^T_{kj}}}_{[B(\vec{x},t)\,B^T(\vec{x},t)]_{ij}} \partial_i \partial_j f(\vec{x}) \Big\} dt + B_{ij}\, \partial_i f(\vec{x})\, dW_j(t).$$

Connection to FP equation:

Consider $\dfrac{\langle df[x(t)] \rangle}{dt} = \dfrac{d}{dt} \langle f[x(t)] \rangle =$

$$= \langle a\, \partial_x f + \frac{b^2}{2} \partial_x^2 f \rangle.$$

$\uparrow \langle b\, \partial_x f\, dW \rangle = 0$

Now,

$$\frac{d}{dt}\langle f \rangle = \int dx\, f(x)\, \partial_t p(x,t \mid x_0,t_0) =$$

$$= \int dx \left[a\, \partial_x f + \frac{b^2}{2} \partial_x^2 f \right] p(x,t \mid x_0,t_0).$$

Integrate by parts:

$$\int dx\, f(x)\, \partial_t p = \int dx\, f(x) \left\{ - \partial_x [a p] + \frac{1}{2} \partial_x^2 [b^2 p] \right\}.$$

Since $f(x)$ is arbitrary,

$$\partial_t p = - \partial_x [\underset{\substack{\uparrow \\ \text{drift} \\ \text{coeff.}}}{a} p] + \frac{1}{2} \partial_x^2 [\underset{\substack{\uparrow \\ \text{diff'n} \\ \text{coeff.}}}{b^2} p].$$

<u>Multi-D</u>:

$dx_i = A_i dt + B_{ij} dW_j$ leads to

$$\partial_t p = - \partial_i [A_i p] + \frac{1}{2} \partial_i \partial_j \{ [B B^T]_{ij} p \}$$

Note that replacing B by BS s.t. $S S^T = \mathbb{I}$ (i.e., S is orthogonal) does not change the FP eq'n.

Now, consider

$$dV_i = \underbrace{S_{ij}}_{\text{no f'n, orthogonal}} dW_j . \qquad \langle dV_i \rangle = 0$$

Then $\langle dV_i dV_j \rangle = S_{il} S_{jm} \overbrace{\langle dW_l dW_m \rangle}^{\delta_{lm} dt} =$

$$= \underbrace{S_{il} S_{jl}}_{\delta_{ij}} dt = \delta_{ij} dt.$$

Thus $V_i(t)$ is also a Wiener process,

and $$dx_i = A_i\,dt + B_{ik} \underbrace{[S^T S]_{kj}}_{= \delta_{kj}} dw_j = A_i\,dt + B_{ik} S^T_{kj} \underbrace{dV_j}_{\text{Wiener process}}$$

or, equivalently,

$$dx_i = A_i\,dt + B_{ik} S^T_{kj}\,dw_j .$$

Orthogonal transformations simply mix up different sample paths of the stochastic process.

Stratonovich Stochastic Integral

$$(S) \int_{t_0}^{t} G[\underbrace{x(t')}_{\text{obeys an SDE}}, t'] \, dW(t') = \lim_{n\to\infty} \sum_{i=1}^{n} G\left[\frac{x(t_i)+x(t_{i-1})}{2}, t_{i-1}\right] \times$$

$$\times [W(t_i) - W(t_{i-1})] .$$

Stratonovich SDE

$$x(t) = x(t_0) + \int_{t_0}^{t} dt' \, \alpha[x(t'), t'] +$$

$$+ (S) \int_{t_0}^{t} dW(t') \, \beta[x(t'), t'] .$$

Change of variables, Stratonovich SDE:

$$(S) \; df[x(t)] = f'[x(t)] \underbrace{\{ \alpha[x(t), t] dt + \beta[x(t), t] dW(t) \}}_{dx(t), \text{ as in regular calculus}} \qquad (*)$$

Consider $\quad dx(t) = \underbrace{B[x(t)]}_{\text{no explicit } t\text{-dep., for simplicity}} dW(t)$

