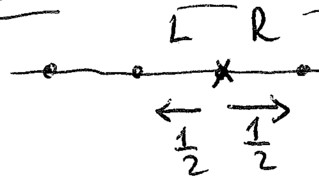
Lecture 2

Consider 1D random walk:



unbiased

After N steps, the particle could be anywhere in $[-N, N]$.

What is $W(m, N)$ = prob. to arrive at $m \in [-N, N]$ after N steps?

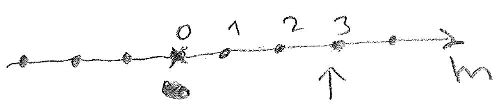
Each individual path has the prob. $\left(\frac{1}{2}\right)^N$, but in general there are multiple paths that arrive at m.

To get to m, we need to make $\frac{N+m}{2}$ steps to the right & $\frac{N-m}{2}$ steps to the left. Indeed,

$$\frac{N+m}{2} + \frac{N-m}{2} = N \; ; \quad \frac{N+m}{2} - \frac{N-m}{2} = m .$$

Note that $\begin{cases} N \text{ even} \to m \text{ even} \\ N \text{ odd} \to m \text{ odd} \end{cases}$

E.g.



$\begin{cases} N=6 \\ m=3 \end{cases}$ impossible to get to $m=3$ in $N=6$ steps

Clearly, $$W(m, N) = \frac{N!}{\frac{N+m}{2}! \, \frac{N-m}{2}!} \left(\frac{1}{2}\right)^N$$

binomial distr'n

Recall that for

$$w(x,n) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$p < 1$, $0 \leq x \leq n$ we have:

$$\begin{cases} \langle x \rangle = np, \\ \langle x^2 \rangle = np + n(n-1)p^2 \end{cases}$$

Then $\mathrm{Var}(x) = \langle x^2 \rangle - \langle x \rangle^2 =$

$$= np + n(n-1)p^2 - n^2p^2 = \underline{\underline{np(1-p)}}. \quad (*)$$

In our case,

$$\left\langle \frac{N+m}{2} \right\rangle = \frac{N}{2} \Rightarrow \underline{\underline{\langle m \rangle = 0}}$$

$$\mathrm{Var}\left(\frac{N+m}{2}\right) = \left\langle \left[\frac{N+m}{2} - \underbrace{\frac{N}{2}}_{\left\langle \frac{N+m}{2} \right\rangle} \right]^2 \right\rangle =$$

$$= \frac{\langle m^2 \rangle}{4} = \frac{N}{4} \Rightarrow \underline{\underline{\langle m^2 \rangle = N}}.$$

from (*)

In other words, RMS displacement is $\sqrt{N}$.

$\boxed{N \text{ large}, m \ll N}$ $\boxed{\text{Random walk, free space}}$

$$\log W(m,N) = N\log\frac{1}{2} + \log(N!) - \log\left(\frac{N+m}{2}!\right) -$$

$$-\log\left(\frac{N-m}{2}!\right) \approx \qquad \boxed{\begin{array}{l}\text{Use } \log n! \simeq (n+\frac{1}{2})\log n - \\ -n + \frac{1}{2}\log 2\pi + O(n^{-1}) \quad \text{Stirling's formula}\end{array}}$$

$$\approx -N\log 2 - \frac{1}{2}\log(2\pi) - \cancelto{0}{\left[N - \frac{N+m}{2} - \frac{N-m}{2}\right]} +$$

$$+ \left(N+\frac{1}{2}\right)\log N - \left(\frac{N+m}{2}+\frac{1}{2}\right)\log\frac{N+m}{2} -$$

$$-\left(\frac{N-m}{2}+\frac{1}{2}\right)\log\frac{N-m}{2} =$$

$$= \left(N+\frac{1}{2}\right)\log N - \frac{N+m+1}{2}\left[\log\frac{N}{2} + \underbrace{\log\left(1+\frac{m}{N}\right)}_{\approx \frac{m}{N} - \frac{m^2}{2N^2}}\right] -$$

$$- \frac{N-m+1}{2}\left[\log\frac{N}{2} + \underbrace{\log\left(1-\frac{m}{N}\right)}_{\approx -\frac{m}{N} - \frac{m^2}{2N^2}}\right] - \frac{\log(2\pi)}{2} - N\log 2 \approx$$

