

# INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2026

## Answers to Questions III. Oct. 20th

1. (a) The eigenvalues are as follows:

$$\begin{aligned} |1_{\mathbf{k}}\rangle &= b_{\mathbf{k}}^{\dagger}|0\rangle, & \mathcal{E}_{1_{\mathbf{k}}} &= \epsilon_{\mathbf{k}} \\ |2_{\mathbf{k}}\rangle &= \frac{1}{\sqrt{2!}}(b_{\mathbf{k}}^{\dagger})^2|0\rangle, & \mathcal{E}_{2_{\mathbf{k}}} &= 2\epsilon_{\mathbf{k}} \\ |3_{\mathbf{k}}\rangle &= \frac{1}{\sqrt{3!}}(b_{\mathbf{k}}^{\dagger})^3|0\rangle, & \mathcal{E}_{3_{\mathbf{k}}} &= 3\epsilon_{\mathbf{k}} + U \end{aligned} \quad (1)$$

Notice that the excitation energies are  $\epsilon_{\mathbf{k}} = \mathcal{E}_{1_{\mathbf{k}}} = \mathcal{E}_{2_{\mathbf{k}}} - \mathcal{E}_{1_{\mathbf{k}}}$  and  $\epsilon_{\mathbf{k}} + U = \mathcal{E}_{3_{\mathbf{k}}} - \mathcal{E}_{2_{\mathbf{k}}}$ .

- (b) If we can ignore occupancies higher than three, then the partition function is

$$Z = \prod_{\mathbf{k}} (1 + e^{-\beta\epsilon_{\mathbf{k}}} + e^{-2\beta\epsilon_{\mathbf{k}}} + e^{-\beta(3\epsilon_{\mathbf{k}}+U)})$$

so that the free energy is

$$F = \sum_{\mathbf{k}} F_{\mathbf{k}} = -k_B T \sum_{\mathbf{k}} \ln \left[ 1 + e^{-\beta\epsilon_{\mathbf{k}}} + e^{-2\beta\epsilon_{\mathbf{k}}} + e^{-\beta(3\epsilon_{\mathbf{k}}+U)} \right]$$

- (c) The occupancy of the  $\mathbf{k}$  state is

$$n_{\mathbf{k}} = \langle n_{\mathbf{k}} \rangle = -\frac{\partial F_{\mathbf{k}}}{\partial \mu} = \frac{e^{-\beta\epsilon_{\mathbf{k}}} + 2e^{-2\beta\epsilon_{\mathbf{k}}} + 3e^{-\beta(3\epsilon_{\mathbf{k}}+U)}}{1 + e^{-\beta\epsilon_{\mathbf{k}}} + e^{-2\beta\epsilon_{\mathbf{k}}} + e^{-\beta(3\epsilon_{\mathbf{k}}+U)}}$$

- (d) Let us plot  $n_{\mathbf{k}}$  at low temperatures. There are three regions to consider:

- $\epsilon_{\mathbf{k}} > 0$ ,  $n_{\mathbf{k}} = 0$ .
- $\epsilon_{\mathbf{k}} < 0$ , but  $\epsilon_{\mathbf{k}} + U > 0$ ,  $n_{\mathbf{k}} = 2$ .
- $\epsilon_{\mathbf{k}} + U < 0$  and  $n_{\mathbf{k}} = 3$ . so that there are two “Fermi surfaces” (see Fig. 1).

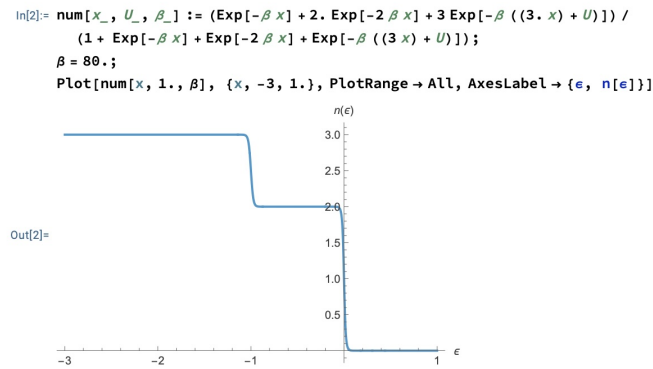


Figure 1: The occupancy versus  $\epsilon_{\mathbf{k}}$ , showing two Fermi surfaces.

2. (a) We may estimate the Bose Einstein transition temperature from

$$T_{BE} = \frac{3.31}{k_B} \left( \frac{\hbar^2 n^{2/3}}{m} \right) = \frac{3.31}{1.38 \times 10^{-23}} \left( \frac{\hbar^2 (10^{21} m^{-3})^{2/3}}{23m_p} \right) \approx 6.9 \mu K.$$

These tiny temperatures are attained by “evaporative cooling” . Sodium atoms are held in a “magneto-optic” trap. Radio waves are used to “evaporate” the most energetic atoms in the trap, leaving behind the cold ones.

- (b) In Helium-4, we may estimate the Bose Einstein transition temperature as

$$T_{BE} = \frac{3.31}{k_B} \left( \frac{\hbar^2 n^{2/3}}{m_{He}} \right) = \frac{3.31}{1.38 \times 10^{-23}} \left( \frac{\hbar^2 ((122/(4m_p)))^{2/3}}{4m_p} \right) \approx 2.76 K.$$

The actual condensation temperature is 2.21K. The difference in condensation temperatures is due to the repulsive interaction between atoms.

