[Ito calculus and stochastic differential equations]

Consider a Langevin equation:

$$\frac{dx}{dt} = \underbrace{a(x,t) + b(x,t)}_{\text{regular functions}} \underbrace{\xi(t)}_{\text{rapidly fluctuating random term}} \qquad (*)$$

$$\begin{cases} \langle \xi(t) \rangle = 0, \\ \langle \xi(t)\xi(t') \rangle = \delta(t-t') \end{cases} \Leftarrow \text{zero correlations, infinite variance}$$

Hard to interpret Eq. (*) b/c $\frac{dx}{dt}$ gets infinite contributions from the noise term. However, we expect Eq. (*) to be integrable and therefore

$$u(t) = \int_0^t dt' \, \xi(t')$$

to be "well-behaved", i.e. continuous

Consider $u(t') = \int_0^t ds\, \xi(s) + \int_t^{t'} ds\, \xi(s) =$

$$= \lim_{\epsilon \to 0} \left[\int_0^{t-\epsilon} ds\, \xi(s) \right] + \int_t^{t'} ds\, \xi(s)$$

independent fluctuations

Thus, $u(t)$ & $u(t') - u(t)$ are statistically independent $\Rightarrow$ $u(t') - u(t)$ is indep. of $u(t'')$ for $\forall\, t'' < t \Rightarrow$
$\Rightarrow u(t')$ depends only on $u(t)$ and <u>not</u> on previous $u(t'')$. Hence $u(t)$ is a <u>Markov process</u>.

What is the FP eq'n for $u(t)$?

$$\langle u(t+\Delta t) - u_0 \rangle \Big|_{u_0,t} = \langle \underbrace{\int_t^{t+\Delta t} ds\, \xi(s) \rangle}_{= \int_t^{t+\Delta t} ds \langle \xi(s) \rangle} = 0,$$

$$\langle [u(t+\Delta t) - u_0]^2 \rangle \Big|_{u_0,t} = \int_t^{t+\Delta t} ds \int_t^{t+\Delta t} ds' \underbrace{\langle \xi(s)\xi(s') \rangle}_{\delta(s-s')} =$$

$$= \Delta t.$$

So,

$$\begin{cases} A(u_0,t) = \lim\limits_{\Delta t \to 0} \dfrac{\langle u(t+\Delta t) - u_0 \rangle|_{u_0,t}}{\Delta t} = 0, \\ \text{drift coefficient} \\ B(u_0,t) = \lim\limits_{\Delta t \to 0} \dfrac{\langle [u(t+\Delta t) - u_0]^2 \rangle|_{u_0,t}}{\Delta t} = 1. \\ \text{diffusion coefficient} \end{cases}$$

Thus, $u(t)$ is the Wiener process:

$$u(t) = \int_0^t dt' \xi(t') = W(t).$$

Eq. (*) can be integrated:

$$x(t) - x(0) = \int_0^t ds\, a[x(s), s] +$$

$$+ \int_0^t \underbrace{ds\, \xi(s)}_{= dW(s)} b[x(s), s]$$

since $dW(s) \equiv W(s+ds) - W(s) = \xi(s) ds$.

stochastic integral with respect to a sample function $W(t)$

Stochastic integration

Consider $\begin{cases} G(t) = \text{arbitrary (potentially stochastic) function of time,} \\ W(t) = \text{Wiener process.} \end{cases}$

We need to define $\int_{t_0}^{t} G(t')\,dW(t')$.

Divide $[t_0, t]$ into n subintervals:

$$t_0 \leq t_1 < t_2 < \dots < t$$

and define intermediate points $t_{i-1} \leq \tau_i \leq t_i$.

Then $\overbrace{\int_{t_0}^{t} G(t')\,dW(t')}^{\equiv I} = \lim_{n\to\infty} S_n$, where

$$S_n = \sum_{i=1}^{n} G(\tau_i)\left[W(t_i) - W(t_{i-1})\right].$$

In general, S_n depends on the particular choice of τ_i's. For example, if $G(\tau_i) = W(\tau_i)$ we obtain:

$$\langle S_n \rangle = \left\langle \sum_{i=1}^{n} W(\tau_i)\left[W(t_i) - W(t_{i-1})\right] \right\rangle =$$

$$= \sum_{i=1}^{n} \left[\min(\tau_i, t_i) - \min(\tau_i, t_{i-1}) \right] =$$

$$= \sum_{i=1}^{n} (\tau_i - t_{i-1}) = \alpha \sum_{i=1}^{n} (t_i - t_{i-1}) = \alpha (t - t_0).$$

$$\tau_i = \alpha t_i + (1-\alpha) t_{i-1}$$
$$0 \leq \alpha \leq 1$$

So, $0 \leq \langle S_n \rangle \leq t - t_0$ depending on α.

