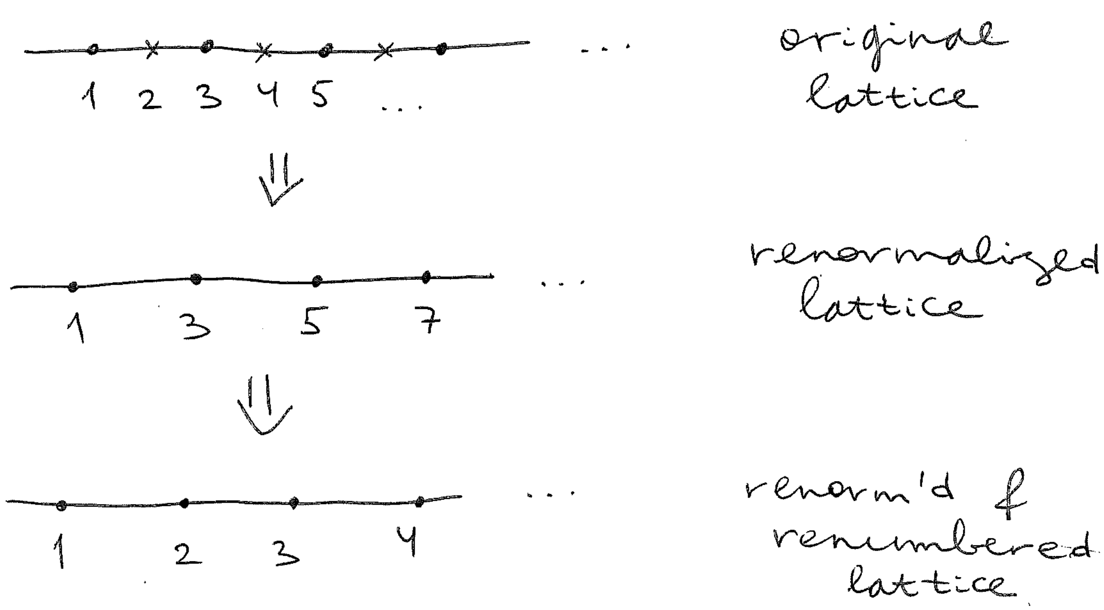
RG in 1D Ising model



1) Write down recursive eq's
2) Extract fixed point structure + linearize around fixed points
3) Get a scaling form of free en.

Consider
$$\mathcal{H} = -\frac{K}{2}\sum_{i=0}^{N-1}\left(s_i s_{i+1} + s_i s_{i-1}\right) - h\sum_{i=0}^{N-1} s_i - \sum_{i=0}^{N-1} C$$
← "β included" → ; some const

(b=2) ⇒ perform the trace in Z over even-numbered spins:

$$Z = \sum_{\{s\}} \prod_{i=\dots 2,4,6,\dots} e^{K s_i (s_{i-1}+s_{i+1}) + h s_i + h\frac{s_{i-1}+s_{i+1}}{2} + 2C}$$

Indeed, consider all coupling terms with s_3:

"old way": $\frac{K}{2} s_2 s_3 + \frac{K}{2}(s_3 s_2 + s_3 s_4) + \frac{K}{2} s_4 s_3$

"new way": $K s_2 s_3 + K s_4 s_3$

Likewise, for s_4:

"old" $\frac{K}{2} s_3 s_4 + \frac{K}{2}(s_4 s_3 + s_4 s_5) + \frac{K}{2} s_4 s_5$

"new" $K(s_4 s_3 + s_4 s_5)$

Now do the partial trace:

$s_i = +1$

$$Z' = \sum_{\ldots s_1, s_3, s_5, \ldots} \prod_{i = \ldots 2,4,6,\ldots} \Big\{ e^{K(s_{i-1}+s_{i+1}) + h + h\frac{s_{i-1}+s_{i+1}}{2} + 2C} + e^{-K(s_{i-1}+s_{i+1}) - h + h\frac{s_{i-1}+s_{i+1}}{2} + 2C} \Big\} =$$

$s_i = -1$

relabel spins consecutively

$$= \sum_{\{s\}} \prod_i \Big\{ e^{(K+\frac{h}{2})(s_i + s_{i+1}) + h + 2C} + e^{-(K-\frac{h}{2})(s_i+s_{i+1}) - h + 2C} \Big\}$$

"old" s_3, s_5 becomes "new" s_2, s_3, etc.