3. (a) If the interaction has the form

$$V(r) = \begin{cases} U, & (r < R), \\ 0, & (r > R), \end{cases} \quad (2)$$

then in second-quantized form, the interaction Hamiltonian is

$$V = \frac{U}{2} \sum_{\sigma, \sigma'} \int d^3x \int_{|\vec{x}' - \vec{x}| < R} d^3x' [\psi^\dagger_\sigma(x) \psi^\dagger_{\sigma'}(x') \psi_{\sigma'}(x') \psi_\sigma(x)]. \quad (3)$$

- (ii) Inverting the Fourier transform, we have  $c_{\vec{k}\sigma} = \int d^3x \psi_\sigma(\vec{x}) e^{-i\vec{k}\cdot\vec{x}}$ , so that

$$\begin{aligned} [c_{\vec{k}\sigma}, c^\dagger_{\vec{k}'\sigma'}]_{\pm} &= \int d^3x d^3x' [\psi_\sigma(x), \psi^\dagger_{\sigma'}(x')]_{\pm} e^{-i(\vec{k}\cdot\vec{x} - \vec{k}'\cdot\vec{x}')} \\ &= \delta_{\sigma\sigma'} \int d^3x e^{-i(\vec{k} - \vec{k}')\cdot\vec{x}} \\ &= \delta_{\sigma\sigma'} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}'). \end{aligned} \quad (4)$$

- (iii) In momentum space, we may write

$$V = \frac{1}{2} \int \frac{d^3k d^3k' d^3q}{(2\pi)^9} V(q) \left[ c^\dagger_{\vec{k}+\vec{q}\sigma} c^\dagger_{\vec{k}'-\vec{q}\sigma'} c_{\vec{k}'\sigma'} c_{\vec{k}\sigma} \right], \quad (5)$$

where

$$V(\vec{q}) = \int d^3x V(\vec{x}) e^{i\vec{q}\cdot\vec{x}} = \frac{4\pi U}{q} \int_0^R dr r \sin(qr) = \left( \frac{4\pi R^3 U}{3} \right) F(qR) \quad (6)$$

and

$$F(x) = \frac{3}{x^2} \left[ \frac{\sin x}{x} - \cos x \right]. \quad (7)$$

The form of the interaction in momentum space is sketched above. The hard core in real space is manifested as a long-range oscillatory component in momentum space.

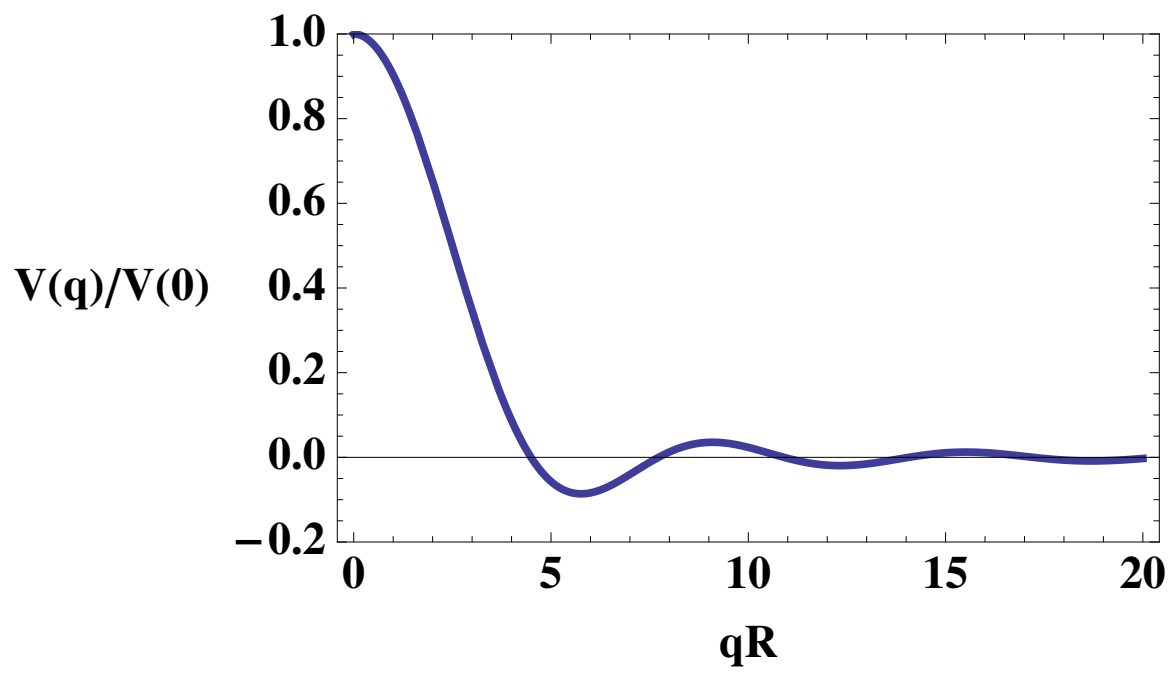


Figure 2: Fourier transformed potential  $V(q)$  for “hard sphere” potential.