$$\approx \left(N+\frac{1}{2}\right)\log N - \frac{N+m+1}{2}\left[\log N - \log 2 + \frac{m}{N} - \frac{m^2}{2N^2}\right] -$$

$$- \frac{N-m+1}{2}\left[\log N - \log 2 - \frac{m}{N} - \frac{m^2}{2N^2}\right] - \frac{\log(2\pi)}{2} -$$

$$-N\log 2 = -\frac{1}{2}\log N + (N+1)\log 2 - \frac{m^2}{2N} - \frac{m^2}{2N} -$$

$$+ 2\,\frac{N+1}{2}\,\frac{m^2}{2N^2} - \frac{\log(2\pi)}{2} - N\log 2 =$$

$$= -\frac{1}{2}\log N + \underbrace{\log 2 - \frac{\log(2\pi)}{2}}_{\frac{1}{2}\log 2 - \frac{1}{2}\log\pi} \underbrace{- \frac{m^2}{N} + \frac{m^2}{2N} + \cancelto{0}{\frac{m^2}{2N^2}}}_{-\frac{m^2}{2N}}$$

$$\Downarrow$$

$$W(m,N) \simeq \sqrt{\frac{2}{\pi N}}\, e^{-\frac{m^2}{2N}}$$

Introduce $\quad x = \underset{\substack{\uparrow \\ \text{lattice coord.}}}{m} \overset{}{\underset{\substack{\uparrow \\ \text{length of a step}}}{l}}$

$$W(x,N) \sim e^{-\frac{x^2}{\underbrace{2Nl^2}_{\text{"}\sigma^2\text{"}}}}$$

$x \in (-\infty, \infty)$

Normalize: $\quad W(x,N) = \dfrac{1}{\sqrt{2\pi N l^2}}\, e^{-\frac{x^2}{2Nl^2}}$.

Suppose that the particle undergoes n displacements per unit time:

$N = nt$.

Then $\quad 2Nl^2 = 2ntl^2 = 4Dt$,

where $D = \dfrac{nl^2}{2}$

We have:

$$W(x,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{x^2}{4Dt}}.$$

$W(x,t)\,\Delta x$ = prob. that the part. is between x & $x+\Delta x$ after time t.

Reflecting barrier, $m = m_1$

assume $m_1 > 0$

Consider $W(m, N; m_1)$ $[m \leq m_1]$ = prob. that particle arrives @ m after N steps

Presence of a reflecting wall gives the particle additional opportunities to arrive at m, s.t.:

$$W(m,N;m_1) = \underbrace{W(m,N) + W(2m_1 - m, N)}_{\text{prob. in the } \underline{\text{absence}} \text{ of the wall}}$$

$m=2,\ t=6$

$m' = 2m_1 - m = 6,\ t=6$ mirror image of the target point

Consider a single-reflection path

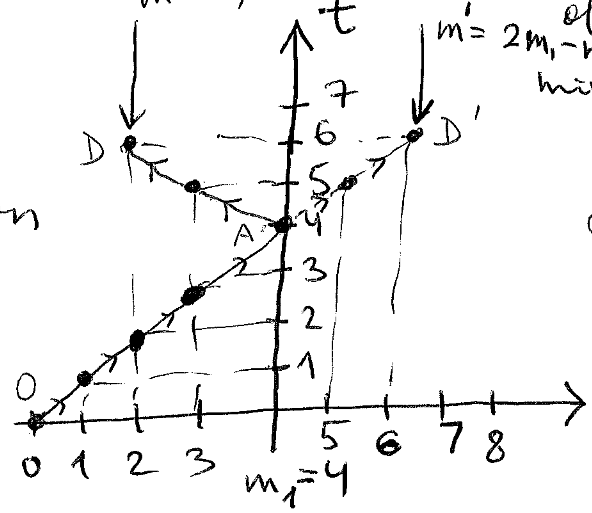


Fig. 1

OAD but also OAD'

So, $2^1 = 2$ "enhancement"

So you have to sum over both paths: the one ending at the target & the other ending at its mirror image.