In discretized form,

$$x_{i+1} = x_i + B\left[\frac{x_{i+1}+x_i}{2}\right](W_{i+1} - W_i) .$$

Recall that

$$f(x+a) = f(x-a) + 2\sum_{n=0}^{\infty} \frac{f^{2n+1}(x)\, a^{2n+1}}{(2n+1)!} \simeq$$

$$\simeq f(x-a) + 2 f'(x)\, a \,.$$

Use $\begin{cases} x+a = x_{i+1}, \\ x-a = x_i \end{cases} \Rightarrow \begin{cases} x = \dfrac{x_{i+1}+x_i}{2}, \\ a = \dfrac{x_{i+1}-x_i}{2}. \end{cases}$

Then $f(x_{i+1}) = f(x_i) + f'\left(\dfrac{x_{i+1}+x_i}{2}\right) \times (x_{i+1} - x_i) =$

$$= f(x_i) + f'\left(\frac{x_{i+1}+x_i}{2}\right) B\left[\frac{x_{i+1}+x_i}{2}\right](W_{i+1} - W_i).$$

This leads to

$$(S)\ df[x(t)] = f'[x(t)] \underbrace{B[x(t)]\, dW(t)}_{dx(t),\ \text{as in ordinary calculus}}$$

generalization to Eq. (*) follows.

<u>Equivalent</u> <u>Itô</u> <u>SDE</u>:

Consider $dx(t) = \underset{\substack{\uparrow \\ \text{Itô SDE}}}{a}[x(t), t]\, dt + b[x(t), t]\, dW(t).$

$$(S) \int_{t_0}^{t} dW(t')\, \beta[x(t'), t'] = \lim_{n\to\infty} \sum_{i=1}^{n} \beta\left[\frac{x(t_i) + x(t_{i-1})}{2}, t_{i-1}\right] \times \underbrace{\Delta W_{i-1}}_{\text{"}\ \Delta W(t_{i-1})}$$

Writing $x(t_i) = x(t_{i-1}) + \overbrace{\Delta x(t_{i-1})}^{\equiv \Delta x_{i-1}}$, we get from Itô SDE:

$$\Delta x(t_{i-1}) = a[x(t_{i-1}), t_{i-1}] \overbrace{\Delta t_{i-1}}^{t_i - t_{i-1}} +$$
$$+ b[x(t_{i-1}), t_{i-1}] \underbrace{\Delta W(t_{i-1})}_{W_i - W_{i-1} = \Delta W_{i-1}} .$$

Finally, $\beta\left[\frac{x_i + x_{i-1}}{2}, t_{i-1}\right] = \beta\left[x_{i-1} + \frac{\Delta x_{i-1}}{2}, t_{i-1}\right] =$

$$= \overbrace{\beta[x_{i-1}, t_{i-1}]}^{=\beta_{i-1}} + \Big\{ a_{i-1}\, \partial_x \beta[x(t_{i-1}), t_{i-1}] +$$
$$+ \frac{b^2_{i-1}}{4} \partial^2_x \beta[x(t_{i-1}), t_{i-1}] \Big\} \frac{\Delta t_{i-1}}{2} + \underbrace{\frac{b_{i-1}}{2} \partial_x \beta[x(t_{i-1}), t_{i-1}] \times \Delta W_{i-1}}.$$

Recall that $df[x(t)] = f[x + \frac{dx}{2}] - f[x] \simeq$

$$\simeq f'[x] \frac{1}{2} \{ \underbrace{a\, dt + b\, dW}_{"dx} \} + \frac{f''[x]}{2!} \frac{1}{4} \underbrace{dx^2}_{b^2 dW^2 = b^2 dt} =$$

$$= \left\{ f'[x] \frac{a}{2} + \frac{f''[x]}{2} \frac{b^2}{4} \right\} dt + f'[x] \frac{b}{2} dW .$$

