Recall the FP eq'n in the presence of an external field:

$$\frac{\partial W}{\partial t} + \vec{u} \cdot \vec{\nabla}_{\vec{r}} W + \vec{K} \cdot \vec{\nabla}_{\vec{u}} W = \beta \vec{\nabla}_{\vec{u}} (W \vec{u}) + q \nabla^2_{\vec{u}} W .$$

If $\vec{K} = 0$ (the particles are free), this eq'n reduces to:

$$\frac{\partial W}{\partial t} + \vec{u} \cdot \vec{\nabla}_{\vec{r}} W = 3\beta W + \beta \vec{u} \cdot \vec{\nabla}_{\vec{u}} W + q \nabla^2_{\vec{u}} W ,$$

which actually has a closed-form solution (cf. Eq. (286) in Chandrasekhar)

Let us consider a more general situation where we expect the system to relax into the Maxwell-Boltzmann (MB) distribution:

$$W \sim e^{-\frac{(m|\vec{u}|^2 + 2m \overbrace{\tilde{U}(\vec{r})}^{\text{potential energy}})}{2k_B T}}$$

Here, $\vec{K} = -\vec{\nabla}_{\vec{r}} \tilde{U}(\vec{r})$.

As before, assume that the stochastic process is Markovian:

$$W(\vec{r}, \vec{u}, t+\Delta t) = \iint d(\Delta\vec{r})\, d(\Delta\vec{u})\, W(\vec{r}-\Delta\vec{r}, \vec{u}-\Delta\vec{u}, t) \times \Psi(\vec{r}-\Delta\vec{r}, \vec{u}-\Delta\vec{u}; \Delta\vec{r}, \Delta\vec{u}),$$

where the transition probability is given by:

$$\Psi(\vec{r},\vec{u};\Delta\vec{r},\Delta\vec{u}) = \psi(\vec{r},\vec{u};\Delta\vec{u})\times$$

$$\times\,\delta(\Delta x - u_1\Delta t)\,\delta(\Delta y - u_2\Delta t)\,\delta(\Delta z - u_3\Delta t).$$

Now

$$W(\vec{r},\vec{u},t+\Delta t) = \int d(\delta\vec{u})\, W(\vec{r}-\vec{u}\Delta t, \vec{u}-\vec{K}\Delta t-\delta\vec{u}, t)\times$$

$$\times\,\psi(\vec{r}-\vec{u}\Delta t,\ \vec{u}-\vec{K}\Delta t-\delta\vec{u};\ \vec{K}\Delta t+\delta\vec{u})$$

Here,
$$\begin{cases} \Delta\vec{u} = \vec{K}\Delta t + \underbrace{\delta\vec{u}(\Delta t)}_{-\beta\vec{u}\Delta t + \vec{B}(\Delta t)}, \\ \Delta\vec{r} = \vec{u}\,\Delta t. \end{cases}$$

stochastic process of the Brownian type

Alternatively,

$$W(\vec{r}+\vec{u}\Delta t, \vec{u}+\vec{K}\Delta t, t+\Delta t) =$$

$$= \int d(\delta\vec{u})\, W(\vec{r},\vec{u}-\delta\vec{u},t)\,\psi(\vec{r},\vec{u}-\delta\vec{u};\delta\vec{u})$$

Now we expand the LHS and the RHS as before, to obtain:

$$\left\{\frac{\partial W}{\partial t} + \vec{u}\cdot\vec{\nabla}_{\vec{r}}W + \vec{K}\cdot\vec{\nabla}_{\vec{u}}W\right\}\Delta t + O(\Delta t^2) =$$

$$= -\sum_i \frac{\partial}{\partial u_i}(W\langle\delta u_i\rangle) + \frac{1}{2}\sum_i \frac{\partial^2}{\partial u_i^2}(W\langle\delta u_i^2\rangle) +$$

$$+\sum_{i<j}\frac{\partial^2}{\partial u_i\partial u_j}(W\langle\delta u_i\,\delta u_j\rangle) + O(\delta u^3),$$

where all moments are defined wrt $\psi(\vec{r},\vec{u};\delta\vec{u})$.