Ito stochastic integral

Choose $\alpha = 0 \Rightarrow \tau_i = t_{i-1}$.

Then $\displaystyle\int_{t_0}^{t} G(t')\,dW(t') = \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} G(t_{i-1})\left[W(t_i) - W(t_{i-1})\right] \right\}$

The limit here is the "mean square" limit, or limit in the mean. We say that X_n converges to X as $n \to \infty$ if

$$\lim_{n\to\infty} \langle (X_n - X)^2 \rangle = 0 \Rightarrow \lim_{n\to\infty} X_n = X.$$

Ex. Consider $\displaystyle\int_{t_0}^{t} W(t')\,dW(t')$.

Using $W_i = W(t_i)$, we obtain:

$$S_n = \sum_{i=1}^{n} W_{i-1} \underbrace{(W_i - W_{i-1})}_{\Delta W_i} = \sum_{i=1}^{n} W_{i-1}\,\Delta W_i =$$

$$= \frac{1}{2} \sum_i \left[(\underbrace{W_{i-1} + \Delta W_i}_{W_i})^2 - W_{i-1}^2 - \Delta W_i^2 \right] =$$

$$= \frac{1}{2}\left[W^2(t) - W^2(t_0) \right] - \frac{1}{2} \sum_i \Delta W_i^2 .$$

Now, $\langle \sum_i \Delta W_i^2 \rangle = \sum_i (W_i - W_{i-1})^2 = \sum_i (t_i - t_{i-1}) =$
$= t - t_0$.

One can show that

$$\left\langle \left[\sum_i (W_i - W_{i-1})^2 - (t - t_0) \right]^2 \right\rangle =$$

$$= 2 \sum_i \underbrace{(t_i - t_{i-1})^2}_{\frac{1}{n^2}} \sim \frac{1}{n} \to 0 \text{ as } n \to \infty .$$

Thus, $\lim_{n\to\infty} \sum_{i=1}^{n} (W_i - W_{i-1})^2 = t - t_0$ and

$$\int_{t_0}^{t} W(t')\, dW(t') = \frac{1}{2}\left[W^2(t) - W^2(t_0) \right] - \frac{1}{2}(t - t_0).$$

The extra term appears b/c $|W(t+\Delta t) - W(t)| \sim \sqrt{t}$ and 2^{nd} order terms need to be included.

Finally, note that

$$\left\langle \int_{t_0}^{t} W(t')\, dW(t') \right\rangle = \frac{1}{2}\left[\langle W^2(t)\rangle - \langle W^2(t_0)\rangle - (t - t_0) \right] = 0 .$$

This is obvious since $\langle W_{i-1} \Delta W_i \rangle \underset{\text{stat. indep.}}{=} \langle W_{i-1}\rangle \langle \Delta W_i \rangle = 0.$

$$\begin{cases} \langle W^2(t) \rangle = \underbrace{\langle W(t) \rangle}_{\text{"}\langle W(t_0)\rangle^2}{}^2 + (t - t_0), \\[2ex] \langle W^2(t_0) \rangle = \langle W(t_0) \rangle^2. \end{cases}$$

Non-anticipating functions

$G(t)$ (regular or stochastic f'n) is non-anticipating (na) if $G(t)$ is statistically indep. of $W(s) - W(t)$ for $\forall s > t$.