↑ use the discrete version of this above

Now,

$$(S) \int_{t_0}^{t} dW(t')\, \beta[x(t'), t'] = \lim_{n\to\infty} \sum_i \beta_{i-1} \Delta W_{i-1} +$$
$$+ \lim_{n\to\infty} \frac{1}{2} \sum_i b_{i-1}\, \partial_x \beta[x(t_{i-1}), t_{i-1}] \underbrace{\Delta t_{i-1}}_{\Delta W^2_{i-1}} \; (=)$$

$$\stackrel{(=)}{} \underbrace{\int_{t_0}^{t} dW(t')\, \beta[x(t'),t']}_{\text{Itô}} + \underbrace{\frac{1}{2}\int_{t_0}^{t} dt'\, b[x(t'),t'] \times \partial_x \beta[x(t'),t']}_{\text{regular } \int}$$

If we choose

$$\begin{cases} \beta(x,t) = b(x,t), \\ \alpha(x,t) = a(x,t) - \frac{1}{2} b(x,t)\, \partial_x b(x,t), \end{cases}$$

we obtain:

$$(S)\; dx(t) = \underbrace{\alpha[x(t),t]\,dt + \beta[x(t),t]\,dW(t)}_{\text{Stratonovich SDE}} =$$

$$= \left(a - \frac{b}{2}\partial_x b\right)dt + b\, dW(t) \quad \text{is the same as}$$

$$dx(t) = \underbrace{a\, dt + b\, dW(t)}_{\text{Itô SDE}}.$$

Indeed, $x(t) = x(t_0) + \int_{t_0}^{t} dt'\, \alpha[x(t'),t'] +$ ← Stratonovich solution

$$+ (S)\int_{t_0}^{t} dW(t')\, \beta[x(t'),t'] =$$

$$= x(t_0) + \int_{t_0}^{t} dt'\, a - \cancel{\frac{1}{2}\int_{t_0}^{t} dt'\, b\, \partial_x b} + \int_{t_0}^{t} dW(t')\, b +$$

$$+ \cancel{\frac{1}{2}\int_{t_0}^{t} dt'\, b\, \partial_x b} = x(t_0) + \int_{t_0}^{t} dt'\, a + \int_{t_0}^{t} dW(t')\, b$$

↑ Itô solution

$x(t)$ is the same in both methods

Likewise, Stratonovich SDE

$$(S)\ dx = \alpha\, dt + \beta\, dw$$

is equivalent to Itô SDE:

$$dx = \underbrace{\left(\alpha + \frac{1}{2}\beta\,\partial_x \beta\right)}_{= a} dt + \underbrace{\beta}_{= b}\, dw .$$

Multi-D

$$\begin{cases} A_i^S = A_i - \frac{1}{2} B_{kj}\, \underbrace{\partial_k}_{= \frac{\partial}{\partial x_k}} B_{ij}, \\ B_{ij}^S = B_{ij}. \end{cases}$$

$$\vec{x} = (x_1, x_2, \ldots, x_n)$$

$\uparrow$ # DoF

$$\text{Itô:}\quad dx_i = A_i(\vec{x}, t)\, dt + B_{ij}(\vec{x}, t)\, dW_j(t).$$

The corresponding FP eq'n is:

$$\partial_t p = -\partial_i [A_i p] + \frac{1}{2}\partial_i \partial_j \{ \underbrace{B_{ik} B_{jk}}_{[BB^T]_{ij}} p \} =$$

$$= -\partial_i \left\{ [A_i^S + \frac{1}{2} B_{kj}^S (\partial_k B_{ij}^S)] p \right\} +$$

$$+ \frac{1}{2}\partial_i \underbrace{\partial_j \{ B_{ik}^S B_{jk}^S p \}}_{B_{ik}^S \times \partial_j (B_{jk}^S p) + (B_{jk}^S p) \times \partial_j B_{ik}^S} \Leftrightarrow$$

$$B_{ik}^S \times \partial_j (B_{jk}^S p) + \underbrace{(B_{jk}^S p) \times \partial_j B_{ik}^S}_{k \leftrightarrow j} =$$

$$= \ldots + B_{kj}^S (\partial_k B_{ij}^S)\, p$$

$$\Leftrightarrow \; - \partial_i \{ A_i^s p \} + \frac{1}{2} \partial_i \{ B_{ik}^s \partial_j (B_{jk}^s p) \} .$$

"Stratonovich form" of
FP equation