Final Exam (2025)

[50 points]

1. Consider a particle which diffuses freely in a finite 1D interval: $[0, a]$

$$(*) \qquad \frac{\partial p(x,t|x_0,t_0)}{\partial t} = D \frac{\partial^2 p(x,t|x_0,t_0)}{\partial x^2}$$

subject to the initial condition

$$p(x,t_0|x_0,t_0) = \delta(x-x_0)$$

and the boundary condition

$$\left.\frac{\partial p}{\partial x}\right|_{x=0,a} = 0 \qquad \text{(reflective boundaries)}$$

Here, $D > 0$ is the diffusion constant.

Solve Eq. (*) using the eigenfunction expansion. What is the spectrum of eigenvalues? Find $\lim_{t\to\infty} p(x,t|x_0,t_0)$ and discuss its properties.

[50 points]

2. Sometimes, it is necessary to keep the inertial term in the Langevin equation:

$$m\ddot{x} + \gamma \dot{x} = F(x) + \xi ,$$

where $F(x)$ is the external force and ξ is the noise term:

$$\langle \xi(\tau) \rangle = 0 , \qquad \langle \xi(\tau)\, \xi(\tau') \rangle = 2\gamma T\, \delta(\tau - \tau')$$

Derive the corresponding forward Fokker-Planck (FP) equation for the probability density $p(\underset{\text{position}}{x}, \underset{\text{velocity}}{v}, \overset{\text{time}}{\tau})$ for a 1D system with an arbitrary initial condition $p(x_0, v_0, 0)$

<u>Note</u>: There is no need to solve the FP equation.

<u>Hint</u>: It may be easiest to use a technique similar to 4.3.5 in Gardiner (4<u>th</u> edition).