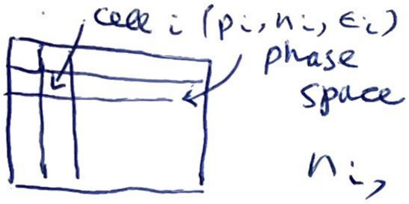

Boltzmann distr'n

n_i, $\sum_i n_i = N$ total # part.

$i = \overline{1, S}$

$$W = \frac{N!}{n_1! \dots n_S!}$$

$$\log W = \log N! - \sum_{i=1}^{S} \log(n_i!) \cong$$

$$\log k! \simeq k \log k - k$$

$$\cong N \log N - N - \sum_i [n_i \log n_i - n_i] =$$

$$= -N \sum_{i=1}^{S} \underbrace{\frac{n_i}{N}}_{p_i} \log\left(\frac{n_i}{N}\right) = -N \underbrace{\sum_i p_i \log p_i}_{\sim \text{entropy}}$$

$$S = -k_B \sum_i p_i \log p_i \sim \log W$$

Maximize entropy subject to two constraints:

$$\begin{cases} \sum_i p_i = 1, \\ \sum_i p_i \epsilon_i = \bar{\epsilon} \end{cases} \Leftarrow \text{constraints}$$

$\begin{cases} p_i \text{ – prob. that a particle is in cell } i \\ \epsilon_i \text{ – part. energy} \end{cases}$

$\bar{\epsilon}$: ~~part.~~ energy per ~~cell~~ particle

Each cell has its own energy

$$F \equiv \frac{S}{k_B} - \beta\left(\sum_i p_i \epsilon_i - \bar{\epsilon}\right) - \alpha\left(\sum_i p_i - 1\right)$$

$$\begin{cases} \dfrac{\partial F}{\partial \beta} = 0 \Rightarrow \sum_i p_i \epsilon_i = \bar{\epsilon} \\ \dfrac{\partial F}{\partial \alpha} = 0 \Rightarrow \sum_i p_i = 1 \end{cases}$$

$$\frac{\partial F}{\partial p_i} = -1 - \log p_i - \beta \epsilon_i - \alpha = 0,$$

$$-\log p_i^* = 1 + \alpha + \beta \epsilon_i, \quad p_i^* = e^{-(1+\alpha)} e^{-\beta \epsilon_i}$$

$$p_i^* \sim e^{\beta \epsilon_i}$$

$$\sum_i p_i = 1 \Rightarrow p_i^* = \frac{e^{-\beta \epsilon_i}}{\sum_j e^{-\beta \epsilon_j}} \qquad \boxed{\text{Boltzmann distr'n}}$$

can get α from here (but won't)

gives α thru β, ϵ_i

β is the inverse T;

$$\frac{\sum e^{-\beta \epsilon_i} \epsilon_i}{\sum_j e^{-\beta \epsilon_j}} = \bar{\epsilon}$$

α & β determined implicitly

gives β thru $\epsilon_i, \bar{\epsilon}$

"Only relevant info used to derive Boltzmann distr'n"

Ensembles vs. dynamical view $\Rightarrow$ ergodicity

Gibbs: $\boxed{E_A}\boxed{E_B}$ $\quad p(A+B) \simeq p(A)p(B) \sim e^{-\theta E} = e^{-\theta(E_A + E_B)}$

$E = E_A + E_B = \text{const}$

Information theory

strings of symbols: M characters, r-letter alphabet

$p_1 \ldots p_r$ letter probs

M large: expect $m_i = M p_i$

\# letters of type i

Then

$$P = p_1^{m_1} \ldots p_r^{m_r} = p_1^{Mp_1} \ldots p_r^{Mp_r}$$

Consider $H = \log\left(\frac{1}{P}\right) = -\log P = -M \sum_{i=1}^{r} p_i \log p_i$

Similar to entropy in stat. phys.

Multiple max. likelihood seqs (same letters in diff. order). How many? $1/P \equiv W$ ← degeneracy

For example, M large, $r=2$: (a,b)

$a: \begin{cases} p_1 = 0.2 \\ p_2 = 0.8 \end{cases}$ b:

~~$2^3 = 8$ seqs:~~ aaa aab aba baa bba bab abb bbb

Very long seqs will have 20% a, 80% b. How many such seqs are there?

$$P = p_1^{Mp_1} p_2^{Mp_2}$$

$p_1 M$ a's, $p_2 M$ b's

Probs of other messages are $\simeq 0$, so that the sum over all high-likelihood messages is 1.

$$\cancel{\left[(Mp_1)!(Mp_2)! \simeq \sqrt{2\pi M p_1}\left(\frac{Mp_1}{e}\right)^{Mp_1} \times \sqrt{2\pi M p_2}\left(\frac{Mp_2}{e}\right)^{Mp_2} = 2\pi M \sqrt{p_1 p_2}\, e^{-M} (Mp_1)^{Mp_1}(Mp_2)^{Mp_2} \right]}$$

Maximize $W \to$ maximize $\frac{1}{P} \to$

$\to$ maximize $H = \log\left(\frac{1}{P}\right)$.