Now, assume that

$$\begin{cases} \langle \delta u_i \rangle = \mu_i \Delta t + \theta(\Delta t^2), \\ \langle \delta u_i^2 \rangle = \mu_{ii} \Delta t + \theta(\Delta t^2), \\ \langle \delta u_i \delta u_j \rangle = \mu_{ij} \Delta t + \theta(\Delta t^2), \end{cases}$$

and that all higher-order moments are at least $\theta(\Delta t^2)$.

Then

$$\frac{\partial W}{\partial t} + \vec{u} \cdot \vec{\nabla}_{\vec{r}} W + \vec{K} \cdot \vec{\nabla}_{\vec{u}} W = -\sum_i \frac{\partial}{\partial u_i} (W \mu_i) + \\ + \frac{1}{2} \sum_i \frac{\partial^2}{\partial u_i^2} (W \mu_{ii}) + \sum_{i<j} \frac{\partial^2}{\partial u_i \partial u_j} (W \mu_{ij}) .$$

FP eq'n

The MB distr'n must satisfy this eq'n identically. Indeed, the LHS is

$$\sim \left(-\frac{m}{k_B T}\right) \vec{u} \cdot \underbrace{\vec{\nabla}_{\vec{r}} \tilde{u}(\vec{r})}_{= -\vec{K}} + \left(-\frac{m}{2k_B T}\right) 2 \vec{K} \cdot \vec{u} = 0,$$

(note that $\frac{\partial W}{\partial t} = 0$ @ equilibrium)

such that

$$-\sum_i \frac{\partial}{\partial u_i} \left[e^{-\frac{m|\vec{u}|^2}{2k_B T}} \mu_i \right] + \frac{1}{2} \sum_i \frac{\partial^2}{\partial u_i^2} \left[e^{-\frac{m|\vec{u}|^2}{2k_B T}} \mu_{ii} \right] + \\ + \sum_{i<j} \frac{\partial^2}{\partial u_i \partial u_j} \left[e^{-\frac{m|\vec{u}|^2}{2k_B T}} \mu_{ij} \right] = 0$$

must hold at equilibrium.

Clearly, if $\delta\vec{u} = -\beta\vec{u}\,\Delta t + \vec{B}(\Delta t)$,

$$\Psi(\vec{u};\delta\vec{u}) = \frac{1}{(4\pi q\,\Delta t)^{3/2}}\, e^{-\frac{|\beta\vec{u}\,\Delta t + \delta\vec{u}|^2}{4q\,\Delta t}}.$$

Therefore,

$$(*)\quad \begin{cases} \mu_i = -\beta u_i, \\ \mu_{ii} = 2q, \\ \mu_{ij} = 0 \end{cases} \quad + \mathcal{O}(\Delta t^2) \text{ terms}$$

$$\beta \sum_i \frac{\partial}{\partial u_i}\left[e^{-\frac{m|\vec{u}|^2}{2k_BT}} u_i\right] + \frac{1}{2}(2q)\sum_i \frac{\partial^2}{\partial u_i^2}\left[e^{-\frac{m|\vec{u}|^2}{2k_BT}}\right] =$$

$$= \ldots + q\left(-\frac{m}{k_BT}\right)\sum_i \frac{\partial}{\partial u_i}\left[e^{-\frac{m|\vec{u}|^2}{2k_BT}} u_i\right] =$$

$$\frac{qm}{k_BT} = \beta$$

$$= \beta \sum_i \frac{\partial}{\partial u_i}[\ldots] - \beta\sum_i \frac{\partial}{\partial u_i}[\ldots] = \underline{\underline{0}}.$$

However, this does not mean that (*) represent the most general set of moments possible, from the point of view of satisfying the equilibrium condition above.