Ex. (1) $W(t)$

(2) $\int_{t_0}^{t} F[W(t')]\,dt'$

(3) $\int_{t_0}^{t} G[W(t')]\,dW(t')$

For an arbitrary non-anticipating function $G(t)$,

$$\int_{t_0}^{t} G(t')\,[dW(t')]^{2+N} = \lim_{n\to\infty} \sum_{i=1}^{n} G_{i-1}\,\Delta W_i^{2+N} =$$

$$= \begin{cases} \int_{t_0}^{t} G(t')\,dt, & N=0 \\ 0, & N>0 \end{cases}$$

Consider $\underline{N=0}$:

$$I = \lim_{n\to\infty} \left\langle \left[\sum_{i=1}^{n} G_{i-1} (\Delta W_i^2 - \Delta t_i) \right]^2 \right\rangle =$$

$$= \lim_{n\to\infty} \Big\{ \Big\langle \sum_i \underbrace{G_{i-1}^2}\,\underbrace{(\Delta W_i^2 - \Delta t_i)^2}_{\text{indep. since } G(t) \text{ is na}} \Big\rangle \oplus$$

$$+\ 2\left\langle \sum_{i>j} \underbrace{G_{i-1} G_{j-1} (\Delta W_j^2 - \Delta t_j)\,(\Delta W_i^2 - \Delta t_i)}_{\text{indep. as well}} \right\rangle \Big\}$$

Now recall that

$$\langle \Delta W_i^2 \rangle = \Delta t_i \implies \langle \Delta W_i^2 - \Delta t_i \rangle = 0,$$

the last term drops out.

Now, $\langle (\Delta W_i^2 - \Delta t_i)^2 \rangle = \underbrace{\langle \Delta W_i^4 \rangle}_{= 3\Delta t_i^2} - 2\Delta t_i \underbrace{\langle \Delta W_i^2 \rangle}_{= \Delta t_i} + \Delta t_i^2 = 2\Delta t_i^2.$

So, $I = 2\lim_{n\to\infty} \left[\sum_i \Delta t_i^2 \langle G_{i-1}^2 \rangle \right] \to 0$

Thus, $\lim_{n\to\infty} \sum_{i=1}^{n} G_{i-1} \Delta W_i^2 = \lim_{n\to\infty} \sum_{i=1}^{n} G_{i-1} \Delta t_i,$ or

$$\int_{t_0}^{t} G(t') \left[dW(t') \right]^2 = \int_{t_0}^{t} G(t')\, dt'.$$

One can also show that

$$\int_{t_0}^{t} G(t') \left[dW(t') \right]^{2+N} = 0 \quad \text{for } N > 0.$$

Hence, for na functions we have:

$$\begin{cases} dW(t)^2 = dt, \\ dW(t)^3 = dW(t)^4 = \dots = 0. \end{cases}$$

Similarly, $\int_{t_0}^{t} G(t')\,dt'\,dW(t') = 0$, and

higher powers of $dt^n\, dW(t)^m$ are all $= 0$ as well.

So, one needs to retain dt, $dW(t)$, and $dW(t)^2$ in calculations.

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(1) <u>polynomials of W(t)</u>:

consider $d[W(t)]^n = [W(t) + dW(t)]^n - W(t)^n =$

$$= \sum_{r=1}^{n} \binom{n}{r} W(t)^{n-r} dW(t)^r$$ = only $r=1,2$ remain

$$= n\,W(t)^{n-1} dW(t) + \frac{n(n-1)}{2} W(t)^{n-2} \underbrace{dt}_{= dW(t)^2} .$$

Substitute $n-1 \to n$, $n \to n+1$ & integrate:

$$(n+1) \int_{t_0}^{t} W(t')^n\, dW(t') + \frac{(n+1)n}{2} \int_{t_0}^{t} W(t')^{n-1} dt' =$$

$$= \int_{t_0}^{t} d[W(t)]^{n+1} \text{, or}$$

$$\int_{t_0}^{t} W(t')^n \, dW(t') = \frac{1}{n+1}\left[W(t)^{n+1} - W(t_0)^{n+1} \right] -$$

$$- \frac{n}{2} \int_{t_0}^{t} W(t')^{n-1} dt' .$$

(2) Function differentials:

$$df[W(t), t] = \frac{\partial f}{\partial t} dt + \cancel{\frac{1}{2} \frac{\partial^2 f}{\partial t^2} (dt)^2} +$$

$$+ \frac{\partial f}{\partial W} dW + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \underbrace{dW^2}_{\text{"}dt} + \cancel{\frac{\partial^2 f}{\partial W \partial t} dt \, dW} + \ldots =$$

$$= \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \right) dt + \frac{\partial f}{\partial W} dW .$$