$$\cancel{F = H - \alpha\left(\sum_i m_i - M\right)}, \quad \cancel{H = -\sum_{i=1} m_i \log p_i}$$

$$\cancel{\frac{\partial F}{\partial m_i} = -\log p_i - \alpha}$$

$$\cancel{F = H - \alpha\left(\sum_i p_i - 1\right)}, \quad \cancel{H = -M \sum_i p_i \log p_i}$$

$$\cancel{\frac{\partial F}{\partial p_i} = -M[1 + \log p_i] - \alpha} = 0$$

$p_i = \frac{m_i}{M}$ ⇐

Indeed,

$$F = -\sum_{i=1}^{r} m_i \log p_i + \alpha\left(\sum_i p_i - 1\right)$$ gives

$$\frac{\partial F}{\partial p_i} = -\frac{m_i}{p_i} + \alpha = 0,$$

or $p_i = \frac{m_i}{\alpha}$.

$\sum_i p_i = 1 \Rightarrow \alpha = M$. $\sum_i m_i = M$

Now, consider probs. $\{p_i\}$, $i = 1 \dots N$
which may <u>not</u> be frequentist

Uncertainty $H(\{p_i\})$ – ?

1. H is a cont. f'n of the p_i's
2. If all $p_i = \frac{1}{N}$, $H \uparrow$ as $N \uparrow$
3. Composition property:

$$H(p_1 \dots p_N) = H(P_1, P_2, \dots) + P_1 H\left(\frac{p_1}{P_1} \dots \frac{p_{n_1}}{P_1}\right) + $$
$$+ P_2 H\left(\frac{p_{n+1}}{P_2} \dots \frac{p_{n_1+n_2}}{P_2}\right) + \dots \quad (*)$$

$$\begin{cases} p_1 + p_2 + \dots + p_{n_1} = P_1, \\ p_{n_1+1} + \dots + p_{n_1+n_2} = P_2, \\ \dots \end{cases}$$ grouped probs

<u>Example</u>

$$\text{kangaroos} \begin{cases} \to \ell & \text{left-handed } p_\ell \\ \to r & \text{right-handed } p_r \end{cases}$$

$$\text{kangaroos} \begin{cases} \to p_b & \text{blue} \\ \to p_g & \text{green} \\ \to p_h & \text{hazelnut} \end{cases}$$ eye color

"subcoutcome" $$\begin{cases} p_\ell + p_r = 1, \\ p_b + p_g + p_h = 1 \end{cases}$$

$$H(p_{\ell b}, p_{rb}, p_{\ell g}, p_{rg}, p_{\ell h}, p_{rh}) =$$
$$= H(p_\ell, p_r) + H(p_g, p_b, p_h)$$

uncertainties combine

Now use $p_r + p_e = 1$:

$$H(p_{eb}, \dots, p_{rh}) = H(p_e, p_r) + p_e H\left(\frac{p_{eg}}{p_e}, \frac{p_{eb}}{p_e}, \frac{p_{eh}}{p_e}\right) + p_r H\left(\frac{p_{rg}}{p_r}, \frac{p_{rb}}{p_r}, \frac{p_{rh}}{p_r}\right)$$

$\frac{p_{eg}}{p_e} = p(g|e)$ in general, $= p(g)$ here

like (*) with $n_1 = n_2 = 3$

using $p_e p_g = p_{eg}$, etc.

$p_e = p_{eg} + p_{eb} + p_{eh}$, etc.

OK, then use (*) to find H:

Consider $p_i = \frac{1}{N} = \frac{1}{\sum_j n_j}$ & $P_i = \frac{n_i}{\underbrace{\sum_j n_j}_{N}} = \frac{n_i}{N}$

$n_i = \#\ p_i$'s in group i

Then $H\left(\frac{1}{N} \dots \frac{1}{N}\right) = H(P_1, P_2, \dots) + \sum_i P_i H\left(\frac{1}{n_i} \dots \frac{1}{n_i}\right)$.

Define $A(m) \equiv H\Big(\underbrace{\frac{1}{m} \dots \frac{1}{m}}_{m \text{ entries}}\Big)$.

We get $A(N) = H(P_1, P_2, \dots) + \sum_i P_i A(n_i)$.

Choose all $n_i = m$: all groups have m members

$$A(N) = A\left(\frac{N}{m}\right) + A(m)$$

$$-\frac{dA(m)}{dm} = -\frac{N}{m^2} \frac{dA(N/m)}{d(N/m)}.$$

$m = 1$: $\underbrace{\frac{dA(m)}{dm}\Big|_{m=1}}_{K} = N \frac{dA(N)}{dN}$

$$A(N) = \underbrace{K}_{>0} \log N + \text{const}$$

$>0 \rightarrow A \uparrow$ as $N \uparrow$

$\underbrace{A(1)}_{H(1)} = 0 \Rightarrow \text{Const} = 0$.

So, $\underbrace{K \log N}_{\sum_i p_i} = H(p_1, p_2, \ldots) + \sum_i p_i K \log n_i$,

$$H(p_1, p_2, \ldots) = -K \sum_i p_i \log \underbrace{p_i}_{n_i/N}$$

holds for irrational #'s as well by continuity

State of max uncertainty:

$$\max\{H(\{p_i\})\} = \max\left(-\sum_i p_i \log p_i\right)$$

[possibly subject to constraints]

at least normal'n