Consider a path with two reflections:

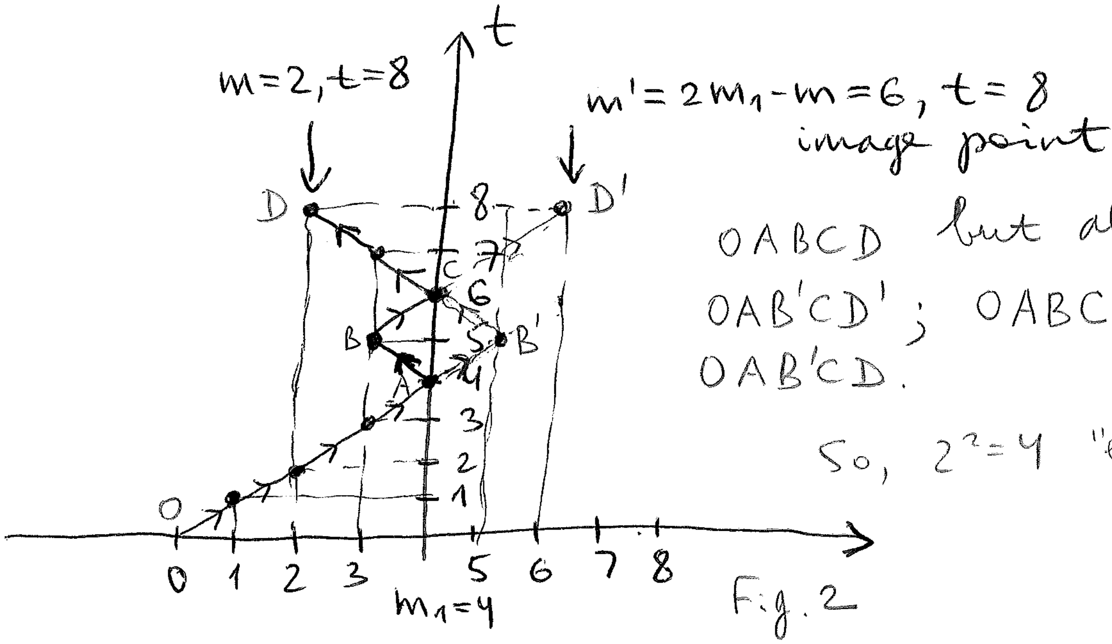


Fig. 2

OABCD but also
OAB'CD' ; OABCD' ;
OAB'CD.

So, $2^2 = 4$ "enhancement"

So, if $N \gg 1$:

$$W(m, N; m_1) = \sqrt{\frac{2}{\pi N}} \left[e^{-m^2/2N} + e^{-(2m_1 - m)^2/2N} \right], \text{ or}$$

barrier pos'n

$$W(x, t; x_1) = \frac{1}{\sqrt{4\pi D t}} \left[e^{-x^2/4Dt} + e^{-(2x_1 - x)^2/4Dt} \right]$$

Note that

$$\left.\frac{\partial W}{\partial x}\right|_{x=x_1} = \frac{1}{\sqrt{4\pi D t}} \left[-\frac{x_1}{2Dt} e^{-x_1^2/4Dt} + \frac{x_1}{2Dt} e^{-x_1^2/4Dt} \right] = 0, \text{ as expected}$$

Absorbing barrier, $m = m_1$

Some paths are no longer accessible since the particle gets absorbed before ever reaching m. How to remove all "forbidden" paths consistently?

Use

$$W(m, N; m_1) = \underbrace{W(m, N) - W(2m_1 - m, N)}_{\text{in the absence of the absorbing wall}}$$

Indeed, in Fig. 1 above OAD is forbidden but its probability is equal to that of OAD', so we can subtract the latter. Likewise, in Fig. 2 OABCD is forbidden (two absorbing events!), but its probability is equal to that of OABCD', so we can subtract that one instead.

Then

$$W(m, N; m_1) = \sqrt{\frac{2}{\pi N}}\left[e^{-m^2/2N} - e^{-(2m_1-m)^2/2N}\right], \text{ or}$$

$$W(x, t; x_1) = \frac{1}{\sqrt{4\pi D t}}\left[e^{-x^2/4Dt} - e^{-(2x_1-x)^2/4Dt}\right]$$

Note that $W(x_1, t; x_1) = 0$, as expected.