Now, we require that

$$Z' = \sum_{\{s\}} \prod_i e^{K' s_i s_{i+1} + h' s_i + C'}$$

Then $h' \sum_i s_i \Rightarrow h' \sum_i \frac{s_i + s_{i+1}}{2}$

$$e^{K' s_i s_{i+1} + h' \frac{s_i + s_{i+1}}{2} + C'} =$$

$$= e^{(K+\frac{h}{2})(s_i+s_{i+1}) + h + 2C} + e^{-(K-\frac{h}{2})(s_i+s_{i+1}) - h + 2C}$$

We have 3 eq's:

$$\begin{cases} S_i=1,\ S_{i+1}=1: & e^{K'+h'+C'} = e^{2K+2h+2C} + e^{-2K+2C}, & (1) \\ S_i=-1,\ S_{i+1}=-1: & e^{K'-h'+C'} = e^{-2K+2C} + e^{2K-2h+2C}, & (2) \\ S_i=1,\ S_{i+1}=-1 \text{ OR } S_i=-1,\ S_{i+1}=1 & e^{-K'+C'} = e^{h+2C} + e^{-h+2C} & (3) \end{cases}$$

Divide (1) by (2):

$$e^{2h'} = \frac{e^{2K+2h+\cancel{2C}} + e^{-2K+\cancel{2C}}}{e^{2K-2h+\cancel{2C}} + e^{-2K+\cancel{2C}}} =$$

$$= \frac{e^{2K+h} + e^{-2K-h}}{e^{2K-h} + e^{-2K+h}} \frac{e^{h}}{e^{-h}} = e^{2h} \frac{\cosh(2K+h)}{\cosh(2K-h)}.$$

Likewise,

$$\begin{cases} e^{4K'} = \dfrac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2 h}, \\ e^{4C'} = e^{8C} \cosh(2K+h)\cosh(2K-h) \times \\ \quad \times \cosh^2(h) \end{cases}$$

Introduce
$$\begin{cases} w = e^{-4C}, \\ x = e^{-4K}, \\ y = e^{-2h} \end{cases}$$

Then e.g.

$$\frac{1}{y'} = \frac{1}{y} \frac{\frac{1}{\sqrt{x}}\frac{1}{\sqrt{y}} + \sqrt{x}\sqrt{y}}{\frac{1}{\sqrt{x}}\sqrt{y} + \sqrt{x}\frac{1}{\sqrt{y}}} =$$

$$= \frac{1}{y} \frac{1+xy}{y+x}, \text{ or}$$

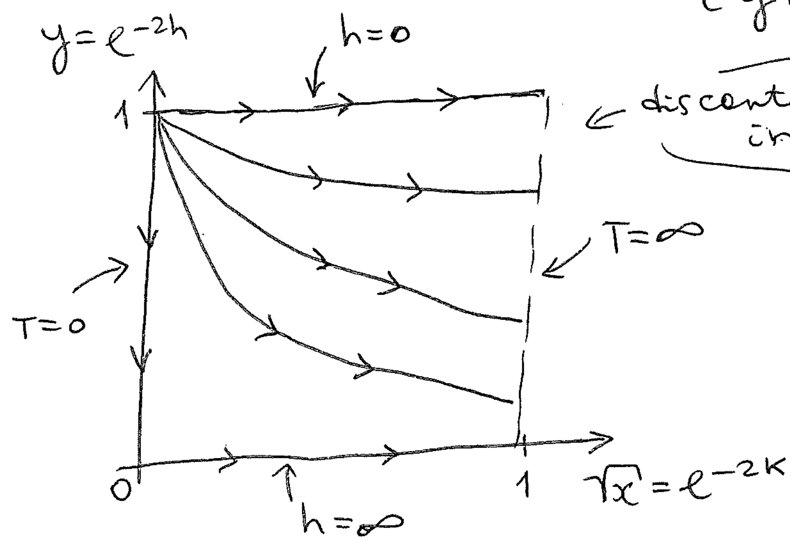
$$\underline{\underline{y' = y\,\frac{x+y}{1+xy}}}.$$

Likewise,
$$\underline{\underline{\begin{cases} w' = \dfrac{w^2 x y^2 (1+y)^2}{(x+y)(1+xy)}, \\ x' = \dfrac{x(1+y)^2}{(x+y)(1+xy)}. \end{cases}}}$$

Note that x' & y' do not depend on w.

So focus on the (x,y) plane first.

Consider $K>0$, $h>0 \Rightarrow \begin{cases} x \in [0,1], \\ y \in [0,1] \end{cases}$



← discontinuous trajectories in reality, w/ ∀ init. cond'n

Indeed,
$$\begin{cases} y^{*\prime} = y^*, \\ x^{*\prime} = x^* = 1. \end{cases}$$

$\nearrow x^* = 1$

Line of fixed points at $\begin{cases} \sqrt{x^*} = 1, \\ 0 \le y^* \le 1 \end{cases}$ ← $T=\infty$ $(\beta = 0)$

Paramagnetic fixed points:

all trajecteries that start at $x>0$ $(T>0)$ flow to them. This is to be expected since $T_c=0$ in 1D Ising model & thus the system looks completely disordered on large enough scales.