(3) Mean value:

for na $G(t)$, $\langle \sum_i G_{i-1} \Delta W_i \rangle =$

$= \sum_i \langle G_{i-1} \rangle \langle \Delta W_i \rangle = 0$, or in

the $n \to \infty$ limit:

$$\left\langle \int_{t_0}^{t} G(t') \, dW(t') \right\rangle = 0 .$$

(4) Correlations:

consider $\langle (\sum_i G_{i-1} \Delta W_i)(\sum_j H_{j-1} \Delta W_j) \rangle =$

$$= \Big\langle \sum_i \underbrace{G_{i-1}\, H_{i-1}}\, \underbrace{(\Delta W_i)^2} \Big\rangle + \Big\langle \sum_{i>j} (G_{i-1} H_{j-1} + G_{j-1} H_{i-1}) \times$$

indep. since G & H are na f's

$$\times \Delta W_j \underbrace{\Delta W_i} \Big\rangle$$

indep. of all other terms:

$\rightarrow \Delta W_j$ indep. ΔW's
$\rightarrow G_{i-1}, H_{i-1}$ na G, H
$\rightarrow G_{j-1}, H_{j-1}$ na G, H and $j<i$

Thus, $2^{\underline{nd}}$ term $\langle \ldots \rangle \sim \langle \Delta W_i \rangle = 0$.

Then $\Big\langle \big(\sum_i G_{i-1} \Delta W_i\big)\big(\sum_j H_{j-1} \Delta W_i\big)\Big\rangle =$

$$= \sum_i \langle G_{i-1} H_{i-1} \rangle \underbrace{\langle \Delta W_i^2 \rangle}_{\text{"}\Delta t_i}, \text{ or}$$

in the $n \to \infty$ limit:

$$\Big\langle \int_{t_0}^{t} G(t')\, dw(t') \times \int_{t_0}^{t} H(t')\, dw(t') \Big\rangle = \int_{t_0}^{t} \langle G(t') H(t') \rangle\, dt'. \qquad (+)$$

Using $dw(t) \Rightarrow \xi(t)\,dt$, we obtain:

$$\int_{t_0}^{t} dt' \int_{t_0}^{t} ds' \langle G(t') H(s') \xi(t') \xi(s') \rangle \underset{\uparrow}{=}$$

ξ indep. of G, H

$$= \int_{t_0}^{t} dt' \int_{t_0}^{t} ds' \langle G(t') H(s') \rangle \underbrace{\langle \xi(t') \xi(s') \rangle}_{\text{"}\delta(t'-s')}.$$

to satisfy Eq. (+) above

Thus, $\xi(t)$ noise is δ-correlated, or white. Now, consider "edge case" integrals:

$$\begin{cases} I_1 = \int_{t_1}^{t_2} dt\, f(t)\, \delta(t - \underline{\underline{t_1}}), \\ I_2 = \int_{t_1}^{t_2} dt\, f(t)\, \delta(t - \underline{\underline{t_2}}). \end{cases}$$

Note that $\langle \int_{t_0}^{t} dW(t') \underbrace{G(t')}_{\text{na by assumption}} \underbrace{[\int_{t_0}^{t'} dW(s')\, H(s')]}_{\text{na if } H \text{ is na by assumption}} \rangle =$

$= 0$ by the mean value formula.

Using $dW(t) = \xi(t)\, dt$, we obtain:

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} ds' \underbrace{\langle G(t') H(s') \rangle}_{\text{"}f(t', s')\text{"}} \underbrace{\delta(t' - s')}_{\delta\text{-function at the upper } \int \text{ limit}} = 0.$$

This is consistent with $I_2 = 0$. Indeed, in Itô calculus fluctuations are in the future of integrands $\Rightarrow$ $\delta(t'-s')$ comes from $\langle \underbrace{\xi(t')\xi(s')}_{\text{"future fluctuations"}} \rangle$ and therefore $\delta(t'-s')$ is "just above" the upper limit in $\int_{t_0}^{t'} ds'$.

Likewise, in I_1 $\delta(t - t_1)$ should be "just inside" the $\int_{t_1}^{t_2} dt$ integral, leading to $I_1 = f(t_1)$.