$T=0,\ h=\infty$

Another fixed point: $(x^*=0,\ y^*=0)$
Indeed, you can never move off it...
Fully aligned configuration...

Finally, $(x^*=0,\ y^*=1) \Leftarrow T=0,\ h=0$

$$\begin{cases} y^{*\prime}=y^{*2}, & \Leftarrow y^{*\prime}=1 \\ x^{*\prime}=x^*\dfrac{(1+y^*)^2}{y^*}=4x^* & \Leftarrow x^{*\prime}=0 \end{cases}$$

Ferromagnetic fixed point...
Long-range correlations, critical behavior
Two variables, h & T, must be set to correct values before the system becomes critical. $\Leftarrow$ "least stable fixed point" since everything flows away from it.

Linearize the flow eq's around $(\overset{x^*}{0},\overset{y^*}{1})$:

$$\begin{cases} x'\sim 4x, \\ (y-1)'\sim 2(y-1) \Leftarrow \epsilon'\sim 2\epsilon \end{cases} \qquad \epsilon=y-y^*=y-1$$

$y'\sim 2y-1$

Indeed, if $\begin{cases} x=\alpha \\ y=1+\epsilon \end{cases} \Rightarrow y'\simeq y(1+\alpha+\epsilon)(1-\alpha)=$

$1+xy\to 1+\alpha$

$\simeq y(1+\alpha+\epsilon-\alpha)=$
$=y(1+\epsilon)=$
$(1+\epsilon)\ =1+2\epsilon.$

$y' = 1+\epsilon' \Rightarrow \quad \epsilon' \approx 2\epsilon$, as above

Total free en. must stay const:

$\bar{f} = \beta f$ (reduced free en. per spin)

transforms as

$$\bar{f}' = b^d \bar{f} \qquad \left(\begin{matrix} b=2, \\ d=1 \end{matrix} \text{ here}\right)$$

More explicitly,

$$\bar{f} = b^{-1} \bar{f}(\underbrace{b^2 x}_{4x \sim x'}, \underbrace{b\epsilon}_{2\epsilon \sim \epsilon'})$$

We can iterate this to make b have any value (powers of 2 + analytic contin'n)

Hence choose $b^2 x = 1$:

$$\bar{f}(x,\epsilon) = \sqrt{x}\, \bar{f}\left(1, \frac{\epsilon}{\sqrt{x}}\right) = \sqrt{x}\, \tilde{f}\left(\frac{\epsilon}{\sqrt{x}}\right) \qquad (*)$$

Recall that

$$f = -k_B T \log \left\{ e^{\beta J} \cosh(\beta H) + \sqrt{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}} \right\}$$

from transfer matrix

expand f in ϵ, x:

$$\bar{f}(x,\epsilon) = -K \underbrace{- \sqrt{x}\sqrt{1+\frac{\epsilon^2}{x}}}$$

singular part,

agrees with $(*)$ if $\tilde{f}(z) = \sqrt{1+z^2}$.

So, $(*)$ predicts a scaling form near T_c

but $\underline{\text{does not}}$ give $\bar{f}(z)$ explicitly.

We can get the free en. as well; consider the $H=0$ case for simplicity.

Consider
$$\underbrace{\bar{f}_{\bullet}(x_0)}_{\text{orig. prms}} - \underbrace{\overset{=0}{C_0}}_{\text{const part split off explicitly}} = 2^{-\ell}[\underbrace{\bar{f}(x_\ell)}_{\text{renormalized prms}} - C_\ell]$$

Note that
$$e^{4C'} = e^{8C} \underbrace{f(K, \overset{=0 \text{ here}}{h})}_{\cosh^2(2K)}$$

$$C_\ell = 2C_{\ell-1} + \underbrace{\tfrac{1}{4}\log f(K_{\ell-1}, \overset{=0}{h_{\ell-1}})}_{\text{"}R(K_{\ell-1})}$$

$$\tfrac{1}{2}C_\ell = \underbrace{C_{\ell-1}}_{2C_{\ell-2} + R(K_{\ell-2})} + \tfrac{1}{2}R(K_{\ell-1}) \quad , \text{ or}$$

$$\overset{2^{-2}}{\widetilde{\tfrac{1}{4}}} C_\ell = C_{\ell-2} + \tfrac{1}{2}R(K_{\ell-2}) + \tfrac{1}{4}R(K_{\ell-1}) \text{ etc.}$$

In general, $2^{-\ell}C_\ell = \underset{\overset{\shortparallel}{0}}{C_0} + \sum_{k=0}^{\ell-1} 2^{-(k+1)} R(\underbrace{K_k}_{\text{"}x_k\text{"}})$

So, $$\bar{f}(x_0) = 2^{-\ell}\underbrace{\bar{f}(x_\ell)}_{\substack{\text{for large } \ell, \text{ end up} \\ \text{at a "trivial" fixed} \\ \text{point where } \bar{f}(x_\ell) \\ \text{can be calculated}}} - \sum_{k=0}^{\ell-1} 2^{-(k+1)} R(x_k)$$

$R(x_k)$ can be also computed along the flow lines ... $\Rightarrow$ get $\bar{f}(x_0)$ in the end.

1D Ising model: flow to $(y^* = 1, x^* = 1)$, $T=\infty$

no field!

a paramagnetic fixed point:

$$\bar{f}_*(x_\ell) = \log 2$$

as previously defined

$$w_\ell = e^{-4C_\ell} \Rightarrow \bar{f}(x_\ell) - C_\ell = \log 2 + \log w_\ell^{1/4} =$$

$$\frac{1}{w_\ell^{\frac{1}{4}}} = e^{C_\ell}$$

$$= \log\left(\frac{w_\ell^{1/4}}{2}\right).$$

$$-\log w_\ell^{\frac{1}{4}} = C_\ell,$$

$$-C_\ell = \log w_\ell^{1/4}$$

Then $\bar{f}(x_0) = \lim_{\ell \to \infty} 2^{-\ell} \log\left(\frac{w_\ell^{1/4}}{2}\right).$

Now, use $(y=1)$

$$\begin{cases} w' = \dfrac{4w^2 x}{(x+1)^2}, \\ x' = \dfrac{4x}{(x+1)^2}. \end{cases}$$

Define

$$\begin{cases} u = \dfrac{(wx)^{1/4}}{2}, \\ v = \dfrac{1-\sqrt{x}}{1+\sqrt{x}} = \tanh(K) \end{cases}$$

$$v^2 = \frac{(1-\sqrt{x})^2}{(1+\sqrt{x})^2} = \frac{1-2\sqrt{x}+x}{1+2\sqrt{x}+x}$$

$$v' = \frac{1-\sqrt{x'}}{1+\sqrt{x'}} = \frac{1-\frac{2\sqrt{x}}{x+1}}{1+\frac{2\sqrt{x}}{x+1}} = \frac{x+1-2\sqrt{x}}{x+1+2\sqrt{x}}$$

So, $v' = v^2$ (*)

Likewise,

$$u' = \frac{u^2(1+v)^2}{1+v^2} \qquad (**)$$

$$l \to \infty \Rightarrow x_l \to 1 \Rightarrow \frac{v_l^{1/4}}{2} \to u_l .$$

$$\text{So,} \quad \bar{f}(x_0) = \lim_{l\to\infty} 2^{-l} \log(u_l)$$

Iterate (*) & (**):

$$\log u_l = 2\log u_{l-1} + 2\log(1+v_{l-1}) -$$

$$-\log(1+v_{l-1}^2) =$$

$$= 2\log\left[\frac{u_{l-2}^2(1+v_{l-2})^2}{1+v_{l-2}^2}\right] +$$

$$+2\log(1+v_{l-2}^2) - \log(1+v_{l-2}^4) =$$

$$= 2^2\log u_{l-2} + 2^2\log(1+v_{l-2}) -$$

$$-\log(1+v_{l-2}^{2^2}) = \dots =$$

$$= 2^l \log u_0 + 2^l \log(1+v_0) - \log(1+v_0^{2^l})$$

Recall that $v = \tanh K < 1$ for finite T:

$$\lim_{l\to\infty} (1+v_0^{2^l}) = 0$$

$$\text{Thus} \quad \bar{f}(x_0) = \log u_0(1+v_0)$$

Now, $u_0 = \frac{(\lambda_0 x_0)^{1/4}}{2} = \frac{1}{2}(\underbrace{e^{-4c_0}}_{1} e^{-4K})^{1/4} =$

$$= \frac{1}{2} e^{-K}.$$

So, $\bar{f}(x_0) = \log\left[\frac{1}{2} e^{-K}\left(1 + \frac{e^{K} - e^{-K}}{e^{K} + e^{-K}}\right)\right] =$

$$= \log\left[\frac{1}{2} \cancel{e^{-K}} \frac{2\cancel{e^{K}}}{\underbrace{e^{K} + e^{-K}}_{\cosh(K)}}\right] = -\log[2\cosh(K)].$$

From transfer matrix:

$$\bar{f} \underset{H=0}{=} -\log\{e^{K} + e^{-K}\} = -\log\{2\cosh(K)\},$$